Physics 8.01T   **Fall Term 2004**

Problem Set 10: Torque, Rotational Dynamics, Physical Pendulum, Angular Momentum; Solutions

Problem 1: (*Moment of Inertia*)

A 1" US Standard Washer has inner radius $r_1 = 1.35 \times 10^{-2}\, m$ and an outer radius $r_2 = 3.10 \times 10^{-2}\, m$. The washer is approximately $d = 4.0 \times 10^{-3}\, m$ thick. The density of the washer is $\rho = 7.8 \times 10^{3}\, kg/m^3$. Calculate the moment of inertia of the washer about an axis passing through the center of mass and show that it is equal to $I_{cm} = \dfrac{1}{2} m_w \left(r_0^2 + r_i^2 \right)$.
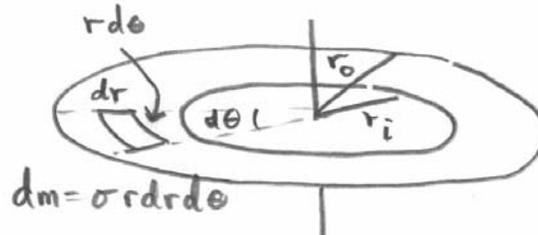
We need to calculate the integral

$$I_{cm} = \int r^2\, dm.$$

There are several ways of approaching this problem. To calculate the integral, we need to choose the mass element, i.e. we need to split the washer into elementary parts. Probably the simplest choice is to take the mass element to be a thin ring of width dr. The mass of this ring is

$$dm = \rho d\, dS = \sigma 2\pi r\, dr,$$

where d is the thickness, $\sigma = \rho d$ is the mass per unit surface area and we used the fact that the ring has small width to approximate its surface area as $dS = 2\pi r\, dr$. Then the integral that we need to calculate is

$$I_{cm} = \int r^2 dm = 2\pi\sigma \int_{r_1}^{r_2} r^2 r\, dr = \frac{\pi\sigma}{2}\left(r_2^4 - r_1^4\right) \approx 4.4 \times 10^{-5} kg \times m^2.$$

To put the answer into the form given in the problem, we need to find the mass of the washer. The mass of the whole is the sum of the masses of the parts, so it is

$$m_w = \int dm = 2\pi\sigma \int_{r_1}^{r_2} r\, dr = \pi\sigma\left(r_2^2 - r_1^2\right) \approx 76\, g.$$

Since $(r_2^4 - r_1^4) = (r_2^2 - r_1^2)(r_1^2 + r_1^2)$, the final answer is

$$I_{cm} = \frac{1}{2} m_w \left(r_1^2 + r_2^2\right).$$

Another way of solving the problem is to choose the mass element to be a small "rectangle" in polar coordinates of radial width dr and angular size $d\theta$. Then the surface area of this element is $dS = r\, d\theta\, dr$ and the moment of inertia is

$$I_{cm} = \int r^2\, dm = \sigma \int_0^{2\pi} d\theta \int_{r_1}^{r_2} dr\, r^2 r$$

The integral over θ is easy to calculate since the integrand does not depend on θ and after that we are left with exactly the same expression as in the previous case.

Yet another way to calculate the moment of inertia is to use the following trick. Consider the washer to be made of two solid disk, one of positive mass and radius r_2 and the other one of negative mass and radius r_1. If you recall that for a solid disk of radius R the moment of inertia is $I = \pi\sigma R^4/2$, you again recover the previous answer.

Problem 2: Experiment 09 Physical Pendulum

Part One: Ruler Pendulum

The ruler has a mass $m_r = 0.159$ kg, a width $a = 0.028$ m, a length $b = 1.00$ m, and the distance from the pivot point to the center of mass is $l = 0.479$ m.
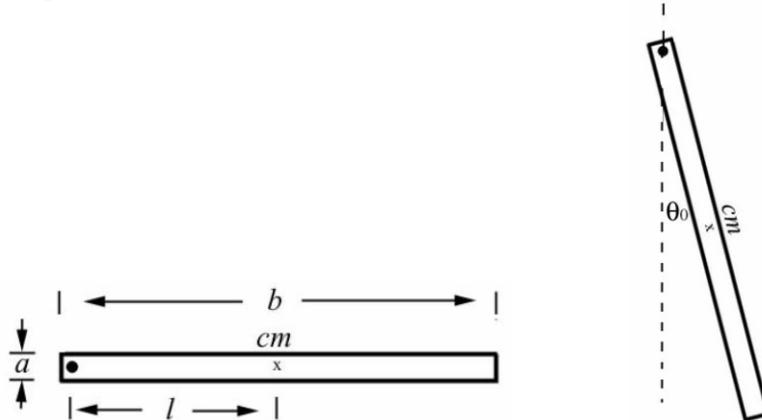
Enter your measured period into the T_{meas} column of the table below and calculate the other entries using the formulas

$$T_{\text{ideal}} = 2\pi\sqrt{l/g}$$

and

$$T_{\text{theory}} = 2\pi\sqrt{l/g}\sqrt{\frac{I}{ml^2}}\left(1+\frac{\theta_0^2}{16}\right)$$

with $g = 9.805$ ms^{-2}.

Solution:

The first part of the analysis of the experiment is to calculate the moment of inertia about an axis passing through the center of mass, perpendicular to the plane formed by the sides of the ruler. In particular, choose Cartesian coordinates with the origin at the center of mass, and the x-axis along the length, and y-axis along the width. The mass per unit area

$$\sigma = \frac{mass}{Area} = \frac{m_1}{ab}.$$

The mass element is rotating about the z-axis in a circular orbit with radius $r_\perp = \left(x^2 + y^2\right)^{1/2}$, so the moment of inertia about the center of mass of the ruler is

$$I_{cm} = \int_{body} (r_{\perp,cm})^2 \, dm = \sigma \int_{x=-b/2}^{x=b/2} \left(\int_{y=-a/2}^{y=a/2} \left(x^2 + y^2 \right) dy \right) dx$$

We first do the integral in the y-direction,

$$I_{cm} = \frac{m_1}{ab} \int_{x=-b/2}^{x=b/2} \left(\left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=-a/2}^{y=a/2} \right) dx = \frac{m_1}{ab} \int_{x=-b/2}^{x=b/2} \left(x^2 a + \frac{a^3}{12} \right) dx$$

We now do the integral in the x-direction

$$I_S = \frac{m_1}{ab} \left(\frac{x^3}{3} a + \frac{a^3}{12} x \right) \Big|_{x=-b/2}^{x=b/2} = \frac{m_1}{ab} \left(\frac{b^3}{12} a + \frac{a^3}{12} b \right) = \frac{m_1}{12} \left(b^2 + a^2 \right)$$

$$I_S = \frac{(0.159kg)}{12} \left((1.000m)^2 + (0.028m)^2 \right) = 1.326 \times 10^{-2} \text{ kg} \cdot \text{m}^2$$

Now use the parallel axis theorem to calculate the moment of inertia about the pivot point,

$$I_S = \frac{m_1}{12} \left(b^2 + a^2 \right) + m_1 l_{cm}{}^2 .$$

Using the data for the ruler, the moment about the pivot point is

$$I_S = 1.326 \times 10^{-2} \text{ kg} \cdot \text{m}^2 + (0.159 \text{ kg})(0.479 \text{ m})^2 = 4.97 \times 10^{-2} \text{ kg} \cdot \text{m}^2$$

θ_0	T_{meas}	T_{theory}	T_{ideal}
0.10	1.622	1.623	1.390
0.25	1.633	1.629	1.390
0.45	1.642	1.643	1.390

Part Two: Added Mass

Consider the effect of a brass weight clipped to the ruler. The weight is shaped like a washer with an outer radius $r_o = 0.016$ m and an inner radius $r_i = 0.002$ m; it has a mass $m_w = 0.050$ kg. It is clipped to the ruler so that the inner hole is over the 0.500 m mark on the ruler, or $l = 0.479$ m from the pivot point. The clip has a mass $m_c = 0.0086$ kg and you may assume its center of mass is also over the 0.500 m mark on the ruler. If you treat the washer and clip as point masses,

then, as was discussed in the notes for Experiment 08, the combined unit (ruler, weight and clip) has a moment of inertia about the pivot point

$$I_P = \frac{m_r}{12}\left(a^2 + b^2\right) + m_r l^2 + (m_c + m_w)d^2$$

where $d = l$ for this situation. The restoring torque that tries to return the pendulum to a vertical position will be

$$\tau = (m_r l + m_c d + m_w d)g \sin\theta \approx (m_r l + m_c d + m_w d)g\theta$$

1. Use these two expressions to derive an equation of motion for the pendulum and calculate its period T in the small amplitude ($\sin\theta \cong \theta$) approximation. Express your answer algebraically in terms of the variables a, b, d, l, m_r, m_w, m_c, and g.

Answer:

a) The equation of motion for the pendulum is

$$\tau = I_p \frac{d^2\theta}{dt^2}$$

or, when we substitute expressions for τ and I_p,

$$-(m_r l + m_c d + m_w d) g \sin\theta = \left(\frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2\right)$$

For small amplitudes $\sin\theta \approx \theta$ and the equation becomes the familiar equation $-kx = m dx^2/dt^2$, so we immediately find the angular frequency

$$\omega = \sqrt{g \frac{m_r l + m_c d + m_w d}{\frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2}}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\frac{m_r}{12}(a^2 + b^2) + m_r l^2 + (m_c + m_w)d^2}{g(m_r l + m_c d + m_w d)}}$$

2. Evaluate your result numerically and compare with the value you measured in your experiment.

Solution

The moment of inertia of the washer and binder clip treated as point masses about the pivot point is

$$I_{S,c,w} = (m_c + m_w)l_{cm}^2 = 0.0586\,\text{kg}(0.479\,\text{m})^2 = 1.345 \times 10^{-2}\,\text{kg}\cdot\text{m}^2$$

Thus the total moment of inertia is

$$I_S{}^{total} = \frac{m}{12}\left(b^2 + a^2\right) + m l_{cm}{}^2 + \left(m_c + m_w\right) l_{cm}{}^2$$

$$I_S{}^{total} = 4.97 \times 10^{-2}\ \text{kg} \cdot \text{m}^2 + 1.345 \times 10^{-2}\ \text{kg} \cdot \text{m}^2 = 6.32 \times 10^{-2}\ \text{kg} \cdot \text{m}^2 \ .$$

So the new period is

$$T_0 = \frac{2\pi}{\omega_p} \cong 2\pi \sqrt{\frac{I_S{}^{total}}{l_{cm} m^{total} g}} = 2\pi \sqrt{\frac{6.32 \times 10^{-2}\ \text{kg} \cdot \text{m}^2}{(0.479\ \text{m})(0.218\ \text{kg})(9.805\ \text{m} \cdot \text{s}^{-2})}} = 1.56\ \text{s}$$

So the approximation gives good agreement with the data.

Displacement $[m]$	Measured Period $[s]$	Calculated Period T_0 $[s]$
0.20	1.585	1.56

3. If you treat the brass object washer as a point mass, its moment of inertia about the pivot point P is $I_{w,P} = m_w l^2$. If the brass object is a washer with an inner radius r_i and outer radius r_o, then moment of inertia about its center of mass given by $I_w = \frac{1}{2} m_w (r_o{}^2 + r_i{}^2)$. If the washer is a solid disc with radius r, the moment of inertia about its center of mass given by $I_w = \frac{1}{2} m_w r^2$. When this is taken into account, what is the new (and more accurate) expression for $I_{w,P}$? How many percent does this differ from the simpler expression $I_{w,P} = m_w l^2$?

Solution:

In Problem 1, you showed that the moment of inertia of a washer about the center of mass is given by the result, $I_{cm} = \frac{1}{2} m_w \left(r_0{}^2 + r_i{}^2\right)$, so the total moment of inertia is now

$$I_S{}^{total} = \frac{m}{12}\left(b^2 + a^2\right) + m l_{cm}{}^2 + \left(m_c + m_w\right) l_{cm}{}^2 + \frac{1}{2} m_w \left(r_0{}^2 + r_i{}^2\right)$$

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The moment of inertia of the washer about the center of mass is

$$I_{cm,w} = \frac{1}{2}m_w\left(r_0^2 + r_i^2\right) = \frac{1}{2}(0.050\text{kg})\left((0.016\text{m})^2 + (0.002\text{m})^2\right) = 6.5\times10^{-6}\text{kg}\cdot\text{m}^2.$$

This is negligible compared to the overall moment of inertia. The ratio of the moment of the washer about its center of mass compared to the moment of the washer treated as a point mass about the pivot point is

$$\frac{I_{cm,w}}{m_w l_{cm}^2} = \frac{6.5\times10^{-6}\text{kg}\cdot\text{m}^2}{1.147\times10^{-2}\text{kg}\cdot\text{m}^2} = 5.7\times10^{-4}$$

Problem 3: *Stall Torque of Motor*

The following simple experiment can measure the stall torque of a motor. (See sketch.) A mass m is attached to one end of a thread. The other end of the thread is attached to the motor shaft so that when the motor turns, the thread will wind around the motor shaft. The motor shaft without thread has radius $r_0 = 1.2 \times 10^{-3} m$. Assume the thread winds evenly effectively increasing the radius of the shaft. Eventually the motor will stall.
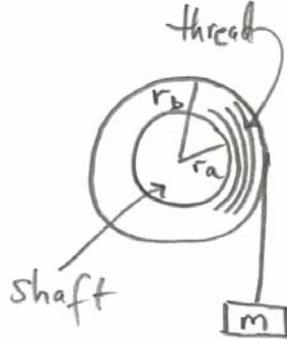
a) Suppose a mass $m = 5.0 \times 10^{-2} kg$ stalls the motor when the wound thread has an outer radius of $r_f = 2.4 \times 10^{-3} m$. Calculate the stall torque.

Answer:

The torque is $\tau_{st} = r_f mg \approx 1.1 \times 10^{-3} N \times m$.

b) Suppose the motor has an unloaded full throttle angular frequency of $\omega_0 = 2\pi f_0 = 2\pi \left(6.0 \times 10^{1} Hz \right)$ (unloaded means that the motor is not applying any torque). Suppose the relation between angular frequency ω and the applied torque τ of the motor is given by the relation

$$\omega = \omega_0 - b\tau$$

where b is a constant with units $s/kg - m^2$. Using your result from part a), calculate the constant b. Make a graph of ω vs. τ.

Answer:

When the motor is stalled, $\omega = 0$. Thus $\omega_0 = b\tau_{st}$ and

$$b = \frac{\omega_0}{\tau_{st}} = \frac{\omega_0}{r_f mg} \approx 3.2 \times 10^5 \frac{s}{kg \times m^2}$$
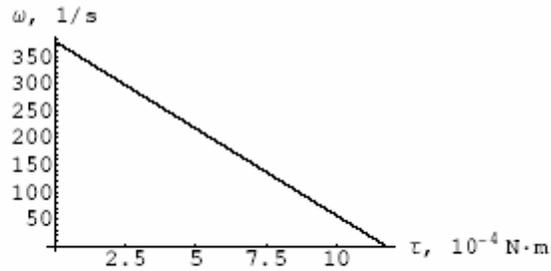
Figure 1: Angular velocity vs. Torque

c) Graph the power output of the motor vs. angular frequency ω. At what angular frequency is the power maximum? What is the power output at that maximum? Briefly explain the shape of your graph. In particular, explain the power output at the extremes $\tau = 0$ and $\tau = \tau_{stall}$.

Answer:

c) The power output is given by

$$P = \tau\omega = \frac{\omega_0 - \omega}{b}\omega.$$

The plot is a parabola (Fig. 2)
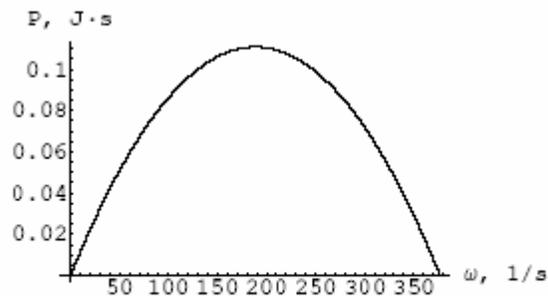
Figure 2: Power output vs. angular velocity

d) What torque does the motor put out at its maximum power output?

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Answer:

d) The maximum is reached at $\omega = \omega_0/2$. The corresponding output power is $\omega_0^2/(4b) \approx 0.1W$.

Problem 4 (*Conservation of Angular Momentum*)

A meteor of mass $m = 2.1 \times 10^{13} \, kg$ is approaching earth as shown on the sketch. The radius of the earth is $r_e = 6.37 \times 10^6 \, m$. The mass of the earth is $m_e = 5.98 \times 10^{24} \, kg$. Suppose the meteor has an initial speed of $v_0 = 1.0 \times 10^1 \, m/s$. Assume that the meteor started very far away from the earth. Suppose the meteor just grazes the earth. The initial moment arm of the meteor (h on the sketch) is called the impact parameter. You may ignore all other gravitational forces except the earth. The effective target size of the earth as initially seen by the meteor is the area πh^2.

a) Draw a force diagram for the forces acting on the meteor.

b) Can you find a point about which the gravitational torque of the earth's force on the meteor is zero for the entire orbit of the meteor?

Answer:

a),b) The meteor moves under the influence of Earth's gravity. The force is directed along the radius vector from the center of the Earth to the meteor, thus the torque created by this force about the center of the Earth is zero.

c) What is the initial angular momentum and final angular momentum (when it just grazes the earth) of the meteor?

Answer:

c) The initial angular momentum is $L = mv_0 h$ and the final angular momentum is $L = mvr_e$.

d) Apply conservation of angular momentum to find a relationship between the meteor's final velocity and the impact parameter h.

Answer:

$$\text{d) Thus } v = v_0 h / r_e.$$

e) Apply the constant energy relation to find a relationship between the final velocity of the meteor and the initial velocity of the meteor.

Answer:

e) The initial energy is $E = \frac{m v_0^2}{2}$ and the final energy is $E = \frac{m v^2}{2} - G \frac{m_e m}{r}$.
Thus
$$v^2 = v_0^2 + 2G \frac{m_e}{r_e}$$

f) Use your results in parts d) and e) to calculate the impact parameter and the effective target size of the earth.

Answer:

f) Combining the results from parts d) and e), we obtain
$$v_0^2 \frac{h^2}{r_e^2} = v_0^2 + 2G \frac{m_e}{r_e}.$$

Thus

$$h = r_e \sqrt{1 + \frac{2G m_e}{r_e v_0^2}} \approx 7.1 \times 10^9 m$$

and the effective scattering cross-section is

$$\sigma = \pi h^2 = \pi r_e^2 \left(1 + \frac{2G m_e}{r_e v_0^2}\right) \approx 1.6 \times 10^{20} m^2$$

Problem 5: Platform Diving

In the 2002 World Cup Trials, Kyle Prandi set up a diving record with a back 3 ½ somersault pike from the 10 m board. He pushed off from the board at an angle of $\theta_0 = 46^0$ with an initial speed $v_0 = 3.3\,\text{m/s}$. You may assume that his body was completely straight with his arms stretched above his head when he jumped.
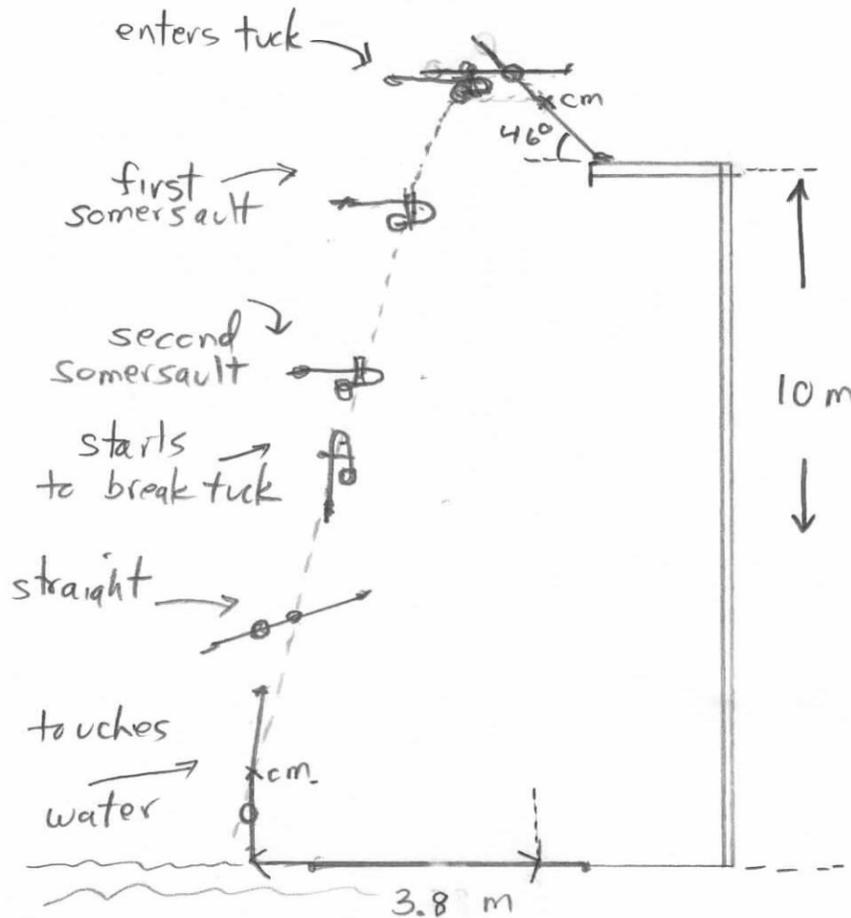
He took .33 seconds to enter into a tuck after completing ½ a rotation. At the .49 second mark he returned to his starting height. Once in a full tuck, he completed 2 revolutions at the 1.1 second mark. At that point he began to straighten out which he finished at the 1.47 second mark after making ¼ rotation. He made 1 more rotation, when his fingers touched the water 1.65 seconds after he left the platform. When he touched the surface his legs were bent but his center of mass was 1.3 m above the surface of the water. Kyle is 1.7 m long and when his arms are straight out above his head, his length is 2.2 m. His mass is 63 kg. His center of mass is 0.9 m above his soles. You may see the jump at

http://www.usadiving.org/USD_03redesign/media/video.htm

Somersaults	Dive details	time	Vertical distance starting point
zero	start	0	0
.25	Enters full pike	.33	
.75	Returns to starting height	.49	
1.25	Completes first turn in full pike	.73	
2.25	Completes second turn in full pike	1.1	
2.5	Starts to straighten	1.2	
3.4	Completely straight	1.47	
3.5	Fingers touch water	1.65	

a) Based on the above information, make a graph of his angular velocity as a function of time. Indicate any assumptions that you have made for the various stages of his motion.

b) Based on the initial conditions, calculate his vertical distance from his starting point at the various times indicated in the table.

c) Explain whether or not you think his final angular velocity is equal to his initial angular velocity?

d) Let I_0 denote his moment of inertia about his center of mass just after he left the board. Let I_1 denote his moment of inertia about his center of mass when he is in a full tuck. Let After he pulled his body into a tuck, by what fraction, $(I_1 - I_0)/I_0$, did his moment of inertia change?

e) Suppose when he goes into a tuck, he has reduced his length by a factor of 2. Does the ratio, I_1/I_0, agree reasonably with your angular velocity data? Explain your answer.

Solution:

a) We are given the sequence t_i of time marks and corresponding number of rotations n_i. To calculating the angular velocity at time t, we divide the difference between the number of rotations at this time and the number of rotations at the previous time by the corresponding difference in time and multiply the result by 2π, i.e. we approximate

$$\omega(t_i) \approx 2\pi \frac{n_i - n_{i-1}}{t_i - t_{i-1}}.$$

In addition, since the angular momentum about the diver's center of mass is conserved, and his moment of inertia at the end is equal to his initial moment of inertia, we take his initial angular velocity to be equal to the final angular velocity. The resulting points are presented in the table below and plotted on Fig. 3

t,s	0	0.33	0.49	0.73	1.1	1.2	1.47	1.65
# of somersaults	0	0.25	0.75	1.25	2.25	2.5	3.4	3.5
ω, 1/s	3.5	4.8	19.6	13.1	17.	15.7	20.9	3.5
Height, m	10.6	10.9	10.6	9.8	7.3	6.4	3.5	1.2

Note that the fluctuations in the angular velocity are likely to be due to inaccuracy in measuring the rotational angle and the associated time. We expect the diver's angular velocity to be constant while in the tuck.
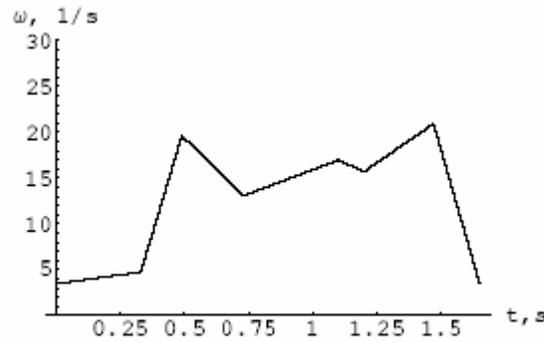
Figure 3: Angular velocity vs. time

b) Consider the motion of the center of mass. The vertical position is

$$y(t) = y_0 + v_{0y} - \frac{gt^2}{2}$$

where $v_{0y} = v_0 \sin 46° \approx 2.4$ m/s and $y_0 = 10 + 0.9 \sin 46° = 10.6$ m. The values calculated using this formula are presented in the table above.

c) His final angular velocity must be equal to his initial angular velocity because his is straighten out both in the beginning and in the end, so his moment of inertia is the same in both cases. In addition, the angular momentum about his center of mass $L_{cm} = I_{cm}\omega$ is conserved, so the angular velocities must be the same. In fact, we have already used this fact to make the plot of angular velocity vs. time.

d) Let us denote by ω_0 his angular velocity when he leaves the ground and by ω_1 his angular velocity while in the tuck. To find ω_1, we average ω over the first two and the last points on the graph and to find ω_1 we average ω over the middle portion of the graph. Thus $\omega_0 \approx 3.9$ 1/s and $\omega_1 \approx 17.3$ 1/s. Using conservation of angular momentum, $L_1 = I_1\omega_1 = L_0 = \omega_0 I_0$, we find

$$\frac{I_1 - I_0}{I_0} = \frac{\omega_0}{\omega_1} - 1 \approx -0.8$$

e) We expect $I \propto l^2$ where l is the diver's length. We find

$$\sqrt{\frac{I_0}{I_1}} = \sqrt{\frac{\omega_1}{\omega_0}} \approx 2.1$$

which is in excellent agreement with the fact that his length has changed by a factor of 2.