

5 SIMILITUDE

5.1 Use of Nondimensional Groups

For a consistent description of physical processes, we require that all terms in an equation must have the same units. On the basis of physical laws, some quantities are dependent on other, independent quantities. We form nondimensional groups out of the dimensional ones in this section, and apply the technique to maneuvering.

The Buckingham π -theorem provides a basis for all nondimensionalization. Let a quantity Q_n be given as a function of a set of $n - 1$ other quantities:

$$Q_n = f_Q(Q_1, Q_2, \dots, Q_{n-1}). \quad (67)$$

There are n variables here, but suppose also that there are only k independent ones; k is equivalent to the number of physical unit types encountered. The theorem asserts that there are $n - k$ dimensionless groups π_i that can be formed, and the functional equivalence is reduced to

$$\pi_{n-k} = f_\pi(\pi_1, \pi_2, \dots, \pi_{n-k-1}). \quad (68)$$

Example. Suppose we have a block of mass m resting on a frictionless horizontal surface. At time zero, a steady force of magnitude F is applied. We want to know $X(T)$, the distance that the block has moved as of time T . The dimensional function is $X(T) = f_Q(m, F, T)$, so $n = 4$. The (MKS) units are

$$\begin{aligned} [X(\cdot)] &= m \\ [m] &= kg \\ [F] &= kgm/s^2 \\ [T] &= s, \end{aligned}$$

and therefore $k = 3$. There is just one nondimensional group in this relationship; π_1 assumes only a constant (but unknown) value. Simple term-cancellation gives $\pi_1 = X(T)m/F T^2$, not far at all from the known result that $X(T) = F T^2 / 2m$.

Example. Consider the flow rate Q of water from an open bucket of height h , through a drain nozzle of diameter d . We have

$$Q = f_Q(h, d, \rho, \mu, g),$$

where the water density is ρ , and its absolute viscosity μ ; g is the acceleration due to gravity. No other parameters affect the flow rate. We have $n = 6$, and the (MKS) units of these quantities are:

$$\begin{aligned} [Q] &= m^3/s \\ [h] &= m \\ [d] &= m \\ [\rho] &= kg/m^3 \\ [\mu] &= kg/ms \\ [g] &= m/s^2 \end{aligned}$$

There are only three units that appear: [length, time, mass], and thus $k = 3$. Hence, only three non-dimensional groups exist, and one is a unique function of the other two. To arrive at a set of valid groups, we must create three nondimensional quantities, making sure that each of the original (dimensional) quantities is represented. Intuition and additional manipulations come in handy, as we now show.

Three plausible first groups are: $\pi_1 = \rho Q / d \mu$, $\pi_2 = d \rho \sqrt{gh} / \mu$, and $\pi_3 = h / d$. Note that all six quantities appear at least once. Since h and d have the same units, they could easily change places in the first two groups. However, π_1 is recognized as a Reynolds number pertaining to the orifice flow. π_2 is more awkward, but products and fractions of groups are themselves valid groups, and we may construct $\pi_4 = \pi_1 / \pi_2 = Q / d^2 \sqrt{gh}$ to nondimensionalize Q with a pressure velocity, and then $\pi_5 = \pi_1 / \pi_4 = \rho d \sqrt{gh} / \mu$ to establish an orifice Reynolds number independent of Q . We finally have the useful result

$$\begin{aligned} \pi_4 &= f_\pi(\pi_5, \pi_2) \rightarrow \\ \frac{Q}{d^2 \sqrt{gh}} &= f_\pi \left(\frac{\rho d \sqrt{gh}}{\mu}, \frac{h}{d} \right). \end{aligned}$$

The uncertainty about where to use h and d , and the questionable importance of h/d as a group are remnants of the theorem. Intuition is that h/d is immaterial, and the other two terms have a nice physical meaning, e.g., π is a Reynolds number. The power of the π -theorem is primarily in reducing the number of parameters which must be considered independently to characterize a process. In the flow example, the theorem reduced the number of independent parameters from five to two, with no constraints about the actual physics taking place.

5.2 Common Groups in Marine Engineering

One frequently encounters the following groups in fluid mechanics and marine engineering:

1. Froude number:

$$Fr = \frac{U}{\sqrt{gL}}, \quad (69)$$

where U is the speed of the vessel, g is the acceleration due to gravity, and L is the waterline length of the vessel. The Froude number appears in problems involving pressure boundary conditions, such as in waves on the ocean surface. Roughly speaking, it relates the vessel speed U to water wave speeds of wavelength L ; the phase speed of a surface wave is $V = \sqrt{\lambda g / 2\pi}$, where λ is the wavelength.

2. Cavitation number:

$$\delta = \frac{P_\infty - P_v}{\frac{1}{2}\rho U^2}, \quad (70)$$

where P_∞ represents the ambient total pressure, P_v the vapor pressure of the fluid, and U the propeller inlet velocity. A low cavitation number means that the Bernoulli pressure loss across the lifting surface will cause the fluid to vaporize, causing bubbles, degradation of performance, and possible deterioration of the material.

3. Reynolds number:

$$Re = \frac{Ul}{\mu/\rho}, \quad (71)$$

where U is velocity, μ is absolute viscosity, and ρ is density. Since Re appears in many applications, l represents one of many length scales. Reynolds number is a ratio of fluid inertial pressures to viscous pressures: When Re is high, viscous effects are negligible. Re can be used to characterize pipe flow, bluff body wakes, and flow across a plate, among others.

4. Weber number:

$$W = \frac{\rho U^2 l}{\sigma}, \quad (72)$$

where σ is the surface tension of a fluid. Given that $[\sigma] = N/m$ (MKS), ρU^2 normalizes pressure, and l normalizes length. The Weber number indicates the importance of surface tension.

To appreciate the origins of these terms from a fluid particle's point of view, consider a box having side lengths $[dx, dy, dz]$. Various forces on the box scale as

$$\begin{aligned} \text{(inertia)} \quad F_i &= \rho \frac{\partial v}{\partial t} dx dy dz + \rho v \frac{\partial v}{\partial x} dx dy dz \simeq \rho U^2 l^3 \\ \text{(gravity)} \quad F_g &= \rho g dx dy dz \simeq \rho g l^3 \\ \text{(pressure)} \quad F_p &= P dx dy \simeq P l^2 \\ \text{(shear)} \quad F_s &= \mu \frac{\partial v}{\partial z} dx dy \simeq \mu U l \\ \text{(surface tension)} \quad F_\sigma &= \sigma dx \simeq \sigma l. \end{aligned}$$

Thus the groups listed above can be written as

$$\begin{aligned} Fr &= \frac{F_i}{F_g} \simeq \frac{U^2}{gl} \\ \delta &= \frac{F_p}{F_i} \simeq \frac{P}{\rho U^2} \\ Re &= \frac{F_i}{F_s} \simeq \frac{\rho U l}{\mu} \\ W &= \frac{F_i}{F_\sigma} \simeq \frac{\rho U^2 l}{\sigma} \end{aligned}$$

When testing models, it is imperative to maintain as many of the nondimensional groups as possible of the full-scale system. This holds for the geometry of the body and the kinematics of the flow, the surface roughness, and the all of the relevant groups governing fluid dynamics. Consider the example of nozzle flow from a bucket. Suppose that we conduct a model test in which Re is abnormally large, i.e., the viscous effects are negligible. Under inviscid conditions, the flow rate is $Q = \pi d^2 \sqrt{2gh}/4$. This rate cannot be achieved for lower- Re conditions because of fluid drag in the orifice, however.

In a vessel, we write the functional relationship for drag as a starting point:

$$\begin{aligned} C_r &= \frac{D}{\frac{1}{2} \rho A U^2} = f_D(\rho, \mu, g, \sigma, U, l) \\ &= f_r(Re, Fr, W). \end{aligned}$$

First, since l is large, W is very large, and hence surface tension plays no role. Next, we look at $Re = Ul/\nu$ and $Fr = U/\sqrt{gl}$, both of which are important for surface vessels. Suppose that $l_{ship} = \lambda l_{model}$, so that usually $\lambda \gg 1$; additionally, we set $g_{model} = g_{ship}$, i.e., the model and the true vessel operate in the same gravity field.

Froude number similitude requires $U_{model} = U_{ship}/\sqrt{\lambda}$. Then Reynolds number scaling implies directly $\nu_{model} = \nu_{ship}/\lambda^{3/2}$. Unfortunately, few fluids with this property are workable in a large testing tank. As a result, accurate scaling of Re for large vessels to model scale is quite difficult. For surface vessels, and submarines near the surface, it is a routine procedure to employ turbulence stimulators to achieve flow that would normally occur with ship-scale Re . Above a critical value $Re \simeq 500,000$, C_r is not sensitive to Re . With this achieved, one then tries to match Fr closely.

5.3 Similitude in Maneuvering

The linear equations of motion for the horizontal yaw/sway problem are:

$$\begin{aligned} (m - Y_{\dot{v}}) \dot{v} - Y_{\dot{v}} v + (mU - Y_r) \dot{r} + (m x_G - Y_r) \dot{r} &= Y \\ (I_{zz} - N_{\dot{r}}) \dot{r} + (m x_G U - N_r) \dot{r} + (m x_G - N_{\dot{v}}) \dot{v} - N_{\dot{v}} v &= N. \end{aligned}$$

These equations can be nondimensionalized in a standard way, by using the quantities $[U, L, \rho]$: these three values provide the necessary units of length, time, and mass, and furthermore are readily accessible to the user. First, we create nondimensional states, denoted with a prime symbol:

$$\begin{aligned}
\dot{w}' &= \frac{L}{U^2} \dot{w} \\
v' &= \frac{1}{U} v \\
\dot{r}' &= \frac{L^2}{U^2} \dot{r} \\
r' &= \frac{L}{U} r.
\end{aligned} \tag{73}$$

We follow a similar procedure for the constant terms as follows, including a factor of $1/2$ with ρ , for consistency with our previous expressions:

$$\begin{aligned}
m' &= \frac{m}{\frac{1}{2}\rho L^3} \\
I'_{zz} &= \frac{I_{zz}}{\frac{1}{2}\rho L^5} \\
x'_G &= \frac{x_G}{L} \\
U' &= \frac{U}{U} = 1 \\
Y'_0 &= \frac{Y_0}{\frac{1}{2}\rho L^3} \\
Y'_0 &= \frac{Y_0}{\frac{1}{2}\rho U L^2} \\
Y'_f &= \frac{Y_f}{\frac{1}{2}\rho L^4} \\
Y'_r &= \frac{Y_r}{\frac{1}{2}\rho U L^3} \\
Y' &= \frac{Y}{\frac{1}{2}\rho U^2 L^2} \\
N'_0 &= \frac{N_0}{\frac{1}{2}\rho L^4} \\
N'_0 &= \frac{N_0}{\frac{1}{2}\rho U L^3} \\
N'_f &= \frac{N_f}{\frac{1}{2}\rho L^5} \\
N'_r &= \frac{N_r}{\frac{1}{2}\rho U L^4} \\
N' &= \frac{N}{\frac{1}{2}\rho U^2 L^3}.
\end{aligned} \tag{74}$$

Note that every force has been normalized with $\frac{1}{2}\rho U^2 L^2$, and every moment with $\frac{1}{2}\rho U^2 L^3$; time has been also nondimensionalized with L/U . Thus we arrive at a completely equivalent set of nondimensional system equations,

$$\begin{aligned}
(m' - Y'_0)\dot{w}' - Y'_0 v' + (m' U' - Y'_f)r' + (m' x'_G - Y'_f)\dot{r}' &= Y' \\
(I'_{zz} - N'_f)\dot{r}' + (m' x'_G U' - N'_r)r' + (m' x'_G - N'_0)\dot{w}' - N'_0 v' &= N'.
\end{aligned} \tag{75}$$

Since fluid forces and moments generally scale with U^2 , the nondimensionalized description holds for a range of velocities.

5.4 Roll Equation Similitude

Certain nondimensional coefficients may arise which depend explicitly on U , and therefore require special attention. Let us carry out a similar normalization of the simplified roll equation

$$(I_{xx} - K_\phi)\ddot{\phi} - K_p\dot{\phi} - K_\phi\phi = K. \quad (76)$$

For a surface vessel, the roll moment K_ϕ is based on metacentric stability, and has the form $K_\phi = -\rho g \nabla (GM)$, where ∇ is the displaced fluid volume of the vessel, and GM is the metacentric height. The nondimensional terms are

$$I'_{xx} = \frac{I_{xx}}{\frac{1}{2}\rho L^3} \quad (77)$$

$$K'_\phi = \frac{K_\phi}{\frac{1}{2}\rho L^3}$$

$$K'_p = \frac{K_p}{\frac{1}{2}\rho UL^4}$$

$$K'_\phi = \frac{K_\phi}{\frac{1}{2}\rho U^2 L^3}$$

$$\dot{\phi}' = \frac{L^2}{U^2} \dot{\phi}$$

$$\phi' = \frac{L}{U^2} \phi,$$

leading to the equivalent system

$$(I'_{xx} - K'_\phi)\ddot{\phi}' - K'_p\dot{\phi}' - \left(\frac{2gL}{U^2}\right) \nabla'(GM')\phi = K'. \quad (78)$$

Note that the roll angle ϕ was not nondimensionalized. The Froude number has a very strong influence on roll stability, since it appears explicitly in the nondimensional righting moment term, and also has a strong influence on K'_p .

In the case of a submarine, the righting moment has the form $K_\phi = -Bh$, where B is the buoyant force, and h is the righting arm. The nondimensional coefficient becomes

$$K'_\phi = -\frac{Bh}{\frac{1}{2}\rho U^2 L^3}.$$

K'_ϕ again depends strongly on U , since B and h are fixed; this K'_ϕ needs to be maintained in model tests.