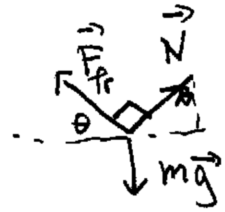
I 9/12

15) Moving Cart Exp't.

free diagram



vertical: $N\cos\theta + F_{fr.}\sin\theta = mg$

horizontal: $N\sin\theta - F_{fr.}\cos\theta = ma$

$$N = m(g\cos\theta + a\sin\theta)$$

$$F_{fr.} = m(g\sin\theta - a\cos\theta)$$

$$F_{fr.} > 0 \quad : \quad g\tan\theta > a$$

$$F_{fr.} < \mu N$$

$$g(\sin\theta - \mu\cos\theta) < a(\cos\theta + \mu\sin\theta)$$

$$(g\tan\theta >)\ a > g\,\frac{\tan\theta - \mu}{1 + \mu\tan\theta}$$

In this band, the glider stays put!

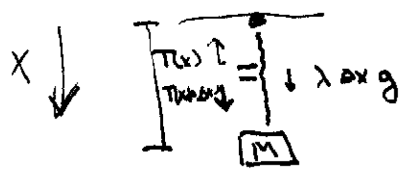
Q: What happens if $\vec{F}_{fr}$ points the other way?
~~Does~~ When does this case apply?

Sept. 12

→ allude to Jean Valjean, SP #2

16) Ropes and Pulleys

Analyze "bit by bit"



mass/unit length = λ

$T(x)$

$$T(x) = T(x+\Delta x) + \lambda \Delta x \, g$$

$$\downarrow$$

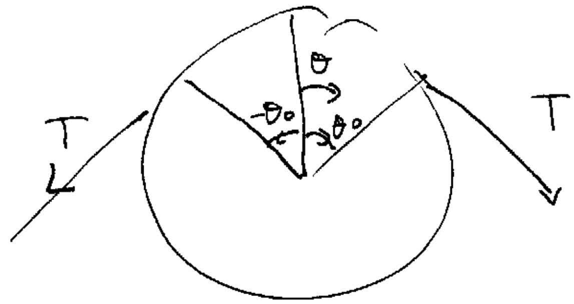
$$\frac{dT}{dx} = -\lambda g$$

$$\left.\begin{aligned} T(x) &= -\lambda g x + \text{const.} \\ \text{block: } T(L) &= Mg \end{aligned}\right\} \Longrightarrow \quad T(x) = \lambda g (L-x) + Mg$$

$$T(0) = (\lambda L + M) g$$

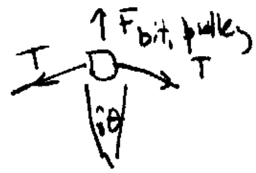
W.B.

Note for $\lambda = 0$ $T = \text{const.} = Mg$; rope "transmits" the force.



constant tension } $\lambda = 0$
neglect gravity }
no friction

forces on bit of rope

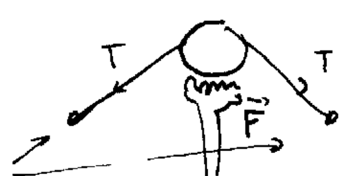


(constraint force, in normal direction — i.e., radially <u>out</u>)

$$F = 2T\frac{\delta\theta}{2} = T\delta\theta$$

$$\left(\vec{F}_{pulley,\, rope}\right)_{down} = \int_{-\theta_0}^{\theta_0} T\cos\theta \, d\theta = 2T\sin\theta_0$$

W.I.S.B.!



total force on system

$$F = 2T\sin\theta_0$$

17) Motion in Polar vector! ~~Component~~ Coordinates

$\hat{r}, \hat{\theta}$ define a non-trivial vector field. That is, their values – the directions they define – depend on where you are! (not what inertial motion gives you!)

In fact
$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

For mechanics:

$$\text{position } \boxed{\vec{r}(t) = r\,\hat{r}}$$

$$\vec{v} = \dot{\vec{r}} \left(\equiv \frac{d\vec{r}}{dt}\right) = \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$

Since $\dot{\hat{r}} = -\sin\theta\,\dot{\theta}\,\hat{x} + \cos\theta\,\dot{\theta}\,\hat{y} = \dot{\theta}\,\hat{\theta}$

$$\boxed{\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{r}\,\hat{r} + \dot{r}\,\dot{\hat{r}} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\,\dot{\hat{\theta}}$$

Since $\dot{\hat{\theta}} = -\cos\theta\,\dot{\theta}\,\hat{x} - \sin\theta\,\dot{\theta}\,\hat{y} = -\dot{\theta}\,\hat{r}$

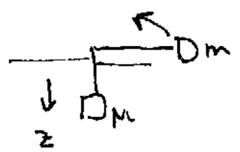
$$\vec{a} = \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} - r\dot{\theta}^2\,\hat{r}$$

$$= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{\text{centrifugal term}}\hat{r} + \underbrace{(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{\text{Coriolis term}}\hat{\theta}$$

also written as $(\ddot{r} - r\omega^2)$, $r\dot{\omega} + 2\dot{r}\omega$

18) Examples

Whirling block



$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \qquad \left(= \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) \right)$$

$$m(\ddot{r} - r\dot{\theta}^2) = -T$$

$$\cancel{Mg - T = -M} \quad M\ddot{z} = Mg - T \;=\; -M\ddot{r}$$

since
$z + r = \text{const.}$
(length of string)

$$\frac{d}{dt} r^2\dot{\theta} = 0 \; ; \; \boxed{mr^2\dot{\theta} \equiv L}$$ independent of time (so = initial value) → angular momentum!

$$m\left(\frac{T}{M} - g - r\left(\frac{L}{mr^2}\right)^2\right) = -T$$

$$\boxed{T\left(\frac{1}{m} + \frac{1}{M}\right) = g + \frac{L^2}{m^2 r^3}}$$

$$M\ddot{r} = \left(\frac{1}{m} + \frac{1}{M}\right)^{-1}\left(g + \frac{L^2}{m^2 r^3}\right) - Mg$$

$$\ddot{r} = \frac{-M}{m+M} g + \cancel{\frac{m}{m+M}} \frac{L^2}{r^3 m(M+m)}$$

↑ dominates for large r, in

↑ dominates for small r, out

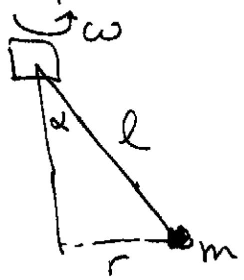
$L = 0$ ✓

settles to $$r_{eq.}^3 = \cancel{\frac{m+M}{L}} \frac{L^2}{mMg}$$

$$\frac{(ML^2/T)^2}{M^2 L/T^2}$ ✓

Sept 19

Conical pendulum



$\ddot{r} - \omega^2 r$

$2\dot{r}\dot{\theta} + r\ddot{\theta}$

vertical : $T\cos\alpha = mg$

horizontal: $T\sin\alpha = mr\omega^2 = ml\sin\alpha\,\omega^2$

sol^{n} 1 : $\sin\alpha = 0$, $T = mg$

sol^{n} 2 : $T = ml\omega^2$

$$\cos\alpha = \frac{g}{l\omega^2}$$

This is only available if $\cos\alpha \le 1$, i.e. $\omega^2 > \frac{g}{l}$

In this regime, sol^{n} 1 is unstable.

Common pendulum

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\sin\theta$$

$$\left[m(\ddot{r} - r\dot{\theta}^2) = -T + mg\cos\theta \right]$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \underset{\text{small }\theta}{\approx} -\frac{g}{l}\theta$$

$$\theta = A\cos(\omega t + \delta) \qquad \text{(harmonic motion)}$$

$$\omega^2 = g/l$$

we'd get the same equations for a spring! $\omega^2 = \frac{k}{m}$

$$T = m(g\cos\theta + l\dot{\theta}^2) = m(g\cos\theta + \omega_{top}^2 l\, \cancel{A\cos^3\theta})$$

$$\cancel{\cos\theta = 0 \underset{\omega^2 = g/l}{\Longrightarrow} mg(1 + A^2)}$$

$$\cancel{g > \cos\theta} \quad g\cos\theta + \omega^2 l < 0$$

if $\cancel{g\cos\theta < l\omega^2}$, rope goes slack!

similarly loop-the-loop

 $\downarrow N \downarrow mg$ $\quad N + mg = m\omega^2/R$ ← radius of curvature

better have $\omega^2 > Rg$!

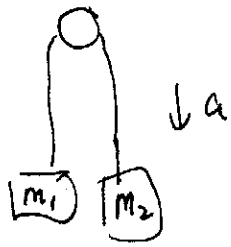
Why the shape is maintained ("walking chain")

$$F \approx \frac{T\Delta s}{R} \qquad m \approx \rho \Delta s \qquad F/\rho\Delta = \frac{T}{R\rho}$$

$$a \approx v^2/R$$

so if $v^2 = T/\rho$, it works!

19] Atwood Machine



$$a = \frac{m_2 g - T}{m_2} = \frac{T - m_1 g}{m_1}$$

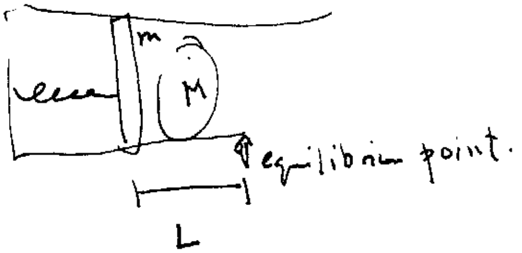
$$\Rightarrow \quad T = \frac{2 m_1 m_2}{m_1 + m_2} g$$

$$a = \frac{m_2 - m_1}{m_2 + m_1} g$$ "slowing it down"

$$S = \frac{1}{2} a t^2 = \frac{m_2 - m_1}{m_2 + m_1} \frac{g t^2}{2}$$

$$\boxed{g = \frac{2S}{t^2} \frac{m_2 + m_1}{m_2 - m_1}}$$

Spring gun



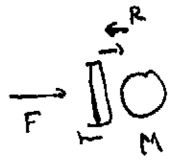
velocity of ball?

$$(m+M)\ddot{x} = -kx \qquad x(0) = -L \quad \text{(origin at equilibrium point)}$$
$$\dot{x}(0) = 0$$

$$x = A\cos\omega t + B\sin\omega t$$
with $\omega^2 = k/_{m+M}$

while in contact

[contact? :



$$\frac{F-R}{m} = \frac{R}{M}, \quad R>0 \quad \Rightarrow F>0 \;]$$

$$-L = A$$

$$\dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$B = 0$$

so $x = -L\cos\omega t$, until $x = 0$ $\left(t = \frac{\pi}{2\omega}\right)$.

At that time $\dot{x} = L\omega\sin\omega t = L\omega = L\sqrt{\frac{k}{m+M}}$

20) Newtonian gravity: statement

$$\vec{F}_{12} = \frac{G m_1 m_2}{|\vec{r}_{12}|^2}\,\hat{r}_{12}$$

force of 2 on 1

vector from 2 to 1 $= (\vec{r}_1 - \vec{r}_2)\hat{}$

21) Newtonian gravity $\overset{\sim}{\rightleftarrows}$ Kepler Laws

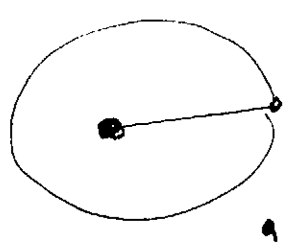
$M \gg m$; neglect motion of M, use it as origin

central force (no F_θ) $\Leftrightarrow$ $\underbrace{m(r\ddot{\theta} + 2\dot{r}\dot{\theta})}_{= \frac{1}{r}\frac{d}{dt}[m(r^2\dot{\theta})]} = 0$

$r^2\dot{\theta} = \text{const.}$

but $\delta A = \frac{1}{2} r^2 \delta\theta$,

so $\boxed{\frac{dA}{dt} = \text{const.}}$



circular orbit $\ddot{r} = 0$

$$-\frac{GMm}{r^2} = -m\omega^2 r$$

$$\Rightarrow \omega^2 = \frac{GM}{r^3}$$

$$\boxed{T = \frac{2\pi}{\omega} \propto r^{3/2}}$$

22) Newton's hard integral

Gravitational force due to a uniform spherical shell.

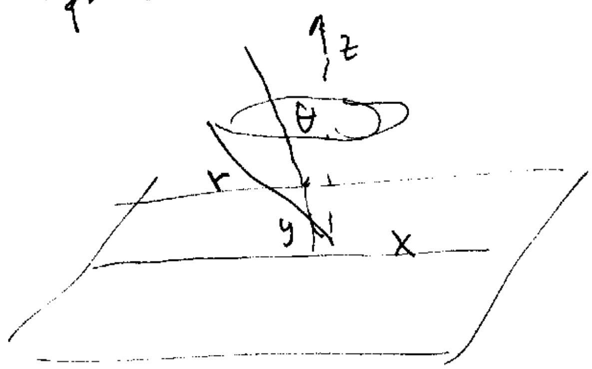
Simple result:

i) Inside the shell, no force!

ii) Outside the shell, force "as if" total mass concentrated at center. 9/24

You should etch this result into your memory; the derivation is for character-building.

Spherical coordinates



$$z = r\cos\theta$$

$$\sqrt{x^2+y^2} = r\sin\theta$$

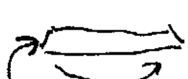
$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

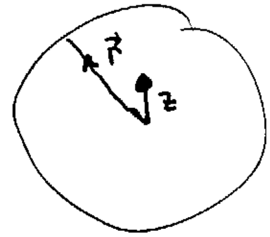
element of area

ring ~~of~~ at radius r,

range of angles $(\theta, \theta+d\theta)$



$r\,d\theta \cdot 2\pi r\sin\theta$ $\qquad dA = 2\pi r^2 \sin\theta\, d\theta$



let density (dm/dA) = σ

vertical component of force / m

$$= 2\pi G\sigma R^2 \int_0^{\pi} d\theta \, \sin\theta \, \frac{R\cos\theta - z}{R^3\left(\sin^2\theta + \left(\cos\theta - \frac{z}{R}\right)^2\right)^{3/2}}$$

$$\left[\int \frac{G\,dM\,(\vec{r} - z\hat{z})_z}{|\vec{r} - z\hat{z}|^3} = 2\pi\sigma R^2 \sin\theta\, d\theta \, \frac{(\vec{r} - z\hat{z})_z}{|\vec{r} - z\hat{z}|^3} \; ; \right.$$

$$\vec{r} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$$

$$\vec{r} - z\hat{z} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta - z)$$

$$(\vec{r} - z\hat{z})_z = R\cos\theta - z$$

$$|\vec{r} - z\hat{z}| = \sqrt{(R\sin\theta\cos\phi)^2 + (R\sin\theta\sin\phi)^2 + (R\cos\theta - z)^2}$$

$$\cancel{= R\sqrt{1 - \frac{2z}{R}\cos\theta}}$$

$$\left. = R\left(\sin^2\theta + \left(\cos\theta - \frac{z}{R}\right)^2\right)^{1/2} \right]$$

$$= 2\pi G\sigma \int_{-1}^{1} du \, \frac{u - \alpha}{(1 + \alpha^2 - 2\alpha u)^{3/2}}$$ with $\alpha \equiv \frac{z}{r}$, $u \equiv \cos\theta$

Since $\dfrac{u-\alpha}{(1+\alpha^2-2\alpha u)^{3/2}} = \dfrac{d}{d\alpha}\dfrac{1}{(1+\alpha^2-2\alpha u)^{1/2}}$

$$I = 2\pi G\sigma \frac{d}{d\alpha}\int_{-1}^{1}\frac{du}{(1+\alpha^2-2\alpha u)^{1/2}}$$

Now using $\displaystyle\int \frac{du}{\sqrt{a-u}} = -2\sqrt{a-u}\ldots$

$$= 2\pi G\sigma \frac{d}{d\alpha}\left[\frac{1}{\sqrt{2\alpha}}\, 2\left[-\sqrt{\frac{1+\alpha^2}{2\alpha}-1} + \sqrt{\frac{1+\alpha^2}{2\alpha}+1}\right]\right]$$

$$= 2\pi G\sigma \frac{d}{d\alpha}\left[\frac{1}{\alpha}\left(\sqrt{(1+\alpha)^2} - \sqrt{(1-\alpha)^2}\right)\right]$$

absolute value! (positive $\sqrt{\ }$)

$$= 2\pi G\sigma \frac{d}{d\alpha}\left[\frac{1}{\alpha}\left(1+\alpha - |1-\alpha|\right)\right]$$

i) if $\alpha < 1$, $|1-\alpha| = 1-\alpha$, quantity in $(\ )= 2\alpha$, $\frac{d}{d\alpha} \to 0$!

This corresponds to $z < R \Leftrightarrow$ inside sphere.

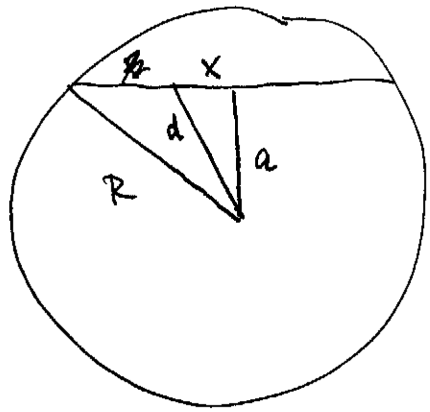
ii) if $\alpha > 1$, $|1-\alpha| = \alpha - 1$, so

$$I = 2\pi G\sigma \frac{d}{d\alpha}\frac{2}{\alpha} = -\frac{4\pi G\sigma}{\alpha^2} = \frac{-4\pi G\sigma R^2}{z^2}$$

$$= \frac{\text{attraction} \cdot GM}{(\text{distance to center})^2}$$

(since $M = 4\pi R^2\sigma$!)

23) The Gravity Express



So gravity at distance r from center, assuming uniform earth $= G\rho \frac{4\pi}{3} r^3 / r^2$ $(\times m)$

toward center

$$M = G\rho \frac{4\pi}{3} R^3$$

so $\text{force}/m = \frac{GMr}{R^3}$

spring-like!
(for diameters)

Now consider tunnel as in drawing

$$\frac{\text{Effective force}}{m} = \frac{(\text{force})_x}{m}$$

$$= \frac{GMd}{R^3} \; \frac{x}{d} = \frac{GMx}{R^3} \quad \text{anyway!}$$

$$\omega^2 = \frac{GM}{R^3} = g/R \qquad T = \frac{2\pi}{\omega} \approx 80 \text{ minutes!}$$

24] Describing Systems: Center of Mass

"Fundamental" laws are for infinitesimal bits of matter. Fortunately, the laws "scale up"!

$$\vec{F}_i = \vec{F}_{i\,ext.} + \sum_{j=1}^{n} \vec{F}_{ij} = m_i \ddot{\vec{r}}_i$$

$$\text{with } \vec{F}_{ii} = 0, \ \vec{F}_{ij} \underset{\uparrow \, 3^{rd}\text{ law}}{=} -\vec{F}_{ji}$$

Add them up

$$\sum m_i \ddot{\vec{r}}_i = \sum_{i=1}^{n} \vec{F}_i = \sum_{i=1}^{n} \vec{F}_{i\,ext.} + \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} \vec{F}_{ij}}_{\to 0}$$

$$\sum_{i=1}^{n} \vec{F}_{i\,ext.} \equiv \vec{F}_{ext\ (total)}$$

$$\sum m_i \ddot{\vec{r}}_i = \left(\sum m_i \vec{r}_i\right)^{\cdot\cdot}$$

$$\sum m_i \vec{r}_i = \sum m_i \frac{\sum m_i \vec{r}_i}{\sum m_i} \equiv M \vec{r}_{CM}$$

$$\vec{r}_{CM} \equiv \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

$$\boxed{M \ddot{\vec{r}}_{CM} = \vec{F}_{ext}}$$