Going back: ("unreducing")

We chose to work in a frame with the center of mass fixed at the origin and axes such that the orbit lies in the x-y plane. One can "put back" these choices trivially. This is important for applications to distant star-systems (double stars, extra-solar planetary systems, satellites, etc.). The relative orientation + velocity must be fit to observation.

Interesting effects:

projection of A (so T ~~does not~~ + "A observed" only bound M+m)

motion of <u>both</u> bodies.

$$\vec{r}_1 = \underbrace{\frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}}_{\text{fixed}} + \frac{m_2}{m_1+m_2}(\vec{r}_1-\vec{r}_2)$$

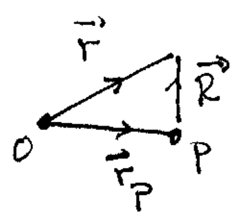
$$\frac{\text{orbit of } 1}{\text{orbit of } 2} \propto \frac{m_2}{m_1}$$

46] Relative Motion: Recollection, and Getting to Rotation

$\vec{F} = m\ddot{\vec{r}}$ in inertial frame. It is often natural, however, to use non-inertial frames – e.g., you may be interested in mechanics in a vehicle (elevator, train, ship, car, rocket) – or on surface of Earth!

It's important to keep straight two cases:

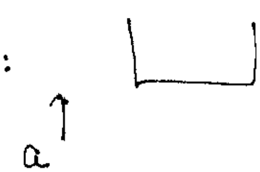
i) moving origin (but fixed orientation)



$$\vec{R} = \vec{r} - \vec{r}_P \quad \text{"take out" translation}$$

$$m\ddot{\vec{R}} = m\ddot{\vec{r}} - m\ddot{\vec{r}}_P = \vec{F} - m\vec{a}_P$$

example:



rising elevator $\Rightarrow$ extra "fictitious" force downward

$\Longrightarrow$ equivalence principle, ...

ii) rotating frame (but fixed origin)

$$\vec{R}(t) = \mathcal{R}(t)^{-1}\,\vec{r}(t) \quad ; \quad \text{so if } \vec{r}(t) = \mathcal{R}(t)\, r_0,$$

inverse of rotation operator

it looks constant

"taking out" rotation

47] Small rotations around an axis

Consider rotation through $\Delta\phi$ around an
$\uparrow$ (assumed small)

axis $\hat{n}$. (The direction is fixed by right-hand rule.)

Then any vector $\vec{S}$ changes by

$$\vec{S} \rightarrow \vec{S} + \Delta\phi\ \hat{n} \times \vec{S} \quad (!)$$

This gives another nice integpretation of the $\times$-product.

Why? $\Delta\vec{S} \perp \vec{S}$ and $\perp \hat{n}$ ✓

the piece $\vec{S} \parallel \hat{n}$ has no effect ✓

if $\vec{S} \perp \vec{n}$, rotation by $\Delta\phi$ ✓

example: $\hat{z}$-axis

$$\Delta\vec{S} = \Delta\phi\ \hat{z} \times (x, y, z) = \Delta\phi\ (-y, x, 0)$$

$$= \Delta\phi\ \underbrace{r_{\cancel{polar}}\ (-\sin\theta, \cos\theta, 0)}_{\text{polar coordinates in x-y plane}}$$

$$= \Delta\phi\ r\ \hat{\theta}$$

With $\hat{n}\,\Delta\phi \equiv \vec{\omega}\,dt$,

$\uparrow$ angular velocity vector: $\vec{\omega} \equiv \hat{n}\,\frac{d\phi}{dt}$

$$\Delta\vec{S} = \vec{\omega} \times \vec{S}\ dt$$

48 | Master formula for dynamical (time-dependent) vectors in rotating frame

$$\boxed{\left(\frac{d\vec{s}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{S}}{dt}\right)_{\text{rot.}} + \vec{\omega}\times\vec{S}}$$

from preceding page!

At one level, very simple: vector changes from change measured <u>within the rotating frame</u>, and from change <u>associated with motion of frame itself.</u>

But puzzling, perhaps. How can $\vec{s}$ have two different time derivatives??

Really: $\vec{s}(t) = \mathcal{R}(t)\vec{S}(t)$

$$\vec{s}(t+\Delta t) = \mathcal{R}(t+\Delta t)\,\vec{S}(t+\Delta t)$$

$$\vec{s}(t) + \frac{d\vec{s}}{dt}\Delta t = \Big[\underbrace{\mathcal{R}(t+\Delta t)\,\mathcal{R}^{-1}(t)}_{\text{small additional rotation}}\;\mathcal{R}(t)\Big]\Big[\vec{S}(t) + \frac{d\vec{S}}{dt}\Delta t\Big]$$

$$= \Big[\mathbb{1} + \vec{\omega}\,\Delta t\;\times\Big]\Big[\mathcal{R}(t)\vec{S}(t) + \mathcal{R}(t)\frac{d\vec{S}}{dt}\Delta t\Big]$$

$$\approx \mathcal{R}(t)\vec{S}(t) + \Big[\vec{\omega}\times\mathcal{R}\vec{S} + \mathcal{R}\frac{d\vec{S}}{dt}\Big]\Delta t$$

So $$\frac{d\vec{S}}{dt} = \vec{\omega} \times R\vec{S} + R\frac{d\vec{S}}{dt}$$

The "master formula" results by picking $R(t_0) = 1$ at the moment of interest. The rotating frame is constantly "reinvented"!

49] Master formula for dynamics

We need to apply this to the vectors of interest for dynamics, i.e. velocity and acceleration.

$$\vec{v}_{in.} = \left(\frac{d\vec{r}}{dt}\right)_{in.} = \left(\frac{d\vec{r}}{dt}\right)_{rot.} + \vec{\omega} \times \vec{r} \qquad (*)$$

$$\vec{a}_{in} = \left(\frac{d\vec{v}_{in}}{dt}\right)_{in} = \left(\frac{d\vec{v}_{in}}{dt}\right)_{rot.} + \vec{\omega} \times \vec{v}_{in.}$$

$$\overset{\text{using } (*)}{=} \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot.} + \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot.}$$

$$+ \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot.} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

[note $\left(\frac{d\vec{\omega}}{dt}\right)_{in.} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot.}$, since $\vec{\omega} \times \vec{\omega} = 0$!]

So, in rotating frame (dropping subscript)

$$\frac{\vec{F}}{m} = \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{\omega}}{dt} \times \vec{r} + 2\,\vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

or

$$\boxed{m\frac{d^2\vec{r}}{dt^2} = \vec{F} \underbrace{- m\frac{d\vec{\omega}}{dt} \times \vec{r}}_{\text{"gyro"}} \underbrace{- m 2\,\vec{\omega} \times \vec{v}}_{\text{Coriolis}} \underbrace{- m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{centrifugal}}}$$

This contains a wealth of phenomena. Let's get oriented with semi-familiar cases:

i) $\vec{v} = 0$, $\vec{\omega} = \text{const.} = (0, 0, \omega)$

$$\vec{\omega} \times \vec{r} = (0,0,\omega) \times (x,y,z) = (-\omega y, \omega x, 0)$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (-\omega^2 x, -\omega^2 y, 0)$$ gives non-vanishing term.

This is the "centrifugal ~~force~~ force" for constant rotation (that is, $-m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2 (x, y, 0)$)

ii) $\dfrac{d\vec{\omega}}{dt} \neq 0$, but still in $\hat{z}$ direction

$$\frac{d\vec{\omega}}{dt} = (0, 0, \alpha)$$ ≡ angular acceleration

New term $-m\dfrac{d\vec{\omega}}{dt} \times \vec{r} = (m\alpha y, -m\alpha x)$

$= \text{"}-mr\alpha\,\hat{\theta}\text{"}$

just corrects for acceleration in $\hat{\theta}$ direction!
[note: r-dependent]

Note that if a body has large $\vec{\omega}$, changing it requires large forces in strange directions. This is the origin of gyroscopic phenomena.

(due to "gyro" term.)