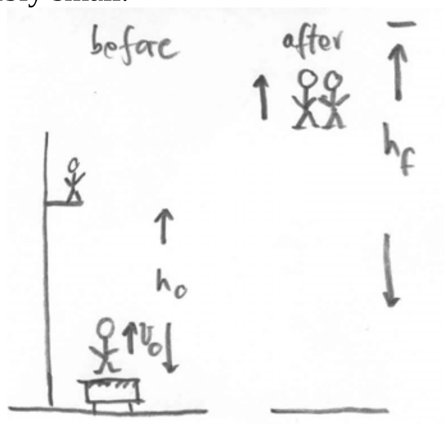
Problem 24: Two acrobats.

An acrobat of mass m_A jumps upwards off a trampoline with an initial velocity v_0. At a height h_0, the acrobat grabs a clown of mass m_B. Assume that the time the acrobat takes to grab the clown is negligibly small.



a) What is the velocity of the acrobat immediately before grabbing the clown?

b) What is the velocity of the acrobat immediately after grabbing the clown?

c) How high do the acrobat and clown rise? How high would the acrobat go in the limit that the mass of the acrobat is much heavier than the mass of the clown?

d) If the acrobat missed the clown, how high would the acrobat go? How does this compare to your limit in part c)?

Solution:

Modeling the problem: The first important observation to make is that there is a collision between the acrobat and the clown. This collision is completely inelastic in that two bodies collide and “stick’ together after the collision. Since we are not given any specific details of the collision, we cannot describe the microdynamical description of how the acrobat slows down and how the clown speeds up. However if we choose as our system the acrobat, clown, and the earth, then the details of the collision are determined by internal forces.

We can identify four states in this problem. We need to specific the position and velocities of the acrobat and the clown in each of these states. Since this is essentially one dimensional motion, let’s choose an origin at the trampoline and the positive y-axis upwards.

State 1: Acrobat A just leaves trampoline with initial velocity $\vec{\mathbf{v}}_{1,A} = v_0\hat{\mathbf{j}}$, and initial position $y_{1,A} = 0$. Denote this time by $t_1 = 0$. Clown B is at rest at the position $y_{1,B} = h_0$.

State 2: Acrobat A just arrives at platform located at $y_{2,A} = y_{2,B} = h_0$ with velocity $\vec{\mathbf{v}}_{2,A} = v_{2,A}\hat{\mathbf{j}}$, immediately before grabbing Clown B. Denote this time by t_2.

State 3: The collision lasts a time Δt_{col}. During this time interval acrobat A grabs Clown B and the two acrobats and at the end of the interval rise together with velocity $\vec{\mathbf{v}}_3 = v_3\hat{\mathbf{j}}$. Denote the time at the end of this interval by $t_3 = t_2 + \Delta t_{col}$.The key assumption is that the collision time is instantaneous $\Delta t_{col} \cong 0$.

State 4: The two acrobats rise to a final height $y_f = h_f$ and hence are at rest, $\vec{\mathbf{v}}_4 = \vec{\mathbf{0}}$.

Model:

Since there are no external forces doing work between states 1 and 2, and states 3 and 4 the mechanical energy is constant between these states.

There are no external forces acting during the time interval between states 2 and 3. Therefore during the collision, the total momentum of the system is constant. The change in the earth's momentum is negligible but there is some slowing down of the acrobat A during the collision. If the collision lasts a significant length of time, then we need to calculate this effect. However by assuming the collision is instantaneous, we can ignore this slowing down, and then the change in momentum of the two acrobats before and after the collision is zero.

State 1 to 2:

Choose zero potential energy $U(y=0)=0$. The mechanical energy in state 1 is

$$E_1 = K_1 + U_1 = \frac{1}{2}m_A v_0^2 + m_B g h_0 .$$

The mechanical energy is state 2 is

$$E_2 = K_2 + U_2 = \frac{1}{2}m_A v_{2,A}^2 + m_A g h_0 + m_B g h_0 .$$

So

$$E_2 - E_1 = \frac{1}{2}m_A v_{2,A}^2 + m_A g h_0 - \frac{1}{2}m_A v_0^2 = 0 .$$

We can solve this for the speed of acrobat A in state2.

$$v_{2,A} = (v_0^2 - 2gh_0)^{1/2} .$$

State 2 to State 3:

Momentum in state 2 is only due to acrobat A

$$\vec{\mathbf{p}}_{2,A} = m_A v_{2,A} \vec{\mathbf{j}} = m_A (v_0^2 - 2gh_0)^{1/2} \vec{\mathbf{j}} .$$

The momentum in state 3 is

$$\vec{\mathbf{p}}_3 = (m_A + m_B) v_3 \vec{\mathbf{j}} .$$

Since momentum is unchanged,

$$m_A (v_0^2 - 2gh_0)^{1/2} \vec{\mathbf{j}} = (m_A + m_B) v_3 \vec{\mathbf{j}} .$$

Wed can solve this equation for the speed of the acrobat and the clown after the collision,

$$v_3 = \frac{m_A}{m_A + m_B} (v_0^2 - 2gh_0)^{1/2} .$$

State 3 to State 4:

The mechanical energy in state 3 is

$$E_3 = K_3 + U_3 = \frac{1}{2}(m_A + m_B) v_3^2 + (m_A + m_B) g h_0 .$$

The mechanical energy is state 4 is

$$E_4 = U_4 = (m_A + m_B) g h_f .$$

So

$$E_4 - E_3 = \frac{1}{2}(m_A + m_B) v_3^2 + (m_A + m_B) g h_0 - (m_A + m_B) g h_f = 0 .$$

We can solve this for the final height of the acrobat and the clown

$$h_f = \frac{1}{2g} v_3^2 + h_0 .$$

We can use our result form state 2 to state 3 for the speed after the collision to find the final height in terms of the speed of acrobat A instate 1 and the initial height of clown B,

$$h_f = \frac{1}{2g} \frac{m_A{}^2}{(m_A + m_B)^2} (v_0^2 - 2gh_0) + h_0 = \frac{1}{2g} \frac{m_A{}^2}{(m_A + m_B)^2} v_0^2 + m_B (2m_A + m_B) gh_0 .$$

When the mass of the acrobat is much greater than the mass of the clown $m_A > m_B$, the mass ratio becomes $\frac{m_A{}^2}{(m_A + m_B)^2} \cong 1$ and so the height becomes

$$h_f = \frac{1}{2g} \frac{m_A{}^2}{(m_A + m_B)^2} (v_0^2 - 2gh_0) + h_0 \cong \frac{1}{2g} (v_0^2 - 2gh_0) + h_0 = \frac{1}{2g} v_0^2$$

d) If the acrobat misses the clown, then the mechanical energy does not change between the ground and the top, so let's choose only the acrobat and earth as our system, then the initial mechanical energy is

$$E_1 = K_1 = \frac{1}{2} m_A v_0^2$$

The final mechanical energy when the acrobat is at the highest point is

$$E_f = U_f = m_A g h_f{}' .$$

Since mechanical energy is constant

$$E_f - E_1 = m_A g h_f{}' - \frac{1}{2} m_A v_0^2 = 0 .$$

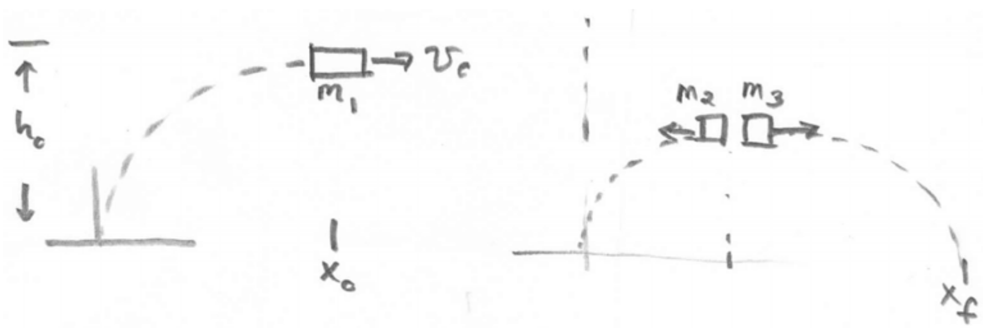
Solve for the final height

$$h_f{}' = \frac{1}{2g} v_0^2 .$$

This agree in the limit when the mass of the acrobat is much greater than the mass of the clown that you found in part c).

Problem 25: Exploding Projectile

An instrument-carrying projectile of mass m_1 accidentally explodes at the top of its trajectory, a height h_0 above the ground. The horizontal distance between launch point and the explosion is x_0. Let v_0 denote the horizontal velocity of the projectile at the top of its trajectory. The projectile breaks into two pieces which fly apart horizontally. The larger piece, m_3, has three times the mass of the smaller piece, m_2. To the surprise of the scientist in charge, the smaller piece returns to earth at the launching station. Neglect air resistance and effects due to the earth's curvature.



a) Find how far the heavier projectile goes using just center of mass arguments.

b) What is the velocity of the projectile at the top of its flight just before the collision?

c) What is the velocity of the smaller piece just after the collision?

d) What is the velocity of the larger piece just after the collision?

e) How far away, x_f, from the original launching point does the larger piece land?

Solution:

a) Since the explosive forces are all internal, the center of mass of the system will travel a distance $x_{cm,f} = 2x_0$ when it hits the ground. The final center of mass is given by

$$x_{cm,f} = \frac{m_2 x_{2,f} + m_3 x_{3,f}}{m_2 + m_3}.$$

Since the smaller object returns to the origin, the final position is given by $x_{2,f} = 0$. Therefore the center of mass becomes

$$2x_0 = \frac{m_3 x_{3,f}}{m_2 + m_3}.$$

Solve for the final position of the heavier object

$$x_{3,f} = \frac{m_2 + m_3}{m_3} 2x_0 .$$

Since mass is conserved, $m_1 = m_2 + m_3$, and the heavier fragment is three times the mass of the lighter piece, $m_3 = 3m_2$. Therefore $m_2 = (1/4)m_1$ and $m_3 = (3/4)m_1$.

Therefore

$$x_{3,f} = \frac{(1/4)m_1 + (3/4)m_1}{(3/4)m_1} 2x_0 = \frac{8}{3} x_0$$

part b)

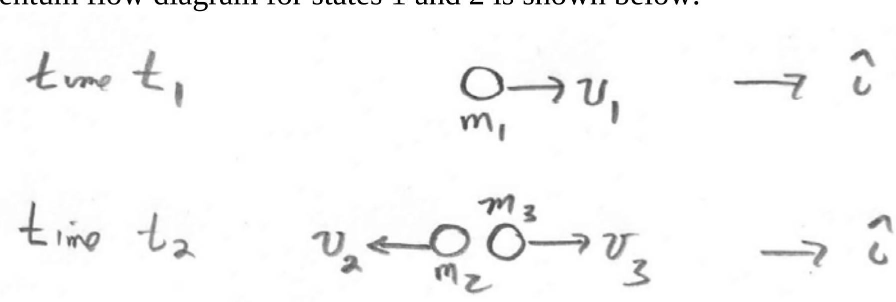
We begin by identifying various states.

State 0, time t_0: The projectile is launched.
State 1 time t_1: The projectile is at the top of it's flight immediately before the explosion. The mass is m_1 and the velocity of the projectile is v_1.

State 2 time t_2: Immediately after the explosion, the projectile breaks into two pieces

State 3 time t_f: The two pieces strike the ground. (They both take the same amount of time to reach the ground since they are falling from the same height and both have no initial velocity in the vertical direction).

The momentum flow diagram for states 1 and 2 is shown below.



The initial momentum before the explosion is

$$p_{x,0}^{total} = m_1 v_1 .$$

The momentum immediately after the explosion is

$$p_{x,f}^{total} = -m_2 v_2 + m_3 v_3 .$$

Since the collision is instantaneous, momentum is conserved in the horizontal direction,

$$p_{x,0}^{total} = p_{x,f}^{total} .$$

If the collision were not instantaneous, then the masses would descend during the explosion and the velocities would no longer have only x-components. So the condition for conservation of momentum in the x-direction is:

$$m_1 v_1 = -m_2 v_2 + m_3 v_3$$

Since mass is conserved, $m_1 = m_2 + m_3$, and the heavier fragment is three times the mass of the lighter piece, $m_3 = 3m_2$. Therefore $m_2 = (1/4)m_1$ and $m_3 = (3/4)m_1$.

There are still two unknown, v_2 and v_3. However there is an additional piece of information: the lighter mass returns exactly to the starting position. This implies that $v_2 = v_1$. Recall from our study of projectile motion, that the horizontal distance is given by $x_0 = v_1 t_1$, independent of the mass. The time that it takes the lighter mass to hit the ground is the same as the time it takes the original projectile to reach the top of its flight (neglecting air resistance). Therefore the velocities must be the same since they traveled the same distance. So the conservation of momentum equation is now,

$$m_1 v_1 = -\frac{1}{4} m_1 v_1 + \frac{3}{4} m_1 v_3 .$$

This equation can now solved for the velocity of the larger piece immediately after the collision,

$$v_3 = \frac{5}{3} v_1 .$$

The larger piece also takes the same amount of time t_1 to hit he ground as the smaller piece. Hence it travels a distance

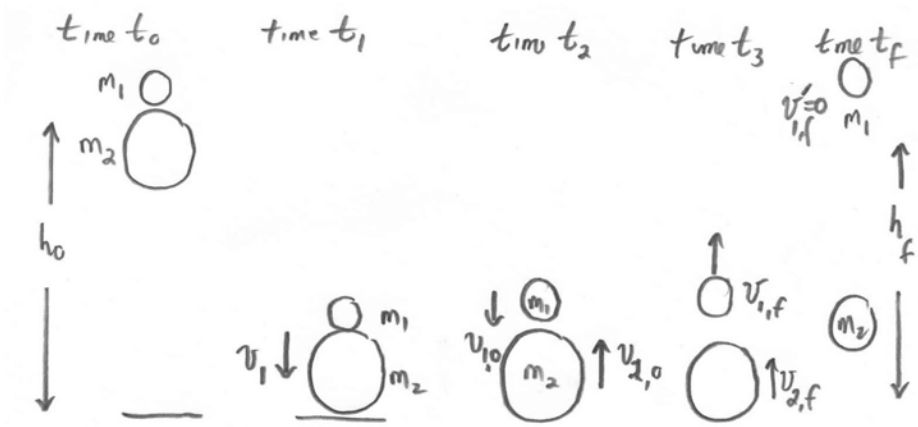
$$x_3 = v_3 t_1 = \frac{5}{3} v_1 t_1 = \frac{5}{3} x_0 .$$

Therefore the total distance the larger piece traveled from the launching station is

$$x_f = x_0 + \frac{5}{3}x_0 = \frac{8}{3}x_0 .$$

Problem 26: Bouncing Superballs

Two superballs are dropped from a height above the ground. The ball on top has a mass m_1. The ball on the bottom has a mass m_2. Assume that the lower ball collides elastically with the ground. Then as the lower ball starts to move upward, it collides elastically with the upper ball that is still moving downwards. How high will the upper ball rebound in the air? Assume that $m_2 >> m_1$. Hint: consider this collision from an inertial reference frame that moves upward with the same speed as the lower ball has after it collides with ground. What speed does the upper ball have in this reference frame after it collides with the lower ball?



Solution: There are five states to this motion.

Initial State time t_0: Superballs are released a height h_0 above the ground

State 1 time t_1: Superballs just reach the ground with velocity $v_{1,0}$

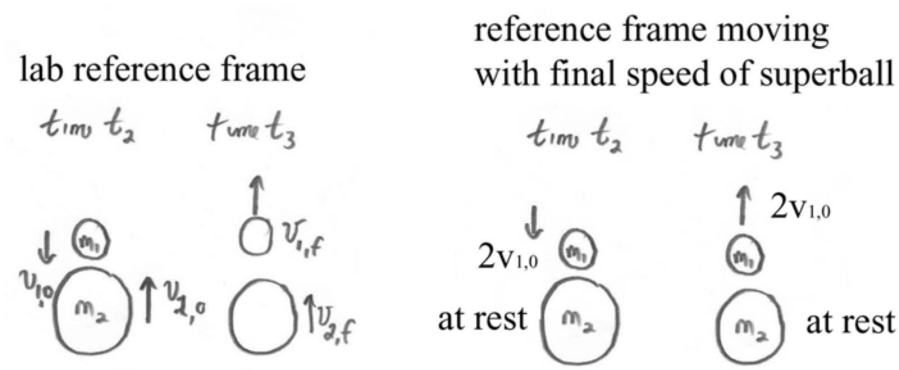
State 2 time t_2: Immediately before collision of large superball and small superball. Large superball collides elastically with ground and reverses direction with same magnitude of velocity, $v_{2,0} = v_{1,0}$. Small superball still moving down with velocity $v_{1,0}$.

State 3 time t_3: Immediately after collision of large superball and small superball. Small superball moves upward with speed $v_{1,f}$. Large superball moves upward with speed $v_{2,f}$

Final State time t_f: Small superball reaches maximum height h_f above the ground

Choice of Reference Frame:

This collision is best analyzed from the reference frame of the observer moving upward with speed $v_{2,0} = v_{1,0}$, the velocity of m_2 just after it rebounded with the ground. In this frame immediately before the collision, the small superball is moving downward with twice the speed as in the lab frame, $v_{1,0}' = 2v_{1,0}$



Assumption: The mass of the large superball is much heavier than the mass of the small superball, $m_2 >> m_1$. This enables us to consider the collision (states 2 and 3) to be equivalent to the small superball bouncing elastically off a hard wall with essentially no recoil of the large superball.

In the large superball reference frame the large superball is at rest after the collision. Before the collision, m_1 has velocity $v_{1,0}' = 2v_{1,0}$. Since the collision between the two superballs is perfectly elastic, m_1 rebounds with velocity $v_{1,f}' = 2v_{1,f}$.

However, in the lab frame, the small superball is moving with speed

$$v_{1,f} = 2v_{1,0} + v_{1,0} = 3v_{1,0}.$$

Therefore, small superball goes upwards to a height h_f given by

$$-\Delta K_{3,f} = \Delta U_{3,f}$$

which is just the condition that

$$\frac{1}{2}m_1\left(3v_{1,0}\right)^2 = m_1 g h_f$$

Recall that we can also use conservation of energy between the Initial State and State 1 to calculate the velocity of the balls just before they hit the ground,

$$\Delta K_{0,1} = -\Delta U_{0,1}\,.$$

Thus

$$\frac{1}{2} m_1 \left(v_{1,0}\right)^2 = m_1 g h_0\,.$$

We can solve for h_f in terms of h_0 since

$$m_1 g h_f = 9\frac{1}{2} m_1 v_{1,0}{}^2 = 9 m_1 g h_0$$

Thus

$$h_f = 9 h_0\,.$$

PRS Question: Exothermic Reaction

Consider the exothermic reaction (final kinetic energy is greater than the initial kinetic energy).

$$H + H \to H_2 + 5ev$$

Two hydrogen atoms collide and produce a diatomic hydrogen molecule. This reaction

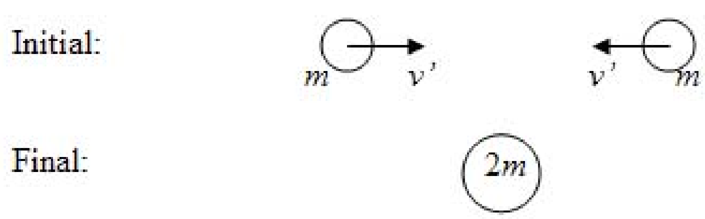
1. is possible.
2. either violates conservation of energy or violates conservation of momentum but not both.
3. conserves conservation of energy and momentum but is not possible for other reasons.

Solution: The reaction

$H + H \to H_2 + 5eV$

is an exothermic reaction. The unit for energy is called the electron-volt. This is the amount of work done by the electric force in moving one electon across a 1 volt electric potential difference. In SI units $1eV = 1.6 \times 10^{-19}\,J$. This means that $K_f > K_0$. Since $\Delta K = K_f - K_0 = W^{nc}$, the non-conservative work is positive, $W^{nc} = 5eV$.

First argument: In the center of mass frame, the total momentum is zero, $\left(\vec{\mathbf{p}}_{total}\right)_{cm} = 0$. Assume that momentum is constant. The momentum flow diagram looks like:



The final velocity in the center of mass frame must be zero because the total momentum is zero. However, $K_f{}' > K_0{}'$ since the reaction is exothermic, so an exothermic reaction in which a two-body collision results in only one final body is impossible unless there is a second final body to carry momentum away.

Similarly if you assume that energy is constant, then the final kinetic energy is greater than the initial kinetic energy so the final molecule must be moving but this violates no change in momentum since the initial momentum is zero in the center of mass frame.

Second argument: In the lab frame, the momentum flow diagram is

Conservation of momentum requires that

$$\vec{\mathbf{p}}_0^{\,total} = \vec{\mathbf{p}}_f^{\,total} \, ,$$

which becomes

$$m\left(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2\right) = 2m\vec{\mathbf{v}}_f \, .$$

Therefore the final velocity is

$$\vec{\mathbf{v}}_f = \frac{1}{2}\left(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2\right).$$

The work-kinetic energy theorem,

$$K_f = K_0 + 5eV$$

becomes

$$\frac{1}{2}(2m)\left(\vec{\mathbf{v}}_f \cdot \vec{\mathbf{v}}_f\right) = \frac{1}{2}m\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1 + \frac{1}{2}m\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2 + 5eV \, .$$

Now substitute for the final velocity, yielding

$$\frac{1}{2}\left(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2\right) \cdot \frac{1}{2}\left(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2\right) = \frac{1}{2}\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1 + \frac{1}{2}\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2 + \frac{5}{m}eV \, .$$

Expanding, this becomes

$$\frac{1}{4}\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1 + \frac{1}{4}\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2 + \frac{1}{2}\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_2 = \frac{1}{2}\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{v}}_1 + \frac{1}{2}\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{v}}_2 + \frac{5}{m}eV \, .$$

A little algebra yields,

$$0=\frac{1}{4}\left(\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_1+\vec{\mathbf{v}}_2\cdot\vec{\mathbf{v}}_2-2\left(\vec{\mathbf{v}}_1\cdot\vec{\mathbf{v}}_2\right)\right)+\frac{5}{m}eV\ .$$

This can be simplified,

$$0=\frac{1}{4}\left(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2\right)\cdot\left(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2\right)+\frac{5}{m}eV\ .$$

Thus the work-kinetic energy theorem requires the condition

$$\frac{1}{4}\left(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2\right)\cdot\left(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2\right)=-\frac{5}{m}eV\ .$$

The left hand side is always positive since dot product of a vector, $\left(\vec{\mathbf{v}}_1-\vec{\mathbf{v}}_2\right)$, with itself is always positive or zero (if the vector is the zero vector). So this is condition is impossible, the reaction as described cannot happen.

Two hydrogen atoms can form a hydrogen molecule provided that there is a third ‘catalyst’ to carry away the momentum (as seen in the center of mass reference frame). Then the reaction can satisfy conservation of energy and momentum.