56) Angular Momentum Recollection + Application

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}$$

$$m \vec{r} \times \frac{d^2 \vec{r}}{dt^2} = \vec{r} \times \vec{F} \equiv \text{torque} \equiv \vec{\tau}$$
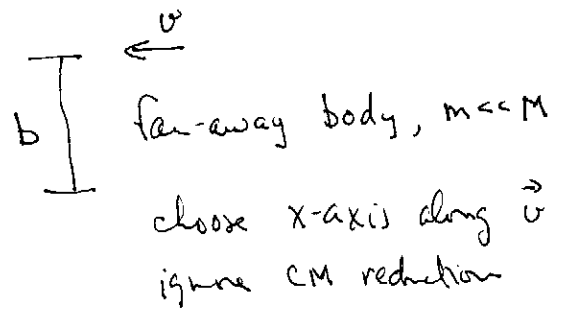
$$\frac{d}{dt} \left(m \vec{r} \times \frac{d\vec{r}}{dt} \right) = \frac{d}{dt} \vec{L} \qquad \Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\vec{L} = m \vec{r} \times \vec{v} = \vec{r} \times \vec{p} = \text{angular momentum}$$

Analogous to $\frac{d\vec{p}}{dt} = \vec{F}$ for linear momentum.

· For central forces $\vec{\tau} = 0 \Rightarrow \vec{L}$ is conserved. We used this, ~~before~~ recently (Kepler's 2nd).

exemple: Gravitational capture

far-away body, $m \ll M$
choose x-axis along $\vec{v}$
ignore CM reduction

$$|L| = m b v$$

$$\mathcal{E} = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2m r^2} - \frac{GMm}{r} = \frac{1}{2} m \dot{r}^2 + \frac{m b^2 v^2}{2 r^2} - \frac{GMm}{r}$$

$$= \mathcal{E}_0 = \frac{1}{2} m v^2$$

At ~~close~~ Is $r = R$ allowed?

Requires $(\dot{r}^2 > 0)$ $\frac{1}{2} m v^2 - \frac{m b^2 v^2}{2 R^2} + \frac{GMm}{R} > 0$

$$\boxed{v^2 \left(\frac{b^2}{R^2} - 1 \right) \leq \frac{2GM}{R}}$$

or $\boxed{\frac{b^2}{R^2} \leq 1 + \frac{2GM}{R v^2}}$

this is $\rightarrow$ $b = R$ as $v \rightarrow \infty$, but is enhanced over "geometrical" capture for finite v.

57) Angular Momentum for Systems

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i + \sum_j \vec{F}_{ij}$$

external forces ← (arrow pointing to $\vec{F}_i$)

internal forces ← (arrow pointing to $\sum_j \vec{F}_{ij}$)

$$\sum_i m_i \vec{r}_i \times \frac{d^2 \vec{r}_i}{dt^2} = \sum_i \vec{r}_i \times \vec{F}_i + \sum_i \sum_j \vec{r}_i \times \vec{F}_{ij}$$

work on this (arrow pointing to last term)

$$\sum_i \sum_j \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2}\left(\sum_i \sum_j + \sum_j \sum_i\right)\left(\vec{r}_i \times \vec{F}_{ij}\right)$$

$$= \frac{1}{2} \sum_i \sum_j \left(\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}\right)$$

$$\underset{3^{rd} \text{ Law}}{=} \frac{1}{2} \sum_i \sum_j \left(\vec{r}_i - \vec{r}_j\right) \times \vec{F}_{ij}$$

Now if $\vec{F}_{ij} \propto \vec{r}_i - \vec{r}_j$ (central forces), this $= 0$!

Then

$$\sum_i m_i \vec{r}_i \times \frac{d^2 \vec{r}_i}{dt^2} = \sum_i \vec{r}_i \times \vec{F}_i$$

$$\parallel \qquad\qquad \parallel$$

$$\sum_i \frac{d}{dt} \vec{L}_i \quad \otimes \quad \sum_i \vec{\tau}_i \; ;$$

or $\frac{d}{dt} \vec{L} = \vec{\tau}$ with $\vec{L} = \sum_i \vec{L}_i$, $\vec{\tau} = \sum_i \vec{\tau}_i$

The result is more general than this derivation.

Example. rod-pendulum

$\lambda = $ mass/length

$$\vec{L} = \int_0^l dm \; \vec{r} \times \vec{v} = -\hat{z} \int_0^l \lambda \, dr \; r \cdot r\dot\theta = -\hat{z} \frac{l^3}{3} \lambda \dot\theta = -\hat{z} \frac{Ml^2}{3} \dot\theta$$

into plane

general theorem for near-Earth gravity

$$\vec{\tau} = \int \vec{r} \times d\vec{F} \; , \quad d\vec{F} = dm \, \vec{g}$$

$$\Downarrow$$

$$\vec{\tau} = \int dm \, \vec{r} \times \vec{g} = M \vec{r}_{CM} \times \vec{g} = \vec{r}_{CM} \times M\vec{g}$$

$$\left\{ \begin{array}{l} = \left(\int dm \, \vec{r} \right) \times \vec{g} \\ = M \vec{r}_{CM} \times \vec{g} \end{array} \right\}$$

$\Rightarrow$ The torque may be calculated as if the force is acting at the CM

NB. Near-Earth gravity only!!

In present example:
$$\vec{\tau} = Mg \frac{l}{2} \sin\theta \, \hat{z} \quad \text{(evaluating × product)}$$

so $\frac{d\vec{L}}{dt} = \vec{\tau}$ reads $-\frac{Ml^2}{3} \ddot\theta = \frac{Mgl}{2} \sin\theta$, or $\ddot\theta = -\frac{3g}{2l} \sin\theta$

Small θ. $\ddot\theta = -\frac{3g}{2l}\theta$; $\omega^2 = \frac{3g}{2l}$ oscillations.

58] Statics

If a rigid body is not moving

$\vec{\tau}$ (no internal motion!)
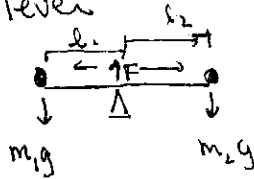
  i) $\Sigma \vec{F}^{ext.} = 0$

  ii) $\Sigma \vec{\tau}^{ext.} = 0$

Since $\frac{d\vec{p}}{dt} = 0$, $\frac{d\vec{L}}{dt} = 0$. And if $\Sigma \vec{F}^{ext.} \neq 0$ or $\Sigma \vec{\tau}^{ext.} \neq 0$,

for a rigid body, you will have $\underset{\downarrow}{\underline{visible}}$ motion. So

(bulk)

i) and ii) are the fundamental equations of statics. This is

a vast subject in engineering (you want buildings, bridges, etc. to
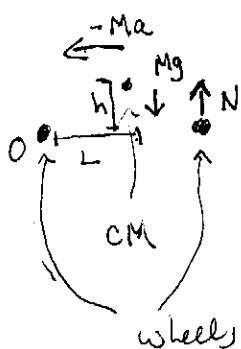
be static!)

example: lever

$$F = m_1 g + m_2 g$$

around fulcrum ~~$l_2 m_2 g$~~ $l_2 m_2 g - l_1 m_1 g = 0$ $\dfrac{l_1}{l_2} = \dfrac{m_2}{m_1}$

example: motorcycle lift-off

torques around O: $hMa - LMg + 2LN = 0$

with $N \geq 0$

possible only for $a < \dfrac{L}{h} g$

with bigger a, "lift off"

Note high CM and limited wheelbase are conducive to liftoff.

—— · ——

Converse? If $\sum F = 0$, $\sum \tau = 0$ we have $\vec{P} = \vec{P}_0$, $\vec{L} = \vec{L}_0$. The first is removed "trivially" by going to CM frame. The 2nd ($\vec{L}_0 \neq 0$) is much more interesting. It does <u>not</u> correspond to a constant rotation, ~~still~~ in general. We'll treat it down the road.

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59] Rigid Body Dynamics: General Setting

The position and orientation of a rigid body
can be specified using 6 quantities. Specifically,
we ~~need to~~ can specify

i) the position of a reference point A (e.g., the CM)

    - 3 coordinates

ii) having fixed A, we need to specify the
position of a second point B. This lies on a
definite sphere around A, since distances are
fixed (the body is rigid). This brings in 2
more coordinates

iii) having fixed A and B, we can still make
a rotation around the axis through them. We
use this to match the actual position of a third
point C (fixed in the body). This brings in 1
more angle.

With three (noncollinear) points fixed, ~~nothing~~ we can
"triangulate" to other points based on the distances to
A,B,C — the position and orientation of the rigid body
are completely specified. It took 6 parameters — three
distances ~~position~~, three angles.

We have 6 equations for how things change in time:

$$\frac{d\vec{p}}{dt} = \vec{F} \quad , \quad \frac{d\vec{L}}{dt} = \vec{\tau}$$

This is just enough so that given initial values of position, orientation, and their rate of change (so we can get $\vec{p}(0)$, $\vec{L}(0)$), we can solve for later values!

60) Rotation Around a Fixed Axis

The phenomena of rigid body motion can be intricate and counter-intuitive, so we'll build up from special cases, that are of importance in themselves. Here: pure rotation around a fixed axis

For concreteness choose the rotation axis to be the z-axis. To get an equation for the rate of rotation, we analyze the z-component of $\frac{d\vec{L}}{dt} = \vec{\tau}$.

$$\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i$$

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \quad , \quad \text{with } \vec{\omega} = (0, 0, \omega), \; \vec{r}_i = (x_i, y_i, z_i)$$

$$\text{so } \vec{\omega} \times \vec{r}_i = (-\omega y_i, \omega x_i, 0)$$

$$\vec{r}_i \times \vec{v}_i = \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = (*, *, \omega(x_i^2 + y_i^2))$$

$$\boxed{L_z = \sum_i m_i (x_i^2 + y_i^2) \omega \equiv I_{zz} \, \omega \;\; (\text{or just } I\omega)}$$

with $\quad I \equiv \sum m_i (x_i^2 + y_i^2) = \sum m_i r_i^2 \longrightarrow \int dm(r)\, r^2$

$\underset{\uparrow}{}$

$\left(\begin{array}{c}\text{distance} \\ \text{from axis}\end{array}\right)^2$

The value of I is a property of the body (and the axis!) that does not change with time, as long as these don't. Specifically, it does not change during rotation around the axis. Thus our dynamical equation becomes

$$\frac{d(I\omega)}{dt} = \vec{\tau}_z$$

$$\|$$

$$I\frac{d\omega}{dt} = I\frac{d^2\theta}{dt^2}$$

$$\underset{\text{since } \omega = \frac{d\theta}{dt}}{\overset{\uparrow}{}}$$

Example: rotating mass arrangement

large I small I