61] Moments of Inertia

These are just exercises in integral calculus, but there are a few special cases and tricks worth knowing.

(uniform) rod about end: $I = \lambda \int_0^l ds\, s^2 = \frac{\lambda L^3}{3} = \frac{ML^2}{3}$

rod about center: $I = 2\lambda \int_0^{l/2} ds\, s^2 = \frac{2\lambda (l/2)^3}{3} = \frac{ML^2}{12}$

hoop 
around center
(⊥ axis)
: $I = MR^2$

disc 
around center
(i.e., axis thru center, ⊥ to disc)
: $I = \sigma \int_0^R 2\pi s\, ds\, s^2 = \frac{\sigma \pi R^4}{2} \underset{\uparrow}{=} \frac{MR^2}{2}$

$M = \sigma \pi R^2$

sphere
around center
: $I = \rho \int_\circ (x^2+y^2)\, dV$

$$= \frac{2}{3}\rho \int (x^2+y^2+z^2)\, dV$$

$$= \frac{2}{3}\rho \int_0^R r^2 \cdot 4\pi r^2\, dr$$

$$= \frac{2}{3}\rho \frac{4\pi R^5}{5} \underset{\uparrow}{=} \frac{2}{5} MR^2$$

$M = \frac{4}{3}\pi R^3 \rho$

disc around diameter : $I = \sigma \int x^2\, dA$

$$= \frac{1}{2}\sigma \int_0^R r^2\, 2\pi r\, dr = \frac{\sigma \pi R^4}{4} = \frac{MR^2}{4}$$

hoop around diameter : $I = \frac{MR^2}{2}$

Parallel Axis Theorem : $I_a = I_{\parallel, cm} + Md^2$

where a = any axis $\parallel, cm$ = axis parallel to a through the cm, d = distance between them.

Proof: $I_a = \int (x-x_a)^2 + (y-y_a)^2 \; dm$

$= \int (x - x_{cm} + x_{cm} - x_a)^2 + (y - y_{cm} + y_{cm} - y_a)^2 \; dm$

$= \int (x-x_{cm})^2 + (y-y_{cm})^2 + \int (x_{cm}-x_a)^2 + (y_{cm}-y_a)^2 \; dm$

$+ 2 \int (x-x_{cm})(x_{cm}-a) + (y-y_{cm})(y_{cm}-a) \; dm$

$= I_{\parallel, cm} + Md^2 + 0.$

(since $\int x \, dm \equiv x_{cm} \int dm$!) etc.

check on rod:

$I_{end} \stackrel{?}{=} I_{center} + M\left(\frac{L}{2}\right)^2$

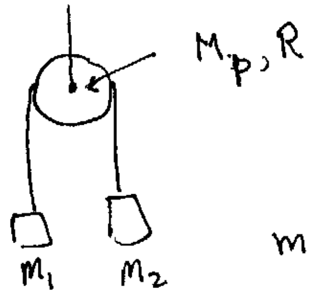
$\frac{ML^2}{3} \stackrel{?}{=} \frac{ML^2}{12} + \frac{ML^2}{4}$ ✓

exercise: disc around bdry. point (axis ⊥ to disc!)

$I_{b.p.} + M(R/2)^2 = MR^2/2$

$I_{b.p.} = \frac{MR^2}{4}$

Application: Physical Atwood's Machine



$m_1 > m_2$

mass 1 : $m_1 a = m_1 g - T_1$ (down)

mass 2 : $-m_2 a = m_2 g - T_2$

pulley : $$\left.\begin{aligned} I \frac{a}{R} &= (T_1 - T_2) R \\ \| \\ \frac{1}{2} M_p R^2 \frac{a}{R} \end{aligned}\right\} \Rightarrow \frac{M_p}{2} a = T_1 - T_2$$

mass 1 - mass 2 + pulley equations

$$\left(m_1 + m_2 + \frac{M_p}{2}\right) a = (m_1 - m_2) g$$

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{M_p}{2}} g$$

62] Non-rigid Bodies

There are interesting phenomena with non-rigid bodies, even under no forces (and even for a fixed axis). This is because I - unlike m - is easy to change.

Example: Skater

$$\frac{dL}{dt} = 0$$

$$\underbrace{L_{final}}_{= I_f \omega_f} = \underbrace{L_{initial}}_{= I_i \omega_i}$$

$$\omega_f = \frac{I_i}{I_f}\omega_i$$ — by making I_f small, make ω_f large

Example: Barstool Shenanigans

$$L = I_1\omega_1 + I_2\omega_2 = 0$$

(1 = me + barstool   2 = arm + weight)

$$\Delta\theta_1 = \int \omega_1\, dt = -\int \frac{I_2\omega_2}{I_1}\, dt \underset{\text{constant } I_1, I_2}{=} -\frac{I_2}{I_1}\Delta\theta_2$$

$\Delta\theta_1 \neq 0$ for $\Delta\theta_2 = 2\pi$!

No angular momentum, start at rest, end at rest — but you've moved + reoriented!

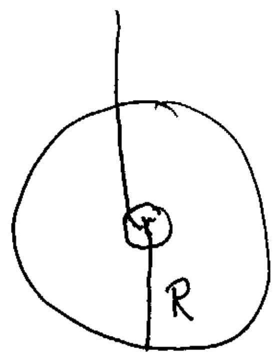
63] Energy in Single-Axis Rotations

$$\mathcal{E}_{rot.} = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i (\vec{\omega}\times\vec{r}_i)^2$$

with $\vec{\omega} = (0,0,\omega)$: $\vec{\omega}\times\vec{r}_i = (-\omega x_i, \omega y_i, 0)$; $(\vec{\omega}\times\vec{r}_i)^2 = \omega^2(x_i^2+y_i^2)$

so $\mathcal{E}_{rot.} = \frac{1}{2}\sum_i m_i (x_i^2+y_i^2)\,\omega^2 = \frac{1}{2} I \omega^2$

Example: Flywheel yoyo

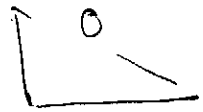


$\omega = v/r$; $I \approxeq \frac{MR^2}{2}$. $\Rightarrow \mathcal{E}_{rot.} = \frac{Mv^2R^2}{4r^2} \underset{R\gg r}{\gg} \frac{1}{2}Mv^2$

so $Mgh \approx \frac{Mv^2R^2}{4r^2}$ for fall from rest

$$\text{"}g_{eff.}\text{"} = \frac{2gr^2}{R^2}$$

Example: Rolling cylinders experiments



larger $I \Rightarrow$ slower roll
(more energy in rotation)

We'll do this quantitatively next.

$g_{eff} \sim \frac{MR^2}{I}$