A 24-DIMENSIONAL SPIN MANIFOLD

by

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ABSTRACT

A 24-Dimensional Spin Manifold
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A brief review of the results of Anderson, Brown and Peterson on the structure of the spin cobordism ring shows that there is a 24-dimensional class for which no representative manifold was previously known. This thesis presents such a manifold.

The manifold is the "Grassmanification" of a certain vector bundle (the tangent bundle with a trivial line bundle split off and discarded) over an orientable 9-manifold X characterized by the non-vanishing of its Stiefel-Whitney number $w_3w_2w_2w_2(X)$. Grassmanification of a vector bundle $E \to X$ is a generalization of projectification of a vector bundle, namely instead of considering the set of lines within E one considers the set of, say, m-planes. The resulting set which we denote $E^{m,n}$ (if E has dimension m + n) is a compact manifold, provided X is.

Writing $\tau(M)$ for the tangent bundle of any manifold M, we compute $H^*(E^{m,n})$, a module over $H^*(X)$, and a basis of it over $H^*(X)$; the tangent bundle $\tau(E^{m,n})$, which equals the Whitney sum of $\tau(X)$ (pulled back to $E^{m,n}$) and the tensor product of the canonical m- and n-plane bundles on $E^{m,n}$; and thus, the Stiefel-Whitney class of $E^{m,n}$.

It is shown that in case the Stiefel-Whitney number of the orientable manifold X above does not vanish, then for the 8-bundle E indicated above, $E^{3,5}$ is a spinor manifold such that $w_6^{\,4}(E^{3,5}) \neq 0$, a condition which implies that $E^{3,5}$ is a representative of the 24-dimensional spin cobordism class.

Various results appear along the way, such as a method of computing E. Thomas' function $\phi_{m,n}$ which gives the Stiefel-Whitney class of the tensor product of bundles. The method involves a formula by which Milnor's symmetric polynomials s_ω may be calculated. Obtaining the Stiefel-Whitney class of a specific tensor product then becomes a straightforward, though tedious, calculation.

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A 24-DIMENSIONAL SPIN MANIFOLD

1 Introduction

Thom [10] invented the study of manifolds by cobordism and determined the structure of the unoriented cobordism ring. Milnor [6], Wall [12] and others determined the oriented cobordism ring Ω_*, and Anderson, Brown and Peterson [2] described the additive structure of the spin cobordism ring Ω_*^{Spin}, as well as most of its multiplicative structure. Manifolds representing many generating classes in Ω_*^{Spin} remain unknown, however. Every spin manifold of dimension <24 is unoriented cobordant to the square of an orientable manifold; this is also true in dimensions 25,26,27, 28,30, and 31 (see [8]). There is however a 24-dimensional spin cobordism class for which no representative manifold was previously known. This thesis presents such a manifold.

2 Spin Cobordism

2.1 KO Characterictic classes

To give an understanding of the place of the manifold in Ω_*^{Spin}, we give here part of the description in [2] of Ω_*^{Spin}: let BO be the classifying space for the orthogonal group. Let $p: BO\langle n\rangle \to BO$ be the fibre space such that $\pi_i(BO\langle n\rangle) = 0$ for $i<n$ and $p_*:\pi_i(BO\langle n\rangle)\to\pi_i(BO)$ is an isomorphism if $i\geq n$. Let $\alpha_n \in H^n(BO\langle n\rangle)$ be the generator

when $n \equiv 0,2 \pmod 8$.

Let $\xi \in \widetilde{KO}^0(X)$ be of filtration n, i.e. ξ is to be trivial on the $(n-1)$-skeleton of X. Then there is a map $f: X \to BO\langle n\rangle$ such that $pf_\xi \simeq \xi: X \to BO$. We define $[\xi] \in H^n(X)$ by

$$[\xi] = \{f_\xi^*(\alpha_n)\}$$

for all f such that $pf_\xi \simeq \xi$.

Define K-theory Pontrjagin classes [1] as follows: let T^m be the maximal torus in $SO(2m)$ (and $SO(2m+1)$). Now $K^0(BT^m) \simeq Z[[x_1,\ldots,x_m]]$ where each x_i has dimension 1. Both $KO^0(BSO(2m))$ and $KO^0(BSO(2m+1))$ are injected into $K^0(BT^m)$ under the map χ which is the composition of the homomorphism "complexification of a bundle", with the K-theory homomorphism associated to $BT^m \to BSO(2m)$. Their common image is the invariants of the Weyl group of T^m in $SO(2m)$ or $SO(2m+1)$. Let $\overline{x}_i = -x_i/(1-x_i)$. Then in $K^0(BT^m)[t]$, $\Pi_{i=1}^m (1 + t(x_i + \overline{x}_i))$ is a polynomial in t and we denote the coefficient of t^ℓ by $\pi^\ell \in K^0(BT^m)$. lies in the image of χ; pulling back, one also writes $\pi^\ell \in KO^0(BSO(2m))$ or $KO^0(BSpin)$. If $J = (j_1,\ldots,j_k)$ is a sequence of integers such that $k \geq 0$ and each $j_i > 1$, let $\pi^J = \pi^{j_1}\pi^{j_2}\ldots\pi^{j_k} \in KO^0(BSpin)$, and let $n(J) = \Sigma j_i$.

2.2 Theorem

The filtration of π^J in $KO^0(BSpin)$ is $4n(J)$ if $n(J)$

is even and is $4n(J) - 2$ if $n(J)$ is odd (see [2]).

Let k be large and let MSpin(8k) be the Thom space of the classifying bundle over BSpin(8k). Let $\phi: H^*(BSpin(8k)) \to \overline{H}^*(MSpin(8k))$ and $\phi: KO^0(BSpin(8k)) \to \overline{KO}^{8k}(MSpin(8k))$ denote the Thom isomorphisms. ϕ raises filtration by precisely 8k, and $\phi([\xi]) = [\phi(\xi)] \in \overline{H}^*(MSpin(8k))$ [5]. Let $\underset{\sim}{MSpin}$ denote the spectrum associated to MSpin(8k). We state the results of [3] in the language of spectra where the Thom isomorphism has degree 0. Let $\underset{\sim}{BO}\langle n\rangle$ be the Ω-spectrum whose 0^{th} term is $BO\langle n\rangle$.

If $n(J)$ is even (respectively, odd), let $f_J: \underset{\sim}{MSpin} \to \underset{\sim}{BO}\langle 4n(J)\rangle$ (respectively $BO\langle 4n(J) - 2\rangle$) be a map corresponding to π^J. If $z \in \overline{H}^*(MSpin)$, let $f_z: \underset{\sim}{MSpin} \to \underset{\sim}{K}(Z_2, \dim z)$ denote the corresponding map, where $\underset{\sim}{K}(Z_2, n)$ denotes the spectrum whose 0^{th} term is $K(Z_2, n)$.

2.3 Theorem

There is a collection of elements $z_i \in H^*(\underset{\sim}{MSpin})$ such that the map

$$F: \underset{\sim}{MSpin} \to \prod_{n(J)\ \mathrm{even}} \underset{\sim}{BO}\langle 4n(J)\rangle \times \prod_{n(J)\ \mathrm{odd}} \underset{\sim}{BO}\langle 4n(J) - 2\rangle \times \prod_i \underset{\sim}{K}(Z_2, \dim z_i)$$

given by $F = \prod f_J \times \prod f_{z_i}$ induces an isomorphism on cohomology with Z_2 coefficients. Hence F induces a C_2-isomorphism on homotopy groups, where C_2 is the class of finite groups of odd order.

Since $\pi_*(\text{MSpin}) \approx \Omega_*^{\text{Spin}}$ has no p-torsion for odd primes p [6], the above theorem allows one to compute the additive structure of Ω_*^{Spin}. (In [2] is given a complicated counting procedure for the number of z_i's in each dimension.) $\pi_*(BO\langle n\rangle)$ is periodic of period 8 in dimensions $\geq n$, the sequence, starting in dimensions $\equiv 0 \pmod 8$, being Z, Z_2, Z_2, 0, Z, 0, 0, 0.

2.4 Generators of Ω_*^{Spin}

The above shows that the classes $[\pi^J] = \{f_J^*(\alpha_{4n(J)})\}$ or $\{f_J^*(\alpha_{4n(J)-2})\}$ in $H^*(\text{MSpin}) \approx \Omega_*^{\text{Spin}}$ are of interest as generators of Ω_*^{Spin}. Manifolds M_J representing $[\pi^J]$ are known in case all j_i are even (the product of quaternionic projective spaces) or in case only one is odd [4].

If M is an n-dimensional spin manifold, denote by $\pi^J(M) \in KO^{-n}(\text{pt.})$ the characteristic number defined by π^J [1]. By 2.2, a representative M_J of $[\pi^J]$ is a spin manifold of dimension $4n(J)$ (or $4n(J) - 2$ if $n(J)$ is odd) such that $\pi^J(M_J) \neq 0$. In [2] it is shown that $\pi^J(M) = P^J(M)$ where $P^J = p_{j_1} \cdots p_{j_k}$ and $p_j \in H^{4j}(\text{BSpin})$ is the Pontrjagin class. Since the reduction mod 2 of the Pontrjagin class p_i of any bundle equals w_{2i}^2 of that bundle, where w_{2i} is the Stiefel-Whitney class,

$$(1) \qquad w_{2j_1}^2 \cdots w_{2j_k}^2(M_J) \neq 0$$

will guarantee that $0 \neq P^J(M) = \pi^J(M)$.

3 Grassmanification of a vector bundle

Real projective space is a compact k-dimensional manifold which can be described as the set of lines (i.e. 1-planes) through 0 in a real (k + 1)-dimensional vector space. The Grassman manifold $G_{m,n}$, a compact mn-dimensional manifold, is the set of m-planes (or n-planes, taking orthogonal complements) through 0 in an (m + n)-dimensional space. Manifolds representing generators for the unoriented and oriented cobordism rings have been described which involve projectification of a vector bundle [4], which is a special case of "Grassmanification" of a bundle.

If $E \to X$ is an (m + n)-dimensional vector bundle let $E^{m,n}$ be the set of m-planes in E, each within a fibre and through the 0-section. There is a fibration $G_{m,n} \to E^{m,n} \to X$, and if X is a compact manifold of dimension k, $E^{m,n}$ is a compact manifold of dimension mn + k, naturally.

Note that orthogonality is established in fibres of E by choosing a Riemannian metric; an m-plane in E then determines the orthogonal n-plane and conversely, so $E^{m,n}$ may also be regarded as the set of n-planes in E.

3.1 $H^*(E^{m,n})$

Let all cohomology in the sequel have Z_2 coefficients.

Theorem

$$H^*(E^{m,n}) \approx H^*(X)[u,v]/(uv = w(E))$$

where $u = 1 + u_1 + u_2 + \dots + u_m$, $u_i = w_i(\gamma_m)$,

$v = 1 + v_1 + v_2 + \dots + v_m$, $v_i = w_i(\gamma_n)$,

γ_m = the canonical m-plane bundle over $E^{m,n}$ whose fibre over an m-plane in E (i.e. point of $E^{m,n}$) consists of the points in that m-plane,

γ_n = the analogous canonical n-plane bundle,

and by abuse of language we write $H^*(X)[u,v]$ for the polynomial algebra $H^*(X)[u_1,\dots,u_m,v_1,\dots,v_n]$ (in the sequel we often abbreviate this list of arguments by u,v).

Proof First suppose X is a point. Then $E^{m,n} = G_{m,n}$ and $w(E) = 1$; the result in this case is well-known. For general X, map $H^*(X)[u,v]/(uv = w(E)) \xrightarrow{f} H^*(E^{m,n})$ by sending u_i to $w_i(\gamma_m)$ and v_j to $w_j(\gamma_n)$.

f is well-defined since $w(\gamma_m)w(\gamma_n) = w(\gamma_m \oplus \gamma_n)$, and an easy argument shows $\gamma_m \oplus \gamma_n = \pi^{-1}(E)$, $\pi: E^{m,n} \to X$ the projection.

f is 1-1 since $G_{m,n} \xrightarrow{i} E^{m,n}$ gives $i^* w_i(\gamma_m) = w_i(\bar{\gamma}_m) \neq 0$ and similarly for γ_n (writing now $\bar{\gamma}_m$ and $\bar{\gamma}_n$ for the canonical bundles on $G_{m,n}$, and $\bar{u}$, $\bar{v}$ for their Stiefel-Whitney classes). In fact the only polynomials in $w_i(\gamma_m)$ and $w_j(\gamma_n)$ carried to 0 by i^* are those given by $i^*(uv) = \bar{u}\cdot\bar{v} = 1 = i^* w(E)$.

f is onto because in the Serre spectral sequence for the fibration $G_{m,n} \to E^{m,n} \to X$, $E_2 = H^*(X) \otimes H^*(G_{m,n})$, and it can be shown that the rank of E_2 and that of $H^*(X)[u,v]/(uv = w(E))$ are equal in each dimension. Since f is injective, E_2, E_∞ and $H^*(E^{m,n})$ must be additively isomorphic, and f must be onto.

3.2 A basis of $H^*(E^{m,n})$ over $H^*(X)$

We wish to find an additive basis for $H^*(E^{m,n})$ over $H^*(X)$ among the monomials in the u_i and v_j. Write $w_i(E) = E_i$ and similarly for other bundles. Since $uv = w(E)$,

(1) $$u_k + u_{k-1}v_1 + \ldots + v_k = E_k$$

for $k = 0, 1, \ldots, n$. We can write $v_1 = u_1 + E_1$, and inductively express $v_1, \ldots, v_n$ in terms of the u_i (and $H^*(X)$). Defining $u_i = 0$ for $i > m$ and $v_j = 0$ for $j > n$, one has (1) also for $n < k \leq m + n$. Then substitution for $v_1, \ldots, v_n$ gives relations among polynomials in the u_i. To express these relations we define polynomials P_k and P_k' by

(2) $$P_0(u) = 1, \; P_k(u) = \sum_{|J|=k} u_J$$

where J stands for a sequence of positive integers $(j_1, \ldots, j_r)$ for some r and we use the notation for any sequence, $|J| = \Sigma j_i$; and

(3) $$P_k'(u) = \sum_{i=0}^{k-1} P_i(u)(u_{k-i} + E_{k-i}).$$

It is easy to show that

(4) $$P_k(u) = \sum_{i=0}^{k-1} P_i(u)u_{k-i}$$

and

(5) $$v_k = P_k'(u), \; k + 1, \ldots, n,$$

using (1). Substituting in (1) we then find

(6) $$P_{n+j}(u) = \sum_{i=0}^{n+j-1} P_i(u)E_{n+j-i} = \sum_{i=0}^{n} P_i(u)\sum_{k=1}^{j} E_{n+k-i}P_{j-k}(E)$$

where E stands for the E_i and $P_k(E)$ is defined by a formula similar to (2).

Further if $j > m + n$ then $0 = P_j'(u) = \sum_{i=0}^{j} u_i v_{j-i}$ is already implied by $0 = v_{n+1} = \ldots = v_{n+m}$ according to (3), hence $P_j'(u) = 0$ yields no new relations among polynomials in the u_i for $j > m + n$.

3.3 The tangent bundle of $E^{m,n}$

Write $\tau(M)$ for the tangent bundle of a manifold M.

Theorem

$\tau(E^{m,n}) \cong \pi^{-1}\tau(X) \oplus (\gamma_m \otimes \gamma_n)$ where $E^{m,n} \xrightarrow{\pi} X$ is the projection.

This can be deduced from the results of [9]. The idea is that the tangent bundle of the total space of a fibration of manifolds is the sum of the vectors along the fibres with an orthogonal subbundle. The latter is isomorphic to $\pi^{-1}\tau(X)$ and the former in our case can be identified with $\gamma_m \otimes \gamma_n$.

3.4 $w(\gamma_m \otimes \gamma_n)$

To compute the Stiefel-Whitney classes of $E^{m,n}$ we need a result of E. Thomas [11] which we state without proof.

Theorem

If ξ is an m-plane bundle and η is an n-plane bundle over X,

$$w(\gamma_m \otimes \gamma_n) = \phi_{m,n}(w_1(\xi),\ldots,w_m(\xi), w_1(\eta),\ldots,w_n(\eta)),$$

where if σ_i is the i^{th} elementary symmetric function in the s_k and τ_j is the j^{th} elementary symmetric function in the t_k in the ring $Z[s_1,\ldots,s_m,t_1,\ldots,t_n]$,

$$(7) \qquad \phi_{m,n}(\sigma_1,\ldots,\sigma_m,\tau_1,\ldots,\tau_n) = \prod_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} (1 + s_i + t_j)$$

(see the Appendix for definition of σ_i). To compute $\phi_{m,n}$ we express it in terms of Milnor's polynomials s_J (see Appendix). Let S denote the set of all sequences $J = (j_1,\ldots,j_m)$ of m integers between 0 and n. Write $J\uparrow$ if $j_1 \leq \ldots \leq j_m$.

Lemma

$$\phi_{m,n}(\sigma_1,\ldots,\tau_n) = \sum_{J\uparrow\, \epsilon\, S} s_J(\sigma_1,\ldots,\sigma_m) \cdot \sum_{\substack{A\, \epsilon\, S \\ a_i \geq j_i}} \tau_{n-a_1}\cdot\ldots\cdot\tau_{n-a_m}\cdot\binom{a_1}{j_1}\ldots\binom{a_m}{j_m}$$

where $\binom{a}{j}$ is the binomial coefficient.

The proof follows from (7) by expanding the product and collecting monomials in the s_i into groups $\Sigma s_1{}^{k_1}\ldots s_m{}^{k_m} = s_J(\sigma_1,\ldots,\sigma_m)$.

An aid to computation results from noticing that if $(a_1,\ldots,a_m)$ is a permutation of $(b_1,\ldots,b_m)$, then $\tau_{n-a_1}\ldots\tau_{n-a_m} = \tau_{n-b_1}\ldots\tau_{n-b_m}$. We can collect identical monomials in τ and add their coefficients together.

4 The manifold $M_{(3,3)}$

4.1 When is $E^{m,n}$ spinor?

Recall that if $E \to X$ is an $(m+n)$-bundle, $H^*(E^{m,n})$ is generated as a ring over $H^*(X)$ by $u_1, \ldots, u_m, v_1, \ldots, v_n$, subject only to $uv = w(E)$. According to Thomas [11], the terms in $\phi_{m,n}(u,v)$ of degree 0, 1, and 2 are

$$1 + (mv_1 + nu_1) + (\tbinom{m}{2}v_1^2 + \tbinom{n}{2}u_1^2 + mv_2 + nu_2 + (mn-1)u_1v_1).$$

By 3, if $M = E^{m,n}$, using $uv = w(E)$ we get

$$\begin{aligned} w(M) &= w(X)w(\gamma_m \otimes \gamma_n) = w(X)\phi_{m,n}(u,v) \\ &= 1 + ((X_1 + mE_1) + (m+n)u_1) + ((X_2 + \\ &\quad \tbinom{m}{2}E_1^2 + mX_1E_1) + u_1(X_1(m+n) + mE_1 + \\ &\quad (mn-1)E_1) + u_1^2(\tbinom{m}{2} + \tbinom{n}{2} + m + mn + 1) \\ &\quad + u_2(m+n)) + \text{higher terms} \end{aligned}$$

where we write for a manifold X, $w_i(X) = X_i$.

Thus if M is orientable, $M_1 = 0$, or

(1) $$m + n \equiv 0 \pmod 2$$

(2) $$X_1 + mE_1$$

For M to be spinor, $M_2 = 0$ as well, or

(3) $$X_2 + \tbinom{m}{2}E_1^2 + mE_1^2 + mE_2 + m^2E_1^2 = 0$$

(4) $$mE_1 + (m^2 - 1)E_1 = 0$$

(5) $$\tbinom{m}{2} + \tbinom{n}{2} + m^2 + m + 1 \equiv 0 \pmod 2,$$

since u_1, u_1^2, and u_2 are independent in $H^*(M)$ over $H^*(X)$ if $m,n > 1$.

By (4), $E_1 = m(m+1)E_1 = 0$, hence (2) shows $X_1 = 0$.

Then by (3) $X_2 = mE_2$. By (5), $\binom{m}{2} + \binom{n}{2} \equiv 1 \pmod 2$. This implies $(m,n) \equiv (0,2), (1,3), (2,0),$ or $(3,1) \pmod 4$. Collecting the above conditions, we see that M will be spinor if and only if

$$(6) \qquad \begin{cases} m \equiv n + 2 \pmod 4 \\ E_1 = X_1 = 0 \\ X_2 = mE_2 \end{cases}$$

4.2 m or n = 1 or 2 does not work

I calculated M_6^4 for some manifolds M involving successive projectifications of vector bundles in up to three stages, and found that no $M_{(3,3)}$ was among them. For more than 3 stages the calculations seem lengthy and rather than continue them I began looking among manifolds $E^{m,n}$ for $1 < m < n$. By (6) the simplest case is $m = 2$, $n = 4$; but computation shows that no spin manifold $E^{2,4}$ can satisfy $w_6^4(E^{2,4}) \neq 0$.

4.3 Relations in $H^*(E^{3,5})$

The next case is $m = 3$, $n = 5$. Write $u_i u_j \ldots u_k = u_{ij\ldots k}$ and $u_i^j = u_{ij}$ for brevity, and similarly for E. Using the results of §3 one can write the relations $P_k'(u) = 0$, $k = 6,7,8$, as

$$(7) \qquad \begin{cases} u_{33} = u_{222} + u_{21^4} + u_{1^6} + P_6(u) \\ u_{322} = u_{31^4} + u_{2221} + u_{21^5} + u_1 P_6(u) + P_7(u) \\ u_{1^8} = u_2 P_6(u) + P_8(u) \end{cases}$$

where P_6, P_7, and P_8 are to be expanded using 3.2(6). In dimension 10 because $u_{3322} = (u_{322})u_2 = (u_{33})u_{22}$ can be decomposed in two ways, there results a relation which can be written

(8) $u_{25} = u_{2^411} + (u_{31} + u_{1^4})P_6 + (u_3 + u_{21})P_7 + (u_2 + u_{11})P_8$

(This can be further reduced using (7)). Choosing an additive basis of $H^*(E^{3,5})$ over $H^*(X)$ whose elements, monomials in u, have no factor u_{33}, u_{322}, u_{1^8}, or u_{25} leads to $u_{2^41^7}$ as a basis in dimension 15. Let $M = E^{3,5}$.

4.4 $w_6{}^4(M)$

Calculating $w(\gamma_3 \otimes \gamma_5)$ yields

$1 + (v_1 + u_1) + (v_2 + u_2 + v_{11}) + v_3 + v_{111} + u_1v_2 +$
$u_1v_{11} + u_2v_1 + u_3) + v_{211} + v_{22} + v_4 + u_{11}v_2 + u_{1111})$
$+ (v_5 + v_{311} + v_{211} + u_1v_{22} + u_1v_{211} + u_1v_4$
$u_{11}v_3 + u_2v_3 + u_{111}v_2 + u_3v_2 + u_{1111}v_1 + u_{15})$
$+ (v_{33} + v_{411} + v_{222} + u_{11}v_{211} + u_2v_4 + u_2v_{22}$
$+ u_{21}v_{21} + u_3v_{21} + u_{21}v_3 + u_3v_3 + u_{1^4}v_{11} + u_{211}v_2$
$u_{21^4}) +$ higher terms,

and applying relations 3.2(5) one has, assuming M is spinor ($X_2 = E_2$ implies $X_3 = E_3$ by the Wu relations),

$w_6(M) = w(X)\, w(\gamma_3 \otimes \gamma_5)$
$= X_6 + X_4E_2 + E_{42} + u_1E_{32} + u_{11}E_{22} + u_{11}E_4$
$+ u_2E_4 + u_3E_3 + u_{11}E_3 + u_{22}E_2 + u_{211}E_2 + u_{222} + u_{16}$
$u_{2211} + u_{21^4}$.

$H^*(X)$ is 0 above dimension 9 if X is a 9-manifold, so $(\Sigma a_i)^4 = \Sigma a_i^4$, which holds in any Z_2-module, shows that any term in $w_6(M)$ involving $H^*(X)$ in dimension > 2 can be neglected in calculating $w_6^4(M)$. Further, since $E_1 = X_1 = 0$, $H^1(X)$ will never enter the calculation; thus any term containing a factor in $H^8(X)$ must be 0 as well. This leaves

(1) $$w_6^4(M) = u_{2^{12}} + u_{1^{24}} + u_{2^8 1^8} + u_{2^4 1^{16}}.$$

To see if $w_6^4(M) = 0$ we then use 4.3(7) and express (1) in terms of our additive basis for $H^*(M)$ over $H^*(X)$. Fully expanded, 4.3(7) and (8) become

$$u_{33} = u_{222} + u_{21^4} + u_{1^6} + E_5u_1 + E_4(u_{11} + u_2)$$
$$+ E_3(u_{111} + u_3) + E_2(u_{1^4} + u_{211} + u_{22})$$

$$u_{322} = u_{31^4} + u_{2221} + u_{21^5} + E_5u_2 + E_4(u_{31} + u_{211} +$$
$$u_{22}) + E_2(u_{2111} + u_{311})$$

$$u_{1^8} = E_5(u_{111} + u_{21} + u_3) + E_4(u_{1^4}) + E_3(u_{2111}$$
$$+ u_{32} + u_{1^5} + u_{221} + u_{311}) + E_2(u_{21^4} + u_{2211} + u_{222})$$

$$u_{2^5} = u_{2^4 11} + x$$

$$x = E_{54}u_1 + E_{52}u_{111} + E_{42}u_{22} + E_{33}(u_{1^4} + u_{211} +$$
$$u_{22}) + E_5(u_{221} + u_{311}) + E_2(u_{21^4} + u_{2211} +$$
$$u_{222}) + E_4(u_{21^4} + u_{1^6}) + E_3(u_{1^7} + u_{31^4}).$$

It is convenient to express u_{2^k} for $k > 5$ by using 4.3(8) repeatedly: $u_{2^k} = u_{2^4 1^{2k-8}} + Q_{k-5}x$ $(k > 5)$, where $Q_0 = 1$, $Q_k = u_{1^{2k}} + u_{1^{2k-2}}u_2 + \ldots + u_{2^k}$, $k > 0$. Thus $Q_{k+5} =$

$u_{1^{2k+10}} + \ldots + u_{1^{2k+2}}u_{2^4}) + (k+1)u_{2^4 1^{2k+2}} + R_k x$, where $R_{2k} = Q_k^{\,2}$ and $R_{2k+1} = u_2 R_{2k}$.

Expanding the last 2 terms of (1) we find

$$w_6^{\,4}(M) = u_{1^{24}} + u_{2^4 1^{16}} + u_{1^8}(u_{2^4 1^8} + (u_{1^6}$$

$$+ u_{21^4} + u_{2211} + u_{222})x) + (u_{2^4 1^{16}} + x(u_{1^{14}}$$

$$+ u_{21^{12}} + u_{221^{10}} + u_{2221^8} + u_{2^4 1^6} + u_{2^4 1^6} + u_{1^4}$$

$$+ u_{22})x)).$$

Now use 4.3(7):

$$= u_{1^{24}} + u_{2^4}(u_{22}P_6^{\,2} + P_8^{\,2}) + (u_{1^4} + u_{22})x^2$$

$$= u_{1^{24}} + P_6^{\,2}(u_{2^4 1^4} + x(u_{11} + u_2)) + u_{2^4}P_8^{\,2}$$

$$+ (u_{1^4} + u_{22})x^2.$$

It helps to calculate $P_6^{\,2}$, $P_7^{\,2}$, $P_8^{\,2}$, $u_{1^{24}}$, and x^2 separately, and finally combine them. This can take about 11 pages. the result is $w_6^{\,4}(M) = u_{2^4 1^7}E_{3222}$. Recall that we assumed that M is a spin manifold, 4.1(6).

4.5 The base manifold X

We must find a 9-manifold X and bundle $E \to X$ satisfying

(1) $$E_1 = X_1 = 0$$

(2) $$X_2 = E_2$$

(3) $$E_{3222} \ (= X_{3222}) \neq 0.$$

For any such bundle, M will be a suitable manifold $M_{(3,3)}$.

The tangent bundle of any orientable (by (1)) 9-manifold X splits off a trivial line bundle. The remaining 8-bundle E will then satisfy (1) and (2), so we need only make sure that $X_{3222} \neq 0$. This we do following the construction of orientable manifolds in [4]. X will be a product of complex projective space CP^2, of dimension 4, and a manifold Y (Y_5 or $M(3,2)$ in [4]): let $F \to RP^2$ be the bundle $H \otimes 5T$ where H is the canonical line bundle on RP^2 and T is a trivial line bundle. Then we put $Y = F^{1,5}$. It is easy to show that $X_{3222} = Y_{32}(CP^2)_{22} \neq 0$ using §3, so X satisfies (1), (2), and (3).

APPENDIX: SYMMETRIC POLYNOMIALS

It is well-known [12] that in the graded polynomial ring R_m in m variables $x_1,\ldots,x_m$, each of dimension 1, the symmetric polynomials (those invariant under permutations of the x_i) form a subring S_m which is a polynomial ring on generators $\sigma_1,\ldots,\sigma_m$ of dimension $\sigma_i = i$, where

(1) $$1 + \sigma_1 + \ldots + \sigma_m = \prod_{i=1}^{m} (1 + x_i).$$

Also well-known is their usefulness in the study of characteristic classes, where the Stiefel-Whitney classes of an m-bundle which splits into m line bundles are the elementary symmetric functions of the first Stiefel-Whitney classes of the line bundles, by the Whitney product theorem and (1). Thomas' result in 3.4 uses them, too.

Milnor [7] defines the following additive basis for S_m: call two monomials equivalent if some permutation of the x_i carries one into the other. If $J = (j_1,\ldots,j_m)$ define s_J by the equation in R_m,

$$s_J(\sigma_1,\ldots,\sigma_m) = \Sigma\, x_1^{k_1}\ldots x_m^{k_m},$$

summing over all monomials equivalent to $x_1^{j_1}\ldots x_m^{j_m}$. s_J is a polynomial (homogeneous of dimension $|J| = j_1 + \ldots + j_m$) since S_m is the polynomial ring on the σ_i. If J satisfies $0 \leq j_1 \leq \ldots \leq j_m$ write $J\uparrow$. It is obvious that $\{s_J(\sigma) \mid J\uparrow\}$(abbreviating as usual $\sigma_1,\ldots,\sigma_m$ by σ)

forms an additive basis for S_m. Another additive basis consists of monomials $\sigma^J = \sigma_1^{j_1} \ldots \sigma_m^{j_m}$ for all J such that $j_i \geq 0$, $i = 1,\ldots,m$. The lemma of 3.4 shows knowledge of the polynomials s_J is useful in computing $\phi_{m,n}$.

It seems to be easier to calculate σ^K in terms of various $s_J(\sigma)$ and then invert the transformation, than to attack the problem directly. This can be done inductively: $\sigma_1 = s_{(1,0,\ldots,0)}(\sigma)$, and once an expression for each σ^J with $|J| < q$ is known, if $|J| = q$ we can write $\sigma^J = \sigma^{J'} \cdot \sigma_i$ with $|J'| < q$ for some i, and use the expression for $\sigma^{J'}$ to find that for σ^J, by the lemma we shall shortly state. The sequences we speak of below will all be ordered sequences of m non-negative integers. If $N = (n_1,\ldots,n_m)$ denotes a sequence we write N(j) for the number of j's appearing in N, and x^N for $x_1^{n_1} \ldots x_m^{n_m}$.

If J is a sequence and $i \geq 0$ an integer, we define a sequence K to be a (J,i)-sequence, if one can obtain K by increasing each of i entries in J by unity. If K is a (J,i)-sequence, choose a suitable set S of i entries of J to be thus increased. (S may contain several copies of any integer). Let h(j) be the number of j's in S. I claim h(0), h(1),... are all fixed by J and K. For from the definitions we find that

$$K(j) = J(j) - h(j) + h(j-1), \ j > 0$$

since increasing a j in J takes away one j from K and increasing a (j-1) in J adds one. Transposing,

(1) $$h(j-1) = K(j) - J(j) + h(j),\ j > 0.$$

Since S if finite, there is a largest $j = j_0$ for which $h(j_0) \neq 0$. Thus using (1) for $j = j_0 + 1, j_0, \ldots, 1$ in turn gives a proof by decreasing induction that K and J determine the $h(j)$.

Lemma

If J is a sequence and $i > 0$ an integer,

(2) $$s_J(\sigma)\cdot\sigma_i = \Sigma c_K s_K(\sigma)$$

summed over all (J,i)-sequences K where c_K is the integer

$$c_K = \prod_{j \geq 0} \frac{K(j)!}{J(j)!} \binom{J(j)}{h(j)}$$

Proof Recall $s_J(\sigma) = \Sigma\ x^T$ summed over monomials x^T equivalent to x^J, and $\sigma_i = \Sigma\ x^D$ summed over sequences D containing i ones and m-i zeroes, so

(3) $$s_J(\sigma)\cdot\sigma_i = \Sigma\ x^T\ \Sigma\ x^D = \Sigma\ x^{T+D}$$

(adding sequences entrywise). It turns out that each T+D is a (T,i)-sequence, and hence x^{T+D} is equivalent to some x^K where K is a (J,i)-sequence. Since (3) is a symmetric polynomial, each monomial occurs together with all equivalent monomials, and there exists a formula (2) in which only (J,i)-sequences occur. If we know the number of monomials x^{T+D} in the sum (3) which are equivalent to x^K, we can then find the coefficient c_K of s_K by dividing by the number of monomials in $s_K(\sigma)$. The latter is $m!/\prod_{j\geq 0} J(j)!$.

Let K be a fixed (J,i)-sequence. How many monomials in

(3) are equivalent to x^K? There are $m!/\prod_{j\geq 0} J(j)!$ different monomials x^T equivalent to x_J, and for each of them $x^T \cdot \sigma_i$ contain the same number of monomials equivalent to x^K. We might as well use x^J to compute this number.

(4) $$x^J \cdot \sigma_i = \sum x^{J+D}$$

where D runs over sequences of i ones and (m−i) zeroes, and x^{J+D} is equivalent to x^K if and only if adding D to J increases exactly h(j) of the j's in J, for each j. There are $\binom{J(j)}{h(j)}$ ways to choose h(j) entries from J(j) candidates, and every possible selection of i unit increases occurs for some D, hence (4) contains $\prod_{j\geq 0} \binom{J(j)}{h(j)}$ monomials equivalent to x^K. Combining the above we have

$$c_K = \frac{\prod_{j\geq 0} \binom{K(j)}{h(j)} \dfrac{m!}{\prod_{j\geq 0} J(j)!}}{\dfrac{m!}{\prod_{j\geq 0} K(j)!}}$$

which gives the formula of the lemma.

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BIOGRAPHICAL NOTE

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