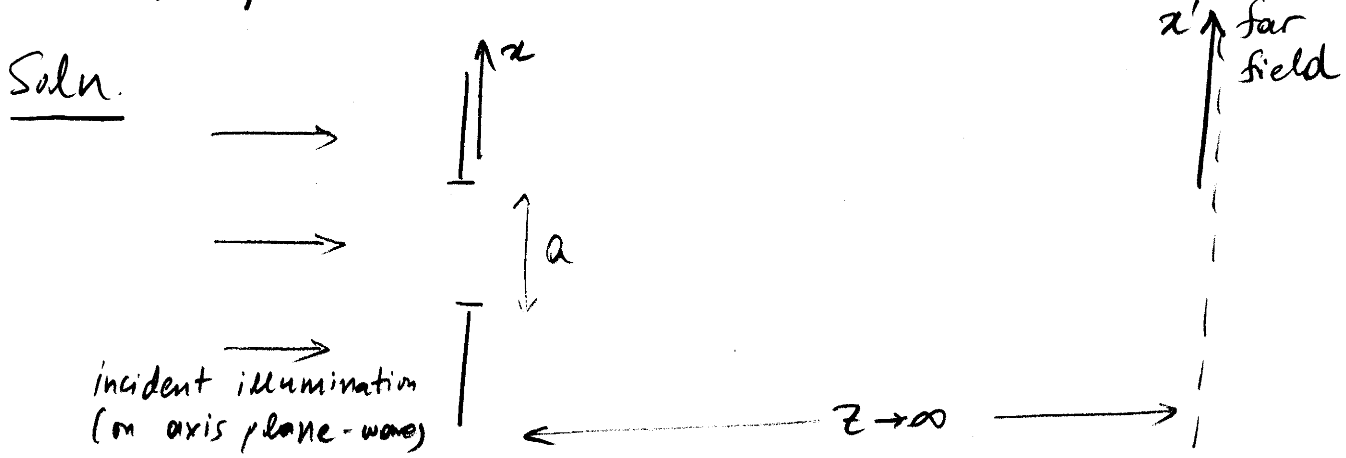
Practice Problem Set #2

① What is the Fraunhofer diffraction pattern of a 1-D slit of size a?

Soln.



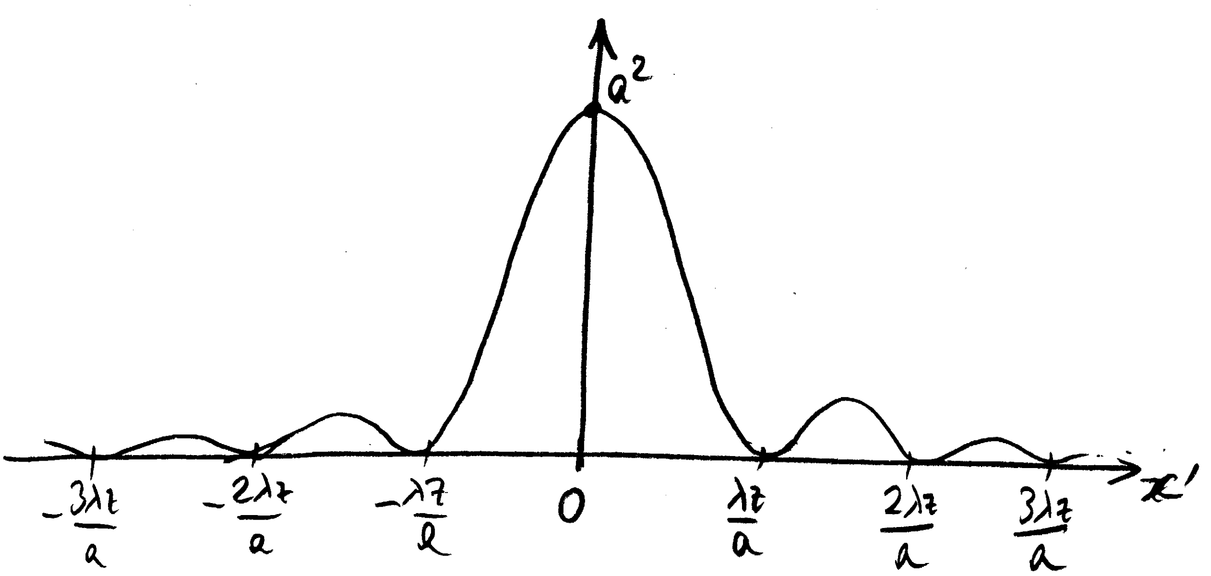
slit description (1D) $\qquad f(x) = \mathrm{rect}\left(\frac{x}{a}\right)$

Fourier transform of slit: $\quad F(u) = a\,\mathrm{sinc}(au)$

Diffracted far field : $\qquad g(x') = e^{i\pi\frac{x'^2+y'^2}{\lambda z}} F\left(\frac{x'}{\lambda z}\right)$

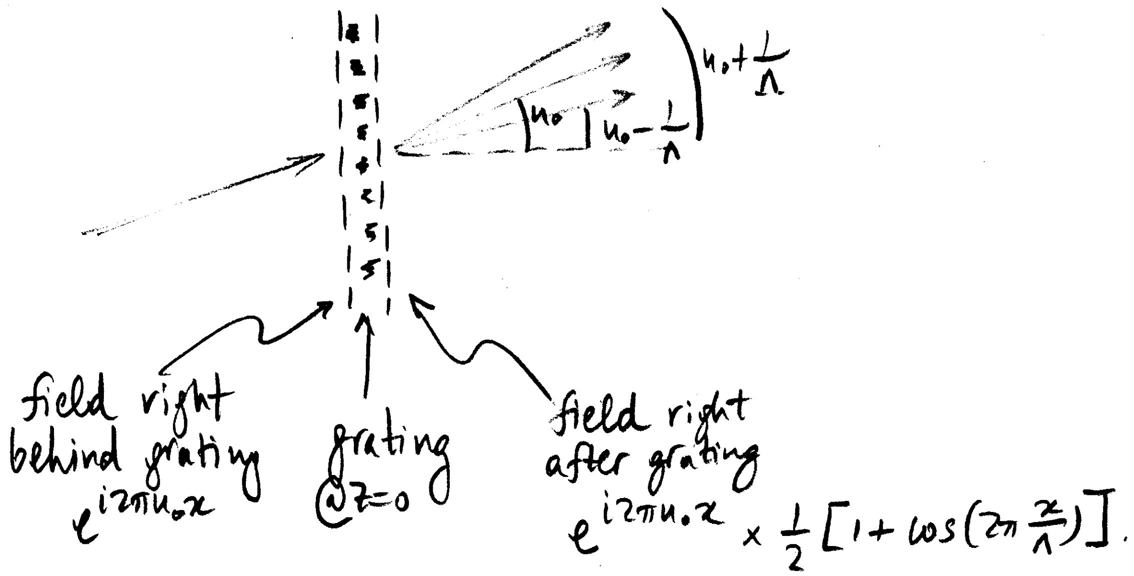
Fraunhofer diffraction pattern (Intensity)

$$|g(x')|^2 = a^2\,\mathrm{sinc}^2\left(\frac{a x'}{\lambda z}\right)$$



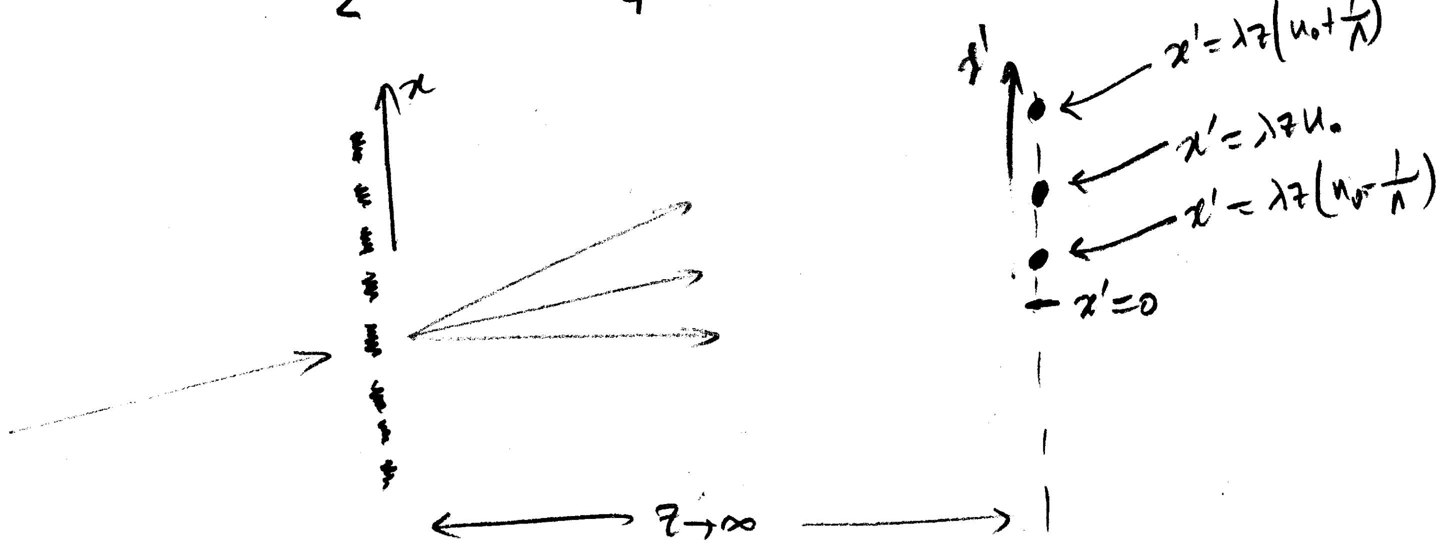
③ How does the result of problem #2 change if the illumination is a plane wave incident at angle θ_0 wrt the optical axis? $(\theta_0 \ll 1)$.

<u>Soln.</u> Let $u_0 = \frac{\sin\theta_0}{\lambda}$ $\rightarrow$ plane wave is $e^{i2\pi u_0 x}$ (@ $z=0$).



$$\mathcal{F}\left\{ e^{i2\pi u_0 x} \times \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\right\} =$$

$$= \frac{1}{2}\delta(u-u_0) + \frac{1}{4}\delta\left(u-u_0-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u-u_0+\frac{1}{\Lambda}\right)$$



② What is the Fraunhofer diffraction pattern of a sinusoidal amplitude grating

$$f(x) = \frac{1}{2}\left[1 + \cos\left(2\pi \frac{x}{\Lambda}\right)\right]$$

Λ is the grating period.

<u>Soln</u>

$$f(x) = \frac{1}{2} + \frac{1}{2}\cos\left(2\pi \frac{x}{\Lambda}\right)$$

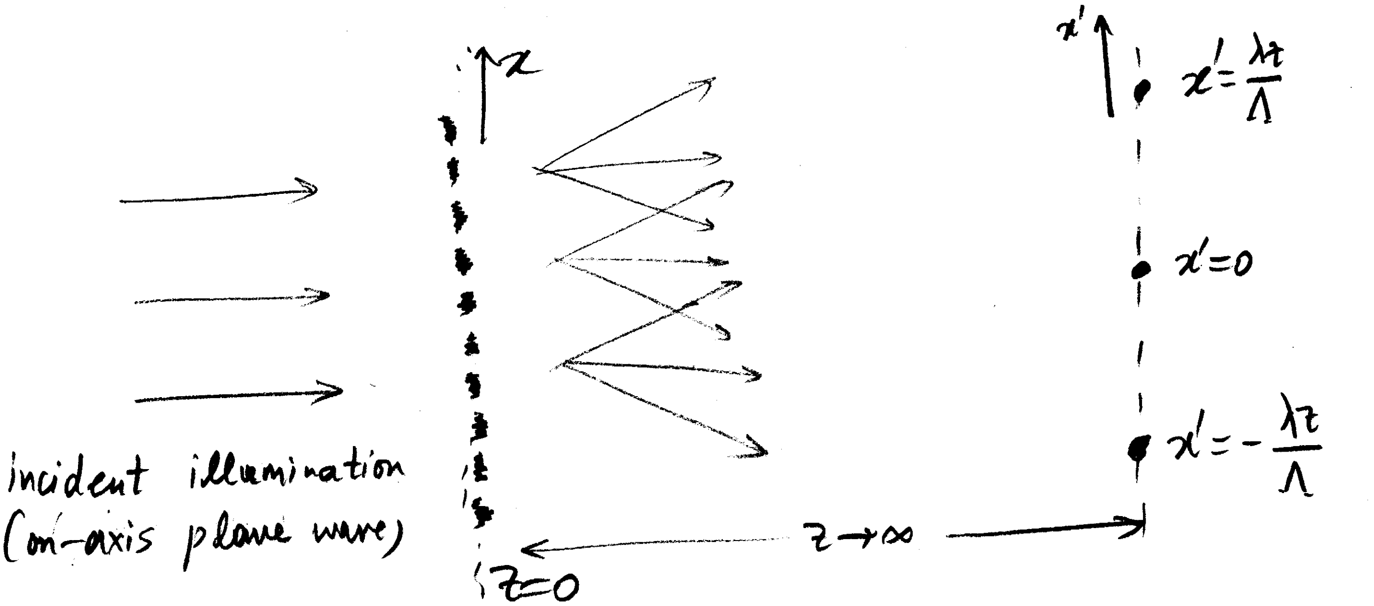
$\boxed{\text{DC-term or 0-th order}}$ plane wave $u=0$ plane wave, $u=\frac{1}{\Lambda}$ plane wave, $u=-\frac{1}{\Lambda}$ $\boxed{\text{diffracted orders}}$

$$= \frac{1}{2} + \frac{1}{4} e^{i2\pi \frac{x}{\Lambda}} + \frac{1}{4} e^{-i2\pi \frac{x}{\Lambda}}$$

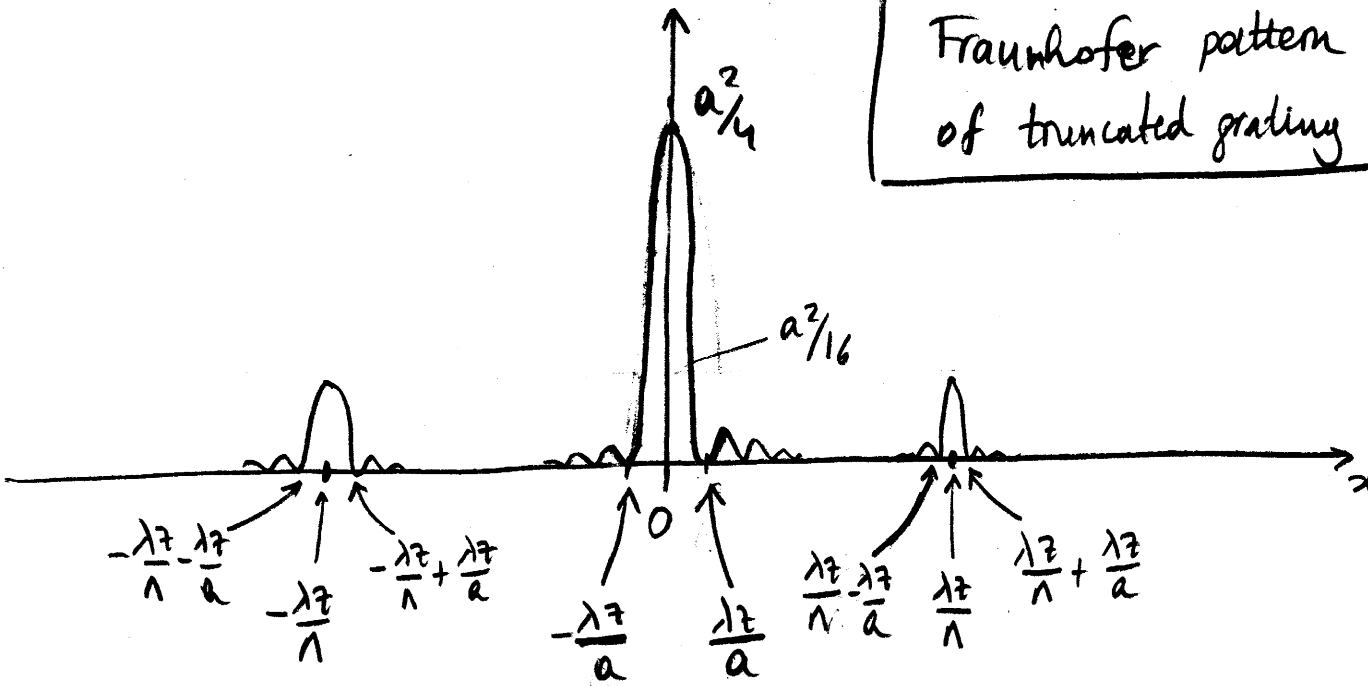
$$\Rightarrow F(u) = \frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u - \frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u + \frac{1}{\Lambda}\right)$$

$$\Rightarrow g(x') = e^{i\pi \frac{x'^2 + y'^2}{\lambda z}}\left[\frac{1}{2}\delta\left(\frac{x'}{\lambda z}\right) + \frac{1}{4}\delta\left(\frac{x'}{\lambda z} - \frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(\frac{x'}{\lambda z} + \frac{1}{\Lambda}\right)\right]$$

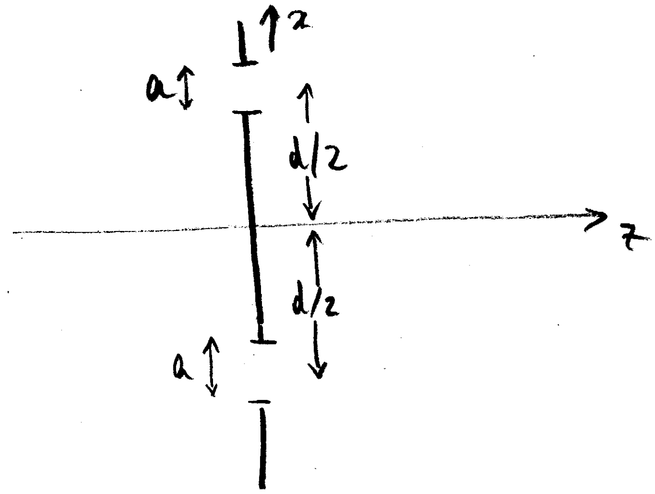
// **<u>Note</u>** without being too rigorous mathematically, we treat the intensity corresponding to the δ-function field as a "very bright & sharp" spot. //



Fraunhofer pattern of truncated grating



(5) What is the Fraunhofer diffraction pattern of two identical slits (width a) separated by a distance $d \gg a$?



$$f(x) = \text{rect}\left(\frac{x-\frac{d}{2}}{a}\right) + \text{rect}\left(\frac{x+\frac{d}{2}}{a}\right)$$

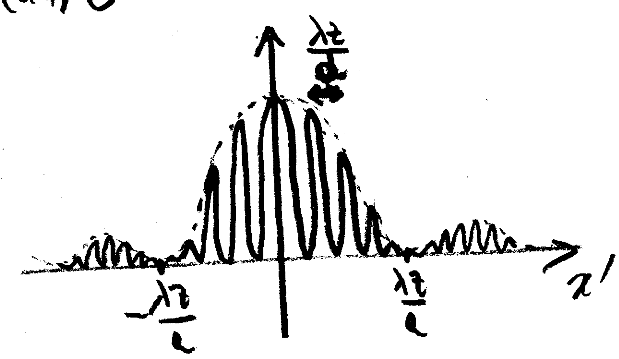
use scaling & shift thms + linearity

$$F(u) = a\,\text{sinc}(au)\,e^{-i2\pi u\frac{d}{2}} + a\,\text{sinc}(au)\,e^{i2\pi u\frac{d}{2}}$$

$$= 2a\,\text{sinc}(au)\cos(\pi u d)$$

$$|g(x')|^2 = 4a^2\,\text{sinc}^2\left(\frac{ax'}{\lambda z}\right)\cos\left(\frac{\pi x' d}{\lambda z}\right)$$

("modulated" sinc pattern)



④ What is the Fraunhofer pattern of a truncated sinusoidal amplitude grating

$$f(x) = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\mathrm{rect}\left(\frac{x}{a}\right)$$

Assume $a \gg \Lambda$.

<u>Soln</u>

Let $f_1(x) = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right] \Rightarrow F_1(u) = \frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)$

$f_2(x) = \mathrm{rect}\left(\frac{x}{a}\right) \Rightarrow F_2(u) = a\,\mathrm{sinc}(au).$

According to the convolution theorem

$$\mathcal{F}\{f_1(x)\, f_2(x)\} = F_1(u) * F_2(u)$$

recall $\quad \delta(u-u_0) * A(u) = \int_{-\infty}^{\infty} \delta(u-u_0)\, A(u'-u)\, du = A(u'-u_0)$

$$= \left[\frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)\right] * a\,\mathrm{sinc}(au)$$

$$= \frac{1}{2}a\,\mathrm{sinc}(au) + \frac{1}{4}a\,\mathrm{sinc}\left(a\left(u-\frac{1}{\Lambda}\right)\right) + \frac{1}{4}a\,\mathrm{sinc}\left(a\left(u+\frac{1}{\Lambda}\right)\right)$$

<u>Note</u>: When you take $|\,|^2$, cross-terms can be ignored. Why?

$$|y(x')|^2 \simeq \frac{a^2}{4}\mathrm{sinc}^2\left(\frac{ax'}{\lambda z}\right) + \frac{a^2}{16}\mathrm{sinc}^2\left(a\left(\frac{x'}{\lambda z}-\frac{1}{\Lambda}\right)\right) + \frac{a^2}{16}\mathrm{sinc}^2\left(a\left(\frac{x'}{\lambda z}+\frac{1}{\Lambda}\right)\right)$$

↑

Fraunhofer pattern (intensity)