~ 10/8

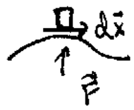
generally, central forces

$$\vec{F}(\vec{r}) = F(r)\,\hat{r}$$

$$\vec{F}(\vec{r}) \cdot d\vec{r} = -\frac{dV}{dr}\, dr$$

with $V(r) = -\int^{r} F(u)\,du$

iii) fixed forces of constraint do no work



$$\vec{F} \cdot d\vec{x} = 0$$

normal   tangential

~~moving forces of constraint, for motion~~

~~iv) fasteners, ropes~~

example: loop-the-loop

iv) forces $\perp$ to the motion do no work

ex: ball on suspended on rope

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mgl(\cos\theta_0 - \cos\theta)$$

pendulum demo

v) static friction does no work at point of contact.

(can do work as "external force" on system!!)

rolling w/o slipping - train

vi). Dynamic friction does negative work

$$\vec{F}\cdot d\vec{x} = \underbrace{\vec{F}\cdot\vec{v}}_{-\mu N}\, dt$$

~~34) (Comment.) Conservation~~

34) Energy for Systems

$$m_i \ddot{\vec{r}}_i = \vec{F}_i^{\,ext.} + \sum_j \vec{F}_{ij}$$

drop this for now

$\vdots$

$$\frac{d}{dt}\sum_i \frac{1}{2} m_i v_i^2 = \sum_i \sum_j \dot{\vec{r}}_i \cdot \vec{F}_{ij}$$

$$\overset{3rd}{=} \frac{1}{2}\sum_i\sum_j (\dot{\vec{r}}_i - \dot{\vec{r}}_j)\cdot \vec{F}_{ij}$$

i) If ~~$\vec{F}_{ij} = (\vec{r}_i - \vec{r}_j)\hat{} F(|\vec{r}_i - \vec{r}_j|)$ ("central")~~

$(\vec{r}_i - \vec{r}_j)^{2\,\cdot} = 0$, no work!

$\Rightarrow$ for rigid bodies, ~~forces of constrain~~ internal forces do no work!

ex: "executive toy"

44a

ii) If $\vec{F}_{ij}[\vec{r}_i - \vec{r}_j] = F[|\vec{r}_i - \vec{r}_j|]\,(\vec{r}_i - \vec{r}_j)\hat{}$

("central")

then $\frac{d}{dt}$ $(\vec{r}_i - \vec{r}_j)^{\cdot}\cdot(\hat{r}_i - \hat{r}_j)\, F[|\vec{r}_i - \vec{r}_j|]$

$= \frac{dr_{ij}}{dt} F(r_{ij})$ with $r_{ij} = |\vec{r}_i - \vec{r}_j|$, so

$$= -\frac{d}{dt} V(r_{ij}) \quad \text{with} \quad V(r) = -\int_{r_0}^{r} F(u)\,du$$

So we get

$$\frac{d}{dt}\left(\sum_i \frac{1}{2} m_i v_i^2 + \frac{1}{2}\sum_i \sum_j V(r_{ij})\right) = 0$$

$$\Leftrightarrow \quad \frac{d}{dt}\left(\sum_i \frac{1}{2} m_i v_i^2 + \sum_{\text{pairs}} V(r_{ij})\right) \overset{\shortparallel}{}$$

fundamental

ex: binary star

$$\frac{1}{2} M_A v_A^2 + \frac{1}{2} M_B v_B^2 - \frac{G M_A M_B}{r_{AB}} = \text{const.}$$

ex: spring masses

"conservation of energy"

(*) 45a

iii) for systems, no net work done by

a) even moving constraints
$(d\vec{x}_a \cdot (\vec{F}_{ab} + \vec{F}_{ba}) = 0)$
$+ d\vec{x}_b \cdot \vec{F}_{ba}$ since $d\vec{x}_a = d\vec{x}_b$ + Newton's 3rd)

b) {inextensible, light rope connections or cable (when taut, $d\vec{x}_\alpha \cdot$

$$\vec{F}_A = T\,\widehat{(x_B - x_A)} \qquad \frac{\vec{x}_B - \vec{x}_A}{|\vec{x}_B - \vec{x}_A|} \qquad \vec{F}_B = T\,\widehat{(x_A - x_B)}$$

otherwise $\vec{F} = 0$!

$$\vec{F}_A \cdot d\vec{x}_A + \vec{F}_B \cdot d\vec{x}_B$$

$$= d(\vec{x}_A - \vec{x}_B) \cdot (\vec{x}_A - \vec{x}_B) = 0!$$

$$= \frac{1}{2} d(x_A - x_B)^2$$

important example: 2 gravitating bodies, more explicitly

$$m_1 \ddot{\vec{r}}_1 = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$m_2 \ddot{\vec{r}}_2 = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_1 - \vec{r}_2|}$$

Consider $m_1 \dot{\vec{r}}_1 \cdot \ddot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 \cdot \ddot{\vec{r}}_2$ $\quad (\sum \vec{F}_i \cdot \vec{v}_i = \text{power})$

i) $= \frac{1}{2} m_1 (\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1)^{\cdot} + \frac{1}{2} m_2 (\dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2)^{\cdot} = \frac{d}{dt}\left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right)$

ii) $= -\frac{G m_1 m_2}{r_{12}^3} \left(\dot{\vec{r}}_1 \cdot (\vec{r}_1 - \vec{r}_2) + \dot{\vec{r}}_2 \cdot (\vec{r}_2 - \vec{r}_1) \right)$ $\quad \left(\begin{array}{c} \vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2 \\ r_{12} \equiv |\vec{r}_{12}| \end{array} \right)$

$$= -\frac{G m_1 m_2}{r_{12}^3} (\vec{r}_1 - \vec{r}_2)^{\cdot} \cdot (\vec{r}_1 - \vec{r}_2)$$

$$= -\frac{G m_1 m_2}{r_{12}^3} \frac{1}{2} \frac{d}{dt} \left\{ (\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2) \right\} = -\frac{G m_1 m_2}{2 r_{12}^3} \frac{d}{dt} r_{12}^2$$

$$= -\frac{G m_1 m_2}{r_{12}^2} \frac{d r_{12}}{dt}$$

$$= \frac{d}{dt} \left(\frac{G m_1 m_2}{r_{12}} \right)$$

so $$\frac{d}{dt}\left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{Gm_1m_2}{r_{12}}\right) = 0$$

$$\mathcal{E} \equiv \Sigma K.E. - \frac{Gm_1m_2}{r_{12}} = \text{constant!}$$

generalization 1: n bodies

$$\mathcal{E} = \sum_{i=1}^{N}(K.E.)_i - \sum_{\substack{\text{pairs}\\ ij}} \frac{Gm_im_j}{r_{ij}} = \text{const.}$$

generalization 2: central forces

$$\vec{F}_{ij} = f_{ij}(r_{ij})\frac{\vec{r}_{ij}}{r_{ij}} \qquad (= -\vec{F}_{ji})$$

so $f_{ij} = f_{ji}$

$$U_{ij}(r_{ij}) = -\int^{r_{ij}} f_{ij}(u)\,du$$

$$\mathcal{E} = \sum_{i=1}^{N}(K.E.)_i + \sum_{\text{pairs}} U_{ij}(r_{ij}) = \text{const.}$$

"examples": crushing a can : $K \to U$

origin of friction : $U \to K$

important example : rigid body

power from internal forces

$$= \sum_i \sum_j \dot{\vec{r}}_i \cdot \vec{F}_{ij}$$

$$= \frac{1}{2} \sum_i \sum_j \left(\dot{\vec{r}}_i \cdot F_{ij} + \dot{\vec{r}}_j \cdot \vec{F}_{ji} \right)$$

suppose $\vec{F}_{ij}(r_{ij}) = f_{ij}(r_{ij}) \frac{\vec{r}_{ij}}{r_{ij}}$

$\Downarrow$

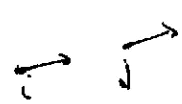
($f_{ij} = f_{ji}$ by 3^{rd} law)

$$\dot{\vec{r}}_i \cdot \vec{F}_{ij} + \dot{\vec{r}}_j \cdot \vec{F}_{ji} = \frac{f_{ij}}{r_{ij}} \left(\dot{\vec{r}}_i - \dot{\vec{r}}_j \right) \cdot \left(\vec{r}_i - \vec{r}_j \right)$$

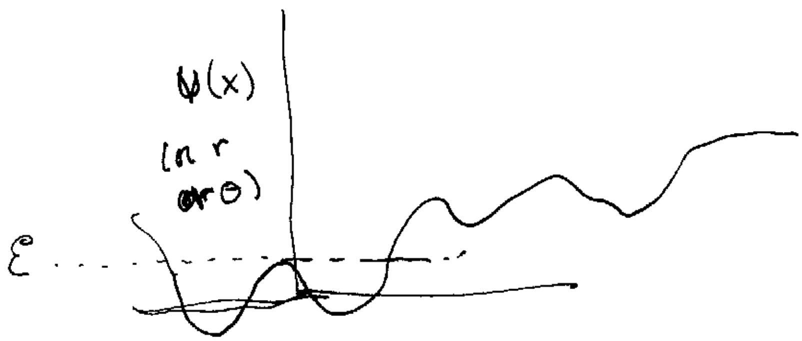
$$= \frac{f_{ij}}{r_{ij}} \frac{1}{2} \frac{d}{dt} r_{ij}^2 = 0$$

since r_{ij} = const.

$\boxed{\text{For a rigid body, internal forces do no work!}}$

intuitively:  equal $\vec{v}$
opposite $\vec{F}$!

35] Energy Diagrams



$$F = -\frac{dV}{dx}\ ,\ \text{so}$$

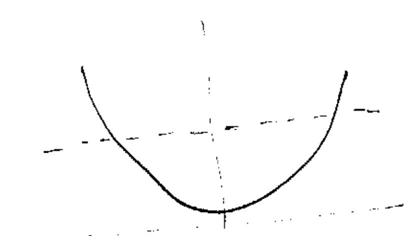
With $\mathcal{E} = \frac{1}{2}mv^2 + U(x)$ conserved

$$v = \pm\sqrt{\frac{2}{m}(\mathcal{E} - U)}$$

$v = 0$ at $\mathcal{E} = U$; "turning points"

oscillations between x_A and x_B

harmonic oscillator



$$U = \frac{1}{2}kx^2$$

$$\dot{x} = \sqrt{\frac{2}{m}\left(\mathcal{E} - \frac{1}{2}kx^2\right)}$$

$$\doteq \cancel{\sqrt{a}\sqrt{b - x^2}}$$

$$\frac{dx}{\sqrt{\frac{2}{m}(\mathcal{E} - \frac{1}{2}kx^2)}} = dt \quad \ldots$$

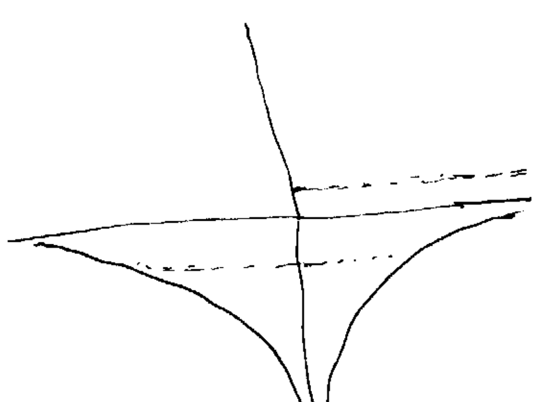
$$\frac{1}{2}kx^2 = \mathcal{E}\bar{x}^2$$

$$\bar{x} = \sqrt{\frac{k}{2\mathcal{E}}}\ x$$

$$\frac{\sqrt{\frac{2\mathcal{E}}{k}}\ d\bar{x}}{\sqrt{\frac{2\mathcal{E}}{m}}\sqrt{1-\bar{x}^2}}$$

$$\sin^{-1}\sqrt{\frac{k}{2\mathcal{E}}}\ x = \sqrt{\frac{m}{k}}\ t$$

gravity - radial motion (only)

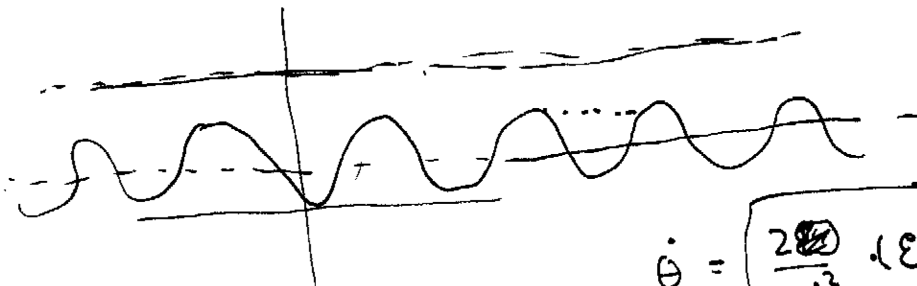


$$U = -\frac{GMm}{r}$$

$$\dot{r} = \sqrt{\frac{2}{m}\left(\mathcal{E} + \frac{GMm}{r}\right)}$$

pendulum

$$U = mgl(1-\cos\theta) \qquad K.E. = \frac{1}{2} m l^2 \dot\theta^2$$



critical $\mathcal{E} = 2mgl$

$$\dot\theta = \sqrt{\frac{2}{ml^2}(\mathcal{E} - U)}$$

periods get very long!...

near $\theta = \pm\pi$, ~~cos~~

$\xrightarrow{\text{critical}} \dot\theta = \sqrt{\frac{2}{ml^2} mgl(1+\cos\theta)}$

near $\theta = \pi$ $\quad u = \pi - \theta$

$\cos\theta = -1 + \frac{u^2}{2}$

$\dot u = -\sqrt{g/l}\, u$

$\ln 1/u \sim \sqrt{g/l}\, t$

$u \sim e^{-\sqrt{g/l}\, t}$

In higher dimensions, "energy landscape"

36) Equilibria

$$F = 0 \iff \frac{dU}{dx} = 0$$

max : unstable

min : stable

near min. : $$U(x) = U(x_0) + \frac{1}{2} \left.\frac{d^2U}{dx^2}\right|_{x_0} (x - x_0)^2$$

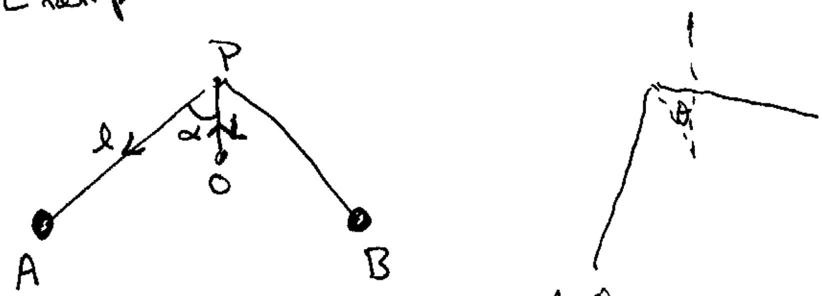
$$F = -\frac{dU}{dx} = -\left.\frac{d^2U}{dx^2}\right|_{x_0} (x - x_0)$$

$\Rightarrow$ looks like spring with $k = \left.\frac{d^2U}{dx^2}\right|_{x_0}$!

frequency $\omega^2 = k/m$; $\omega = \sqrt{\frac{d^2U/dx^2}{m}}$

with friction, settle down to some local minimum

37) Example: teeter-totter



need Σmgy as function of θ
do this as an exercise in vectors.

Step 1 : ~~at~~ $\theta = 0$

$$\vec{OP} = (0, L)$$

$$\vec{PA} = (-\ell\sin\alpha, -\ell\cos\alpha)$$

$$\vec{PB} = (\ell\sin\alpha, -\ell\cos\alpha)$$

$$\vec{OA} = \vec{OP} + \vec{PA} = (-\ell\sin\alpha, L - \ell\cos\alpha) \quad \ldots$$

Step2 : rotating vectors; ~~a~~ general formula

basis:
$$\hat{x}_\theta = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{y}_\theta = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

general: $\vec{v} = (v_x, v_y) = v_x\hat{x} + v_y\hat{y}$

$$\Rightarrow \vec{v}_\theta = v_x\hat{x}_\theta + v_y\hat{y}_\theta = v_x(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) + v_y(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})$$

$$= (\cos\theta\, v_x - \sin\theta\, v_y, \sin\theta\, v_x + \cos\theta\, v_y)$$

Step 3 : rotating the vectors in question

$$\vec{OP}_\theta = \cancel{(L\sin\theta, L\cos\theta)} \quad (L\sin\theta,\ L\cos\theta)$$

$$\vec{PA}_\theta = \ell \times (-\cos\theta\sin\alpha + \sin\theta\cos\alpha,\ -\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

$$\vec{PB}_\theta = \ell \times (+\cos\theta\sin\theta + \sin\theta\sin\alpha,\ -\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

Step 4 : gather y-components of what we want

$$\Sigma mgy = \left[(\vec{OP}_\theta + \vec{PA}_\theta) + (\vec{OP}_\theta + \vec{PB}_\theta)\right]_y mg$$

$$= (2\,\vec{OP}_{\theta\,y} + \vec{PA}_{\theta\,y} + \vec{PB}_{\theta\,y})\, mg$$

$$= mg\,(2L\cos\theta - 2\ell\cos\alpha\cos\theta)$$

$$= 2mg\,(L - \ell\cos\alpha)\cos\theta$$

Step 5 : enjoy!

near $\theta = 0$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

$$U(\theta) = \text{const.} + (\ell\cos\alpha - L)\,mg\,\theta^2$$

stable for $\ell\cos\alpha - L > 0$

~~period of oscillations~~ : $\frac{2\pi}{\omega}$, with $\omega^2 = \dfrac{\cancel{2}mg(\ell\cos\alpha - L)}{\cancel{2m}}$

$= mg\,(\ell\cos\alpha -$

$$\text{K.E.} = 2\times\tfrac{1}{2}\, m\ell^2\dot\theta^2 = m\ell^2\dot\theta^2$$

~~period~~ frequency : $\omega^2 = \dfrac{\text{"}k\text{"}}{\text{"}m\text{"}} = \dfrac{2mg\,(\ell\cos\alpha - L)}{2m\ell^2}$

$$= \frac{g\,(\ell\cos\alpha - L)}{\ell^2}$$

~~They can be~~ .

$T = \frac{2\pi}{\omega}$ can be big!

38] Collisions, revisited

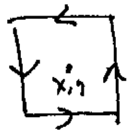
One can go to the CM frame, as we discussed before, to simplify the analysis. The momenta are $\vec{p}, -\vec{p}$. Since $\vec{p} = m_1 \vec{v}_1$, and $(K.E)_1 = \frac{1}{2} m \vec{v}_1^{\,2}$, we have $(K.E.)_1 = \frac{p^2}{2m_1}$, and for the total energy $\mathcal{E} = p^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$. After the collision ~~collision, if it was elastic,~~ the momenta will still be equal and opposite, say $\pm \vec{q}$. Conservation of energy requires $\mathcal{E}_{after} = q^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right) = p^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right) = \mathcal{E}_{before}$, or simply $q^2 = p^2$. So in an energy-conserving ("elastic") collision, the magnitudes of the momenta, and thus of the velocities, do not change. They can only get rotated!

Of course one can ~~analyze~~ go back to the lab frame if desired.

[N.B.: in CM frame only!]

39) When is a force field conservative?

Analyze an infinitesimal loop. Is work around circuit $= 0$?



$$F_x(x, y - \tfrac{dy}{2})\,dx + F_y(x + \tfrac{dx}{2}, y)\,dy - F_x(x, y + \tfrac{dy}{2})\,dx - F_y(x - \tfrac{dx}{2}, y)\,dy \overset{?}{\approx} 0$$

$$-\frac{\partial F_x}{\partial y}\,dy\,dx \quad + \frac{\partial F_y}{\partial x}\,dx\,dy \overset{?}{=} 0$$

$$\Longrightarrow \quad \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0$$

similarly $\dfrac{\partial F_y}{\partial z} - \dfrac{\partial F_z}{\partial y} = \dfrac{\partial F_z}{\partial x} - \dfrac{\partial F_x}{\partial z} = 0.$

Since a big loop can be built up from small ones, this condition is also sufficient.

ex: Central force

$$\vec{F} = \vec{r}\,\frac{f(r)}{r} \equiv g(r)\,(x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$$

$$\frac{\partial F_x}{\partial y} = x\,g'\,\frac{\partial r}{\partial y} = x\,g'\,\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial y} = \frac{xy\,g'}{r}$$

$$\frac{\partial F_y}{\partial x} = \frac{yx\,g'}{r} \quad \text{etc.}$$

$\checkmark$

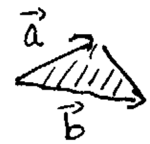
$\Longrightarrow$ conservative

40| Cross-Products

geometrical definition:

$\vec{a} \times \vec{b}$ is a vector

magnitude $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin \underbrace{\angle(\vec{a}, \vec{b})}|$

$= 2 \times$ area  (angle between)

direction: $\perp$ to both $\vec{a}$ and $\vec{b}$,
oriented by right-hand rule

rules: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$(\alpha \vec{a}) \times \vec{b} = \alpha (\vec{a} \times \vec{b})$

these are obvious

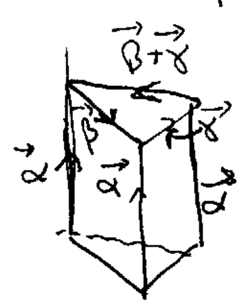
$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

↑ much less obvious!

Proof: i) $(\vec{a} \times \vec{b})_i$ = projected area in i^{th} direction ("shadow")

ii) Σ(project areas) $= 0$ for a closed surface

iii) consider prism



$\vec{\beta} \times \vec{\alpha} + \vec{\gamma} \times \vec{\alpha}$
$+ \vec{\alpha} \times (\vec{\beta} + \vec{\gamma}) + \frac{1}{2} \vec{\beta} \times \vec{\gamma} - \frac{1}{2} \vec{\beta} \times \vec{\gamma} =$

$\Downarrow$!

$\vec{\alpha} \times \vec{\beta} + \vec{\alpha} \times \vec{\gamma} = \vec{\alpha} \times (\vec{\beta} + \vec{\gamma})$

Using this, we can derive the coordinate form

$$(a_x, a_y, a_z) \times (b_x, b_y, b_z)$$

$$\parallel$$

$$(a_x \hat{x} + a_y \hat{y} + a_z \hat{z}) \times (b_x \hat{x} + b_y \hat{y} + b_z \hat{z})$$

$$\parallel$$

$$a_x b_y \, \hat{x} \times \hat{y} + a_x b_z \, \hat{x} \times \hat{z} + a_y b_x \, \hat{y} \times \hat{x} + a_y b_z \, \hat{y} \times \hat{z}$$
$$+ a_z b_x \, \hat{z} \times \hat{x} + a_z b_y \, \hat{z} \times \hat{y}$$

$$= (a_x b_y - a_y b_x)\hat{z} + (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y}$$

$$= (a_y b_z - a_z b_y, \, a_z b_x - a_x b_z, \, a_x b_y - a_y b_x)$$

41] 2-body central force problem: first reduction (CM)

Want to study

$$m_1 \ddot{\vec{r}}_1 = -\frac{G m_1 m_2}{r_{12}^2} \frac{\vec{r}_{12}}{r_{12}} \rightarrow f(r_{12}) \frac{\vec{r}_{12}}{r_{12}}$$

$$m_2 \ddot{\vec{r}}_2 = f(r_{12}) \frac{\vec{r}_{21}}{r_{12}}$$

We have $m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = 0$, so $m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = \text{Const.}$ (constant velocity of CM). By going to CM frame, we can assume $m_1 \dot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 = 0$, so $m_1 \vec{r}_1 + m_2 \vec{r}_2 = \text{Const.}$ By adjusting axes, we can assume $m_1 \vec{r}_1 + m_2 \vec{r}_2 = \vec{0}$.

$$\left(\text{Alternatively, } \vec{r}_{\substack{1\\2}}{}' = \vec{r}_{\substack{1\\2}} - \frac{m_1 \dot{\vec{r}}_1(0) + m_2 \dot{\vec{r}}_2(0)}{m_1+m_2} t - \frac{m_1 \vec{r}_1(0) + m_2 \vec{r}_2(0)}{m_1+m_2} \right)$$

Then $m_1 \ddot{\vec{r}}_1 = \cancel{\frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1+m_2}}$

$$= m_1 \left(\cancel{\frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1+m_2}} + (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) \frac{m_2}{m_1+m_2} \right)$$

$$= \frac{m_1 m_2}{m_1+m_2} \ddot{\vec{r}}_{12}$$

$$\equiv \mu \ddot{\vec{r}}_{12}, \text{ with } \mu \equiv \frac{m_1 m_2}{m_1+m_2} \equiv \text{"reduced mass"}$$

note $\mu \leq m_1, m_2$; $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$; $\mu \xrightarrow[m_2 \to \infty]{} m_1$

So we have

$$\boxed{\mu \ddot{\vec{r}}_{12} = f(r_{12}) \frac{\vec{r}_{12}}{r_{12}}}$$

Now drop the subscripts! – looks like 1-body problem.

42) 2-body central force problem: second reduction ($\vec{L}$)

$$\mu \ddot{\vec{r}} = f(r) \frac{\vec{r}}{r}$$

Take $\vec{r} \times$ both sides:

$$\mu \vec{r} \times \ddot{\vec{r}} = 0$$

$$\parallel$$

$$\frac{d}{dt}(\mu \vec{r} \times \dot{\vec{r}})$$

$$\mu \vec{r} \times \dot{\vec{r}} = \text{const.} \equiv \text{angular momentum (of equivalent problem)}$$

$$\equiv \vec{L}$$

~~Note that~~

$\Rightarrow$ $\vec{r}(t)$ stays in one plane – the plane $\perp$ to the angular momentum; This is the plane determined by $\vec{r}(0)$ and $\dot{\vec{r}}(0)$.

Choose this to be the x-y plane!

$$\vec{r} = r \hat{r}$$

$$\dot{\vec{r}} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

$$\vec{L} = \mu r^2 \dot{\theta} \hat{r} \times \hat{\theta} = \mu r^2 \dot{\theta} \hat{z}$$

$$\mu r^2 \dot\theta \equiv L \quad (\text{"scalar angular momentum"})$$
$$= \text{constant.}$$

43) 2-body central force problem: third reduction ($\mathcal{E}$)

For the effective problem, the conserved energy is

$$\mathcal{E} = \frac{1}{2}\mu v^2 - \int^r f(u)\,du \equiv \frac{1}{2}\mu v^2 + U(r)$$

where for gravity, $f(u) = -\frac{Gm_1m_2}{r^2}$ and $U(r) = \frac{-Gm_1m_2}{r}$.

Since $v^2 = (\dot r \hat r + r\dot\theta\hat\theta)^2 = \dot r^2 + r^2\dot\theta^2$, and $\mu r^2\dot\theta = L$,

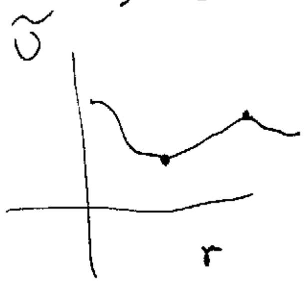
we have $\frac{1}{2}\mu v^2 = \frac{\mu}{2}\dot r^2 + \frac{L^2}{2mr^2}$ Thus

$$\mathcal{E} = \frac{\mu}{2}\dot r^2 + \frac{L^2}{2mr^2} + U(r) \equiv \frac{\mu}{2}\dot r^2 + \tilde U(r)$$

with $\tilde U(r) = \frac{L^2}{2mr^2} + U(r) \equiv$ "effective potential".

44] Using the Effective Potential

i) Circular motion



As in our discussion of equilibrium points, $\dot{r} = 0$ for all time $\Leftrightarrow d\tilde{U}/dr|_{r_0} = 0$.

Here this does not mean a "fixed" particle, however, but just a fixed radius. We have

$$\dot{\theta} = \frac{L}{mr_0^2} = \text{const.} \Rightarrow \text{circular motion at constant rate}$$

Looking at the condition $\left.\frac{d\tilde{U}}{dr}\right|_{r_0} = 0$:

$$\frac{d}{dr}\left(\frac{L^2}{2mr^2} + U(r)\right) = -\frac{L^2}{mr^3} + \left.\frac{dU}{dr}\right|_{r_0} = 0$$

Since

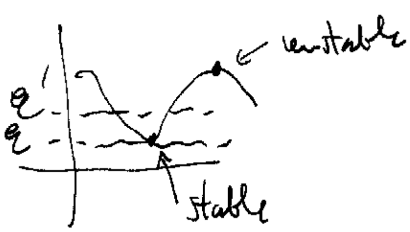
$$\frac{dU}{dr} = -F_r$$

$$L^2 = (mvr)^2$$

this says $\frac{mv^2}{r} = F_r$ — a familiar story!

ii) Stability + Oscillation

Radial impulse changes $\mathcal{E}$ but not L (+ so not $\tilde{U}$). So we analyze like before for oscillations of r.



frequency $\omega^2 = \left.\frac{d^2\tilde{U}}{dr^2}\right|_{r_0} / \mu$

Of course, given r we compute θ using $\dot{\theta} = \frac{L}{mr^2}$.

<u>Angular impulse</u> changes both $\mathcal{E}$ and L, (hence $\tilde{U}$) generally. It is messier to analyze, but no new ideas are required.

45) Orbits and Kepler's Laws

After all the reductions, we have

$$\mathcal{E} = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{GMm}{r}$$

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}}\sqrt{\mathcal{E} - \frac{L^2}{2\mu r^2} + \frac{GMm}{r}}$$

From this we could get $t = t(r)$, but this is awkward.

A better idea is to combine with

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2},$$

by dividing we get

$$\frac{dr}{d\theta} = \frac{\mu r^2}{L}\sqrt{\frac{2}{\mu}}\sqrt{\mathcal{E} - \frac{L^2}{2\mu r^2} + \frac{GMm}{r}}$$

or, with $u \equiv 1/r$

$$\frac{du}{\sqrt{\mathcal{E} - \frac{L^2}{2\mu}u^2 + GMmu}} = \sqrt{\frac{2\mu}{L^2}}\,d\theta.$$

Now we use the integral

$$\int \frac{du}{\sqrt{A + 2Bu - Cu^2}} = \frac{1}{\sqrt{C}} \cos^{-1} \sqrt{\frac{C}{A + \frac{B^2}{C}}} \left(u - \frac{B}{C} \right)$$

with $A \equiv \mathcal{E}$, $B \equiv \frac{GMm}{2}$, $C = \frac{L^2}{2\mu}$ and arrive at

$$\frac{1}{r} - \frac{GMm\mu}{L^2} = \boxed{\cancel{\cos \sqrt{L^2/2m}}} \sqrt{\frac{\mathcal{E} + \frac{(GMm)^2}{2L^2}\mu}{L^2/2\mu}} \cos\theta$$

or $$\frac{1}{r} = Q + \sqrt{Q^2 + R} \cos\theta$$

$$Q \equiv \frac{GMm\mu}{L^2}, \quad R = \frac{2\mu\mathcal{E}}{L^2}$$

Using $\cos\theta = \frac{x}{r}$ and rearranging, this gives

$$2\sqrt{Q^2 + R}\, x + 1 = -R x^2 + Q^2 y^2 \qquad (*)$$

So if
$R < 0$ $(\mathcal{E} < 0)$: ellipse
$R = 0$ $(\mathcal{E} = 0)$: parabola
$R > 0$ $(\mathcal{E} > 0)$: hyperbola

Kepler 1: Ellipse with Sun at Focus

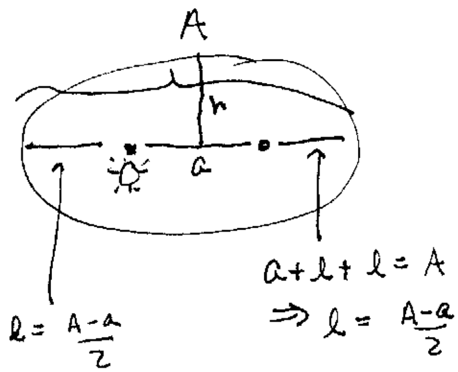
Indeed, we can rewrite (*) as

$$\sqrt{x^2+y^2} + \sqrt{(x-a)^2+y^2} = A$$

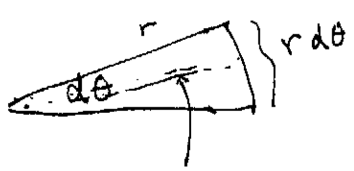
↑ focus at origin!

with $$Q^2 = \frac{4A^2}{(A^2-a^2)^2}$$

$$R = -\frac{4}{A^2-a^2}$$



Kepler 2: Equal areas swept in equal times



$$d\mathcal{a} = 2\times\left(\tfrac{1}{2}\times r\times r\,d\theta\right) = \frac{r^2\,d\theta}{2}$$

since $L = \mu r^2 \frac{d\theta}{dt}$, $\cancel{d\theta = \mu r^2\, dt,\ so\ d\mathcal{a} =}$

$$d\theta = \frac{L}{\mu r^2}\,dt, \text{ so } \boxed{d\mathcal{a} = \frac{L}{2\mu}\,dt}$$

Kepler 3: $(\text{Period})^2 \propto (\text{major axis})^3$

$$\underset{\substack{\parallel\\ T}}{\text{period}} = \frac{\text{area}}{d\,\text{area}/dt} = \frac{\pi \frac{A}{2}\cdot \text{height}}{L/2\mu}$$

to determine height, note $2\sqrt{\left(\frac{a}{2}\right)^2+h^2} = A$, $h^2 = \frac{A^2-a^2}{4}$

so $$T = \frac{\pi A \sqrt{A^2 - a^2}\ \mu}{2L}$$

But $$L = \frac{\sqrt{GMm\mu}}{\sqrt{Q}} = \frac{\sqrt{GMm\mu}\ \sqrt{A^2-a^2}}{\sqrt{2}\ \sqrt{A}},$$ so

$$T = \frac{\pi A^{3/2}}{\sqrt{2}} \sqrt{\frac{\mu}{GMm}} = \frac{\pi A^{3/2}}{\sqrt{2}} \sqrt{\frac{1}{G(M+m)}}$$

since $\mu = \frac{Mm}{M+m}$

$$\boxed{T^2 = \frac{\pi^2 A^3}{2G(M+m)}}$$