For near-Earth gravity, of course,

$$\vec{F}_{ext.} = \sum m_i \vec{g} = M\vec{g}$$

so $$\ddot{\vec{r}}_{CM} = \vec{g} \quad !$$

(exp'ts.)

3/5/4 example: collisions $\quad \vec{F}_{ext.} = 0, \quad \underline{\underline{\vec{v}_{CM} = \text{const.}}}$

A $\qquad$ B

$m_A, v_A \qquad m_B, v_B \qquad$ (v_B is negative)

CM. : $$\frac{m_A\vec{x}_A + m_B\vec{x}_B}{m_A + m_B}$$

its velocity:
$$\frac{m_A\vec{v}_A + m_B\vec{v}_B}{m_A + m_B}$$

after collision: $\vec{v}_A', \vec{v}_B'$

but $$\frac{m_A\vec{v}_A' + m_B\vec{v}_B'}{m_A + m_B} = \frac{m_A\vec{v}_A + m_B\vec{v}_B}{m_A + m_B}$$

$m_A\vec{v}_A + m_B\vec{v}_B$ = conserved. (momentum, total momentum.)

Applications

A) method of measuring mass!

$$m_A v_A' + m_B v_B' = m_A v_A + m_B v_B$$

$$m_A / m_B = \frac{v_B - v_B'}{v_A' - v_A}$$

B) inelastic collisions: All you need...

9/26

25] Reformulation of Newton. Momentum

(Actually, this is Newton's original formulation!)

$$\vec{F} = \frac{d\, m\vec{v}}{dt} \equiv \frac{d\vec{p}}{dt} \qquad \vec{p} = m\vec{v}$$

correction to CM

$$\vec{P} = M\vec{v}_{CM} = \sum m_i \left(\frac{\sum m_i \dot{\vec{x}}_i}{\sum m_i} \right) = \sum m_i \dot{\vec{x}}_i = \sum \vec{p}_i$$

advantages: more advanced theories (relativity)

$$\frac{d \frac{m_0 \vec{v}}{\sqrt{1-v^2/c^2}}}{dt}$$

angular motion

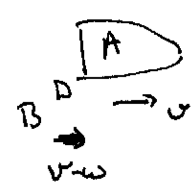
variable mass problems

$$\vec{F}_{ext.} = \frac{d(\vec{P}_A + \vec{P}_B)}{dt} = \frac{d\vec{P}_A}{dt} + \frac{d\vec{P}_B}{dt}$$

can divide it up "ad lib"

"old" exhaust can be ignored!

example: rocket



$$0 = \frac{d\vec{P}_{rocket}}{dt} + \frac{d\vec{P}_{exhaust}}{dt}$$

$$= \frac{d}{dt}(M-kt)\vec{v} + k(\vec{v}-\vec{w})$$

$$= (M-kt)\frac{d\vec{v}}{dt} - k\vec{w}$$

specialise to 1 component

$$dv = \frac{k\omega}{M-kt}\,dt$$

$$v - v_0 = \omega \ln \frac{M}{M-kt} = \omega \ln \frac{1}{1-\frac{kt}{M}}$$

$\omega \cdot 0$ ✓

26) Generalized Conservation

$$\vec{F}_{ext.} = \frac{d}{dt}\vec{P}_{tot.}$$

$$\vec{P}_{tot.} = \sum m_i \vec{v}_i$$

$\Rightarrow$ $\vec{P}_{tot.} = \text{const.}$ if $\vec{F}_{ext.} = 0$

~~also~~ more generally, $(\vec{P}_{tot.})_i = \text{const.}$ if $(\vec{F}_{ext.})_i = 0$

ex: near-earth gravity $\vec{g} = -g\hat{z}$

$(\vec{P}_{tot.})_x$, $(\vec{P}_{tot.})_y$ = conserved.

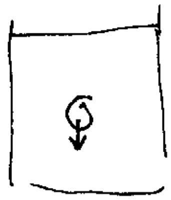
smooth wall ~~(no fric~~ [by the way: wall motion...]

$x=z=0$

$(\vec{P}_{tot})_x$, $(\vec{P}_{tot.})_z$ = const. (neglecting gravity)

$(\vec{P}_{tot.})_x$ = const. (even with gravity)

Example: honey pot



$\rho_{honey} =$

$\rho_{ball} =$, V

$v_{ball} =$ down

momentum of honey?

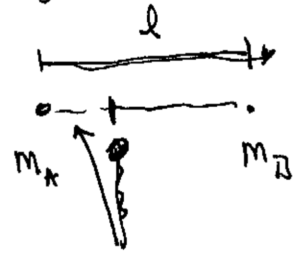
add extra bit $(\rho_h - \rho_b)V, \vec{v}$

then no ~~change~~ in flow, no change in cm position!

so $\rho_b V \vec{v} + (\rho_h - \rho_b) V \vec{v} + \vec{P}_{honey} = M \dot{x}_{cm} = 0$

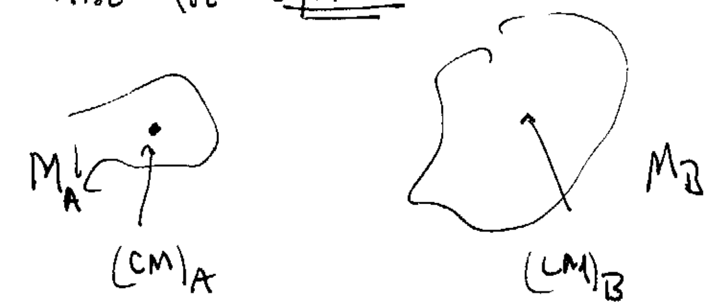
and $\vec{P}_{honey} = -\rho_h V \vec{v}$!

27) Finding the CM



$$\frac{l m_B}{m_A + m_B}$$

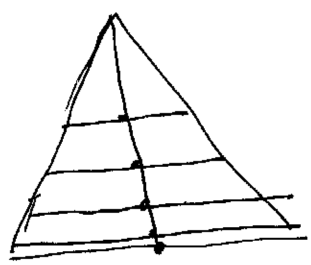
← Also for systems



$$\vec{X}^{A}_{CM} = \frac{\sum_{i\in A} m_i \vec{X}_i}{\sum_{i \in A} m_i} \quad \vec{X}^{B}_{CM} = \frac{\sum_{i\in B} m_i \vec{X}_i}{\sum_{i\in B} m_i}$$

$\sum_{i\in A} m_i \leftarrow M_A$, $\quad \sum_{i\in B} m_i \leftarrow M_B$

uniform triangles
- intersection of bisectors!



$$\vec{X}^{A+B}_{CM} = \frac{\sum_{i\in A+B} m_i \vec{X}_i}{\sum_{i\in A+B} m_i} \quad \leftarrow M_C$$

$$M_A \vec{X}^A_{CM} + M_B \vec{X}^B_{CM} = (M_{A+B}) \vec{X}^{A+B}_{CM}$$

$$\vec{X}^{A+B}_{CM} = \frac{M_A \vec{X}^A_{CM} + M_B \vec{X}^B_{CM}}{M_A + M_B}$$

need not be a point in the system

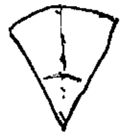
Sectors (chords, ~~edges~~ slices, wedges)

chord

$$M = 2\lambda R\theta$$

$$Mh = \int_{-\theta}^{\theta} \lambda R\cos\theta R d\theta = 2\lambda R^2 \sin\theta \Rightarrow h = \frac{R\sin\theta}{\theta}$$

$\theta = 0, \pi$ ✓

slice  $dM = \sigma 2\theta r\,dr \qquad M = \sigma R^2\theta$

$$h\,dM = \frac{r\sin\theta}{\theta}\,\sigma 2\theta r\,dr = 2\sigma\sin\theta\, r^2\,dr$$

$$\int h\,dM = \frac{2}{3}\sigma R^3\sin\theta$$

$$H = \frac{\frac{2}{3}\sigma R^3\sin\theta}{\sigma R^2\theta} = \frac{2}{3}R\frac{\sin\theta}{\theta}$$

wedge

$$\mathcal{H} \qquad dM = \rho 2\pi r^2 (1-\cos\theta)\,dr$$ 

$$M = \frac{2\pi\rho}{3}(1-\cos\theta)R^3$$

state but don't do

$$h\,dM = \rho 2\pi r^2(1-\cos\theta)\,dr \times \frac{r\sin\theta}{\theta}$$

$$M\mathcal{H} = \frac{\rho\pi}{2}\frac{(1-\cos\theta)\sin\theta}{\theta}$$

$$\mathcal{H} = \frac{3}{4}R\frac{\sin\theta}{\theta}$$

cut-outs

$$M_r\vec{x}_r^{CM} + m_c\vec{x}_c^{CM} = M_{whole}\vec{x}_{whole}^{CM}$$

$$M_r\vec{x}_{CM} = M_{whole}\vec{x}_{wh}^{cm} - \dots$$

Physical determination:

hang it anywhere, plumb line passes through CM

(proof later)

~ 10/1

28) Center of Mass Frame

i) Invariance of the equations : "Galilean relativity"

$$\vec{r}_i{}' = \vec{r}_i - \vec{v}\,t \qquad \vec{F}' = \vec{F}$$

$$t' = t$$

$$\frac{d^2\vec{r}_i}{dt^2} = \vec{F}_{i\,ext.} + \vec{F}_{ij}\left[\vec{r}_i - \vec{r}_j\right]$$

same thing   same thing   same thing !

$$\cancel{\vec{v}_{CM}{}' = \vec{v}_{CM}}$$

ii) $$\vec{r}_{CM}{}' = \vec{r}_{CM} - \vec{v}\,t$$

$$\left(= \frac{\sum m_i \vec{r}_i{}'}{\sum m_i} = \frac{\sum (m_i \vec{r}_i - m_i \vec{v}\,t)}{\sum m_i} = \vec{r}_{CM} - \vec{v}\,t \right)$$

$$\vec{v}_{CM}{}' = \vec{v}_{CM} - \vec{v}$$

choose $\vec{v} = \vec{v}_{CM}$

Things are often simpler to analyze in CM frame.
You can always transform back at the end.

~~29) Elastic~~

29) Elastic Collisions (Slightly Unconventional)

Invariance of the equations

$$\vec{r}_i'(t) = \vec{r}_i(-t) \qquad \vec{F}_i' = \vec{F}$$

$t' = -t$

$$\Rightarrow \frac{d\vec{r}_i'(t)}{dt'} = -\frac{d\vec{r}_i}{dt}(-t)$$

$$\frac{d^2\vec{r}_i'}{dt'^2}(t) = +\frac{d^2\vec{r}_i}{dt^2}(-t)$$

But: friction? inelastic collisions?

Internal Motion

For stiff bodies, we can neglect this. Then in CM, things "bounce back"! (possibly rotated) ↳ same |velocities|.

(If smaller velocities, time-reverse has bigger velocities: too good to be true!)

Start with

lab frame, before ← N.B.: 1d vectors (sign only!)

$m_1, \vec{v}_1 \quad | \quad m_2\ \vec{v}_2$

move to CM frame; use velocity $\vec{v}$ such that

$$m_1(\vec{v}_1 - \vec{v}) + m_2(\vec{v}_2 - \vec{v}) = 0$$

$$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}$$

so $\vec{v}_1 - \vec{v} = \dfrac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1+m_2}$ etc.

CM frame, before

$$m_1,\ \frac{m_2(\vec{v}_1-\vec{v}_2)}{m_1+m_2} \quad \Bigg| \quad m_2,\ \frac{m_1(\vec{v}_2-\vec{v}_1)}{m_1+m_2}$$

$\Downarrow$

CM frame, after

$$m_1,\ \frac{m_2(\vec{v}_2-\vec{v}_1)}{m_1+m_2} \quad \Bigg| \quad m_2,\ \frac{m_1(\vec{v}_1-\vec{v}_2)}{m_1+m_2}$$

now go back to lab frame: <u>add</u> $\vec{v}$

lab frame, after

$$m_1,\ \underbrace{\left(\frac{m_2(\vec{v}_2-\vec{v}_1)}{m_1+m_2} + \frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}\right)}_{} \quad \Bigg| \quad m_2,\ \frac{2m_1\vec{v}_1 + (m_2-m_1)\vec{v}_2}{m_1+m_2}$$

$$\boxed{\text{"}\ \frac{(m_1-m_2)\vec{v}_1 + 2m_2\vec{v}_2}{m_1+m_2}}$$

if $m_2/m_1 \to \infty$

$$m_1, \; -\vec{v}_1 + 2\vec{v}_2 \quad \Big| \quad m_2, \; \vec{v}_2$$

$\vec{v}_2 = 0$ ✓

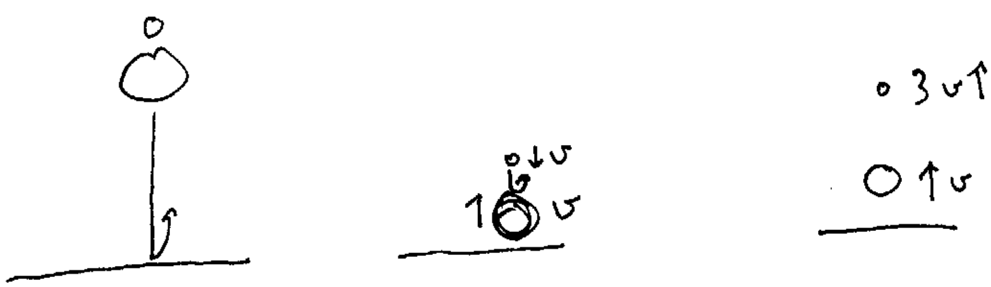
otherwise move with it ... ✓

$\vec{v}_1 - \vec{v}_2 \to \vec{v}_2 - \vec{v}_1 \to 2\vec{v}_2 - \vec{v}_1$

in heavy frame — reversed — back to lab

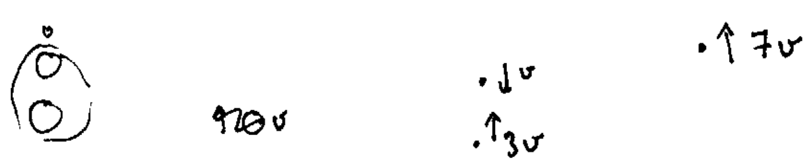
* do executive toy here

Example: Superbounce!



$$S_v = \frac{v^2}{2g} \qquad S_{3v} = 9 S_v \; !!$$

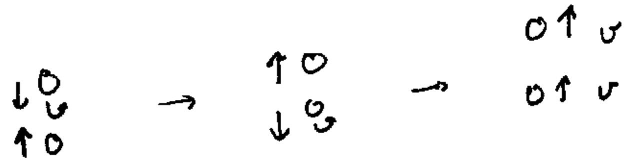
3 balls



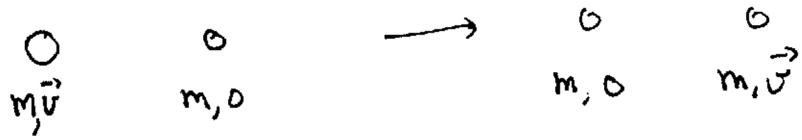
49 times!

4 balls $(15)^2 = 225$ times! n balls $(2^n-1)^2$ times!

equal masses?



"executive toy"



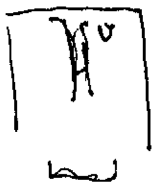
move to *, p.36

30] Impulse

$$\Delta \vec{P} = \int \vec{F}\, dt$$

This can be useful when the LHS is "obvious".

ex:



$$\underbrace{\rho A v}_{\Delta m}\ \underbrace{v}_{v} = \rho A v^2$$

$\rightsquigarrow A_0 v_0$ $\qquad \rho A_0 v_0 v$; $v = \sqrt{v_0^2 - 2gh}$

[$v = v_0 - gt$; $t = \frac{v_0 - v}{g}$

$h = v_0 t - \frac{1}{2} g t^2$

$\rightsquigarrow \frac{1}{g}\left[v_0(v_0 - v) - \frac{1}{2}(v_0 - v)^2\right]$

$= \frac{1}{2g}(v_0^2 - v^2)$]

ex: trampoline

$$\Delta P = 0 \qquad \int_0^T \left(\vec{F}_{\text{man, tramp.}}\, dt + m\vec{g}\, dt\right)$$

$$\int_0^T \vec{F}_{\text{man, tramp}}\, dt = -m\vec{g}\, T$$

$$\frac{1}{T} \int_0^T \vec{F}_{\text{tramp, man.}}\, dt = +m\vec{g}$$

ex: origin of pressure

same idea

31) General Comments on Energy

i) forces + dynamics as function of position, as opposed to time.

ii) like momentum, a much more general principle beyond, classical mechanics

iii) can be "hidden" or dissipated; or stored.

this contrasts with momentum

iv) internal forces do not cancel in general

32) Work-Energy Theorem: Particles

$$m\ddot{\vec{r}} = \vec{F}(\vec{r})$$

dot-product with $d\vec{r} = \dot{\vec{r}}\,dt$

$$m\ddot{\vec{r}}\cdot\dot{\vec{r}}\,dt = \vec{F}\cdot d\vec{r}$$

$$\frac{m}{2}\frac{d}{dt}(\dot{\vec{r}}\cdot\dot{\vec{r}})\,dt = \frac{d}{dt}\left(\frac{m}{2}v^2\right)$$

$$\int_A^B : \quad \underbrace{\frac{m}{2}(v_B^2 - v_A^2)} = \int_A^B \vec{F}\cdot d\vec{r}$$

$$\boxed{\Delta \text{ kinetic energy} = \text{work}}$$

33) Analysis of Cases

The work-energy theorem is ~~most~~ especially useful when one can evaluate the right-hand side without having to solve for the motion! These are several important cases where this it applies:

This is not true in general, of course.

o) If several force-types act, total work = sum of partial works

i) force-fields in 1 dimension

$$\int_A^B \vec{F}(x)\cdot d\vec{x} = \int_A^B F(x)\,dx$$

~~Let x~~ Let $V(x) \equiv -\int_{x_0}^{x} F(x)\,dx$, so $F(x) = -\frac{dV}{dx}$

Then $\text{work}_A^B = -\big(V(B) - V(A)\big) = -\Delta V$

$= -\Delta$ (potential energy)

$\Rightarrow$ Δ (kinetic energy + potential energy)

= work by <u>other</u> forces

Note: df^{n} of V contains an arbitrary constant. Typically, we choose it to vanish at a convenient "natural" place

examples:

spring $\quad F(x) = -k(x - x_0)$

particle attached to

$V(x) = \frac{1}{2}k(x - x_0)^2 \leftarrow V = 0$ at equil. pt.

near-earth gravity, vertical motion

$F(z) = -mg$

$V(z) = mgz \leftarrow$ (to vanish at $z = 0$)

Newtonian gravity, vertical motion

$F(r) = -\frac{GMm}{r^2}$

$V(r) = -\frac{GMm}{r} \leftarrow$ (to vanish at $r = \infty$)

pendulum

$\vec{F} \perp \vec{v}$

$V(\theta) = mg\ell(1 - \cos\theta)$

$K = \frac{1}{2} m\ell^2 \dot{\theta}^2$

$K + V = \text{const.}$; take $\frac{d}{dt} \Rightarrow eq^{n}$ of motion

iii) conservative force-fields in 3d

If $\int_A^B \vec{F}(\vec{r})\,d\vec{r}$ depend only on the endpoints, not on the path, we say the force-field is conservative.

spring (0 size, at origin)

$$\vec{F}(\vec{r}) = -k\vec{r}$$

$$\vec{F}(\vec{r})\cdot d\vec{r} = -k\vec{r}\cdot d\vec{r} = -\tfrac{1}{2}k\,dr^2$$

so $V(\vec{r}) = \tfrac{1}{2}kr^2$

near-earth gravity

$$\vec{F}(\vec{r}) = -mg\hat{z}$$

$$\vec{F}(\vec{r})\cdot d\vec{r} = -mg\,dz$$

$V(\vec{r}) = mgz \implies$ any "1d" force

Newtonian gravity (fixed source)

$$\vec{F}(\vec{r}) = -\frac{GMm\,\hat{r}}{r^2}$$

use $\hat{r}, \hat{\theta}, \hat{\varphi}$ basis [elaborate on argument in plane]

$$\vec{F}(\vec{r})\cdot d\vec{r} = -GMm\,d\left(-\frac{1}{r}\right) = GMm\,d\left(\frac{1}{r}\right)$$

$$V(\vec{r}) = -\frac{GMm}{r}$$

example: escape velocity