16.901: Optimization Project
Design of Film Cooling for Combustor Liners
Sample Solution

1 Background

The temperatures within the primary zone of a combustor are significantly higher than the temperatures which most materials can withstand without significant deterioration. Thus, a critical aspect of the design of a combustor is the development of a method to cool the liner walls of a combustor such that the temperatures which the liner temperatures are well below the limit of the material. A typical method to cool a combustor liner is through film cooling. Film cooling consists of diverting air from the main flow path prior to combustion and then re-introducing this air along the liner surface to provide a film of cooler air to protect the liner.

In this project, you will consider the optimization of a liner film cooling design. The numerical simulation of the liner cooling is based on the finite-difference method developed in Project Two for 16.901. In this project, the cooling air velocity, U_{cool}, and the height of the cooling film, h, are the two design variables. To make certain that the design is realistic, the design variables are bounded by:

$$\begin{aligned} 50 \text{ m/s} \leq \quad & U_{cool} \quad \leq 250 \text{ m/s} \\ 0.001 \text{ m} \leq \quad & h \quad \leq 0.005 \text{ m} \end{aligned}$$

The values (or range of values) of all the parameters are given in Table 1.

Parameter	Definition	Value
k_g	air conductivity	0.1 $W/(m\,K)$
k_w	wall conductivity	26.0 $W/(m\,K)$
h	height of cooling passage	0.001 - 0.005 m
L	axial length between cooling passages	0.3 m
U_{hot}	velocity of hot flow	100 m/sec
U_{cool}	velocity of cool flow	50 - 250 m/sec
T_{hot}	temperature of hot flow	2200 K
T_{cool}	temperature of cool flow	800 K
t_w	thickness of liner wall	0.0015 m

Table 1: Parameter definitions and values

2 Tasks

2.1 Minimization of liner temperature (20%)

In this first task, we consider the minimization of the maximum liner temperature, $T_{\max}$. Note, $T_{\max}$ is assumed to occur on the upper surface at the outlet of the computational domain. The design parameters were non-dimensionalized and mapped such that they both range from -1 to $+1$ as follows:

$$x_1 = -1 + 2\frac{U_{cool} - \min U_{cool}}{\max U_{cool} - \min U_{cool}} \qquad x_2 = -1 + 2\frac{h - \min h}{\max h - \min h}$$

where $\min U_{cool}$ is the lower bound of the design range for U_{cool}, etc. Then, Matlab's **fmincon** optimization routine was used to minimize $T_{\max}$ over the design space. **fmincon** calculated the necessary derivatives through finite differencing for this task. The minimal value was found to be $T_{\max} = 1180.4\,K$ and occured at $U_{cool} = 250$ m/sec and $h = 0.005$ m. This design point is logical since the minimum temperature is expected to occur for the design with the maximum cooling air flow and height. Clearly, U_{cool} and h are at the boundary of the design space.

The optimization for this problem was begun at the center of the design space (i.e. at $U_{cool} = 150$ m/sec and $h = 0.003$ m). The optimization history was

```
UseIter   F-count      f(x)         constraint    Step-size      derivative   Procedure
    1       3     -0.0099065            -1            1           -0.037
    2       7     -0.0235035       -0.8135            1        -1.66e+05    Hessian modified
    3      11     -0.0920041             0            1         -1.8e-09
Optimization terminated successfully:
 Search direction less than 2*options.TolX and
  maximum constraint violation is less than options.TolCon
Active Constraints:
     3
     4
```

The main Matlab source code for the temperature minimization is the **minT.m** script.

2.2 Minimization of cooling mass flow (20%)

In this task, we minimize the mass flow in the cooling film for a given maximum temperature limit. Specifically, we solve the following problem:

$$\min \dot{m} = U_{cool} h \text{ such that } T_{\max} = T_{lim}.$$

In this task, $T_{lim} = 1300\,K$. The temperature constraint is set in the following non-dimensional manner:

$$\frac{T_{\max}}{T_{lim}} - 1 = 0.$$

As in the previous case, finite-difference derivatives have been used. The main script for the optimization is in **optliner.m**. Starting from the center of the design space, the optimal design occurs for $U_{cool} = 189.7$ m/sec and $h = 0.001$m, giving a mass flow of $\dot{m} = 0.192\, m^2/sec$. The iteration history for the case is:

```
                                    max                          Directional
 Iter    F-count       f(x)          constraint     Step-size      derivative   Procedure
    1       3            0.45       0.009906            1          -0.0656
    2       7        0.384226       0.006096            1            0.097    Hessian modified twice
    3      25         0.38422       0.006096     -6.1e-05          -0.0227
    4      29        0.360757       0.002662            1          0.00674
    5      33        0.367434      0.0001362            1         0.000366    Hessian modified
    6      37        0.367799      7.998e-06            1        -0.000152    Hessian modified twice
    7      41        0.367647      2.128e-06            1          -0.0102    Hessian modified twice
    8      45         0.35741      7.668e-05            1          -0.0181
    9      49        0.338858      0.0003395            1            -0.17    Hessian modified twice
   10      53        0.154972        0.02271            1           0.0173
   11      57        0.172227        0.01157            1          0.00926
   12      61        0.181486       0.006055            1          0.00995
   13      65        0.191484      0.0006509            1          0.00118
   14      69        0.192632      8.006e-05            1        -0.000587    Hessian modified twice
   15      73        0.191928      0.0001628            1           0.0003    Hessian modified
   16      77        0.192227      1.209e-06            1        -1.23e-05    Hessian modified
Optimization terminated successfully:
```

2.3 Minimization of cooling mass flow with analytic derivatives (40%)

Analytic sensitivity derivatives were added to the Matlab source code for the objective function (i.e. $\dot{m}$) and the constraint equation with respect to the design variables. The derivatives of $\dot{m}$ are simple as they depend directly on the design variables,

$$\begin{aligned} \frac{d\dot{m}}{dU_{cool}} &= h, \\ \frac{d\dot{m}}{dh} &= U_{cool}. \end{aligned}$$

Since the design variables have been non-dimensionalized, the sensitivity derivatives with respect to the scaled design variables are,

$$\begin{aligned} \frac{d\dot{m}}{dx_1} &= \frac{\max U_{cool} - \min U_{cool}}{2} \frac{d\dot{m}}{dU_{cool}}, \\ \frac{d\dot{m}}{dx_2} &= \frac{\max h - \min h}{2} \frac{d\dot{m}}{dh}. \end{aligned}$$

The constraint equation depends on the states of the finite-difference film-cooling model. Specifically, the value of $T_{\max}$ is the temperature at on the upper surface at the outlet which is found by solving the finite-difference model. For example, consider the dependence of $T_{\max}$ on h, the derivative may be found as follows,

$$\frac{dT_{\max}}{dh} = \frac{\partial T_{\max}}{\partial h} + \frac{\partial T_{\max}}{\partial \vec{T}} \frac{d\vec{T}}{dh}, \qquad (1)$$

where $\vec{T}$ is the vector of temperatures solved in the finite-difference model. Since $T_{\max}$ does not directly depend on the design variables, $\frac{\partial T_{\max}}{\partial h} = 0$. Furthermore, $\frac{\partial T_{\max}}{\partial \vec{T}}$ is a vector of zeros except for the entry corresponding to the upper surface, outlet location where the derivative is one. Thus, the remaining difficulty is to find $\frac{d\vec{T}}{dh}$. Writing the governing equations of the finite-difference model as a set of residual equations we have,

$$\vec{R}\left(\vec{T}, h\right) = 0.$$

Note, the residual equation also depends on U_{cool} but for this derivation, we have not shown this explicitly. Then, if we consider a perturbation in h which will create a perturbation in $\vec{T}$, we have,

$$\vec{R}\left(\vec{T} + d\vec{T}, h + dh\right) = 0.$$

Then, in the limit of small changes, we may Taylor series this result to produce,

$$\vec{R}\left(\vec{T} + d\vec{T}, h + dh\right) \approx \vec{R}\left(\vec{T}, h\right) + \frac{\partial \vec{R}}{\partial \vec{T}} d\vec{T} + \frac{\partial \vec{R}}{\partial h} dh = 0.$$

Or, re-arranging this we find,

$$\frac{\partial \vec{R}}{\partial \vec{T}} \frac{d\vec{T}}{dh} = -\frac{\partial \vec{R}}{\partial h}.$$

Plugging this result into Equation (1) gives,

$$\frac{dT_{\max}}{dh} = -\psi^T \frac{\partial \vec{R}}{\partial h},$$

where the adjoint ψ satisfies,

$$\frac{\partial \vec{R}}{\partial \vec{T}}^T \psi = \frac{\partial T_{\max}}{\partial \vec{T}}^T.$$

A similar result exists for U_{cool},

$$\frac{dT_{\max}}{dU_{cool}} = -\psi^T \frac{\partial \vec{R}}{\partial U_{cool}}.$$

This adjoint-based sensitivity derivative was implemented in **condif.m**.

For the constraint $T_{lim} = 1300\,K$, the optimization was re-run with the analytic derivatives. The optimum occurs for $U_{cool} = 190.5$ m/sec and $h = 0.001$m, giving a mass flow of $\dot{m} = 0.192\ m^2/sec$ and clearly matches the finite-differenced derivative optimization from Section 2.2. The iteration history for the case is:

Iter	F-count	f(x)	max constraint	Step-size	Directional derivative	Procedure
1	1	0.45	0.009906	1	-0.0943	
2	3	0.359221	0.0001474	1	-0.173	
3	6	0.266075	7.981e-06	0.5	-0.0976	Hessian modified twice
4	9	0.215392	0.001534	0.5	-0.0394	
5	12	0.194336	0.002947	0.5	-0.0073	
6	14	0.183999	0.004592	1	0.00811	
7	16	0.192196	0.0001133	1	8.32e-06	Hessian modified
8	18	0.192168	4.955e-05	1	1.9e-05	Hessian modified
9	20	0.192162	3.339e-05	1	6.11e-05	Hessian modified
10	22	0.192224	3.274e-08	1	6e-08	Hessian modified

```
Optimization terminated successfully:
```

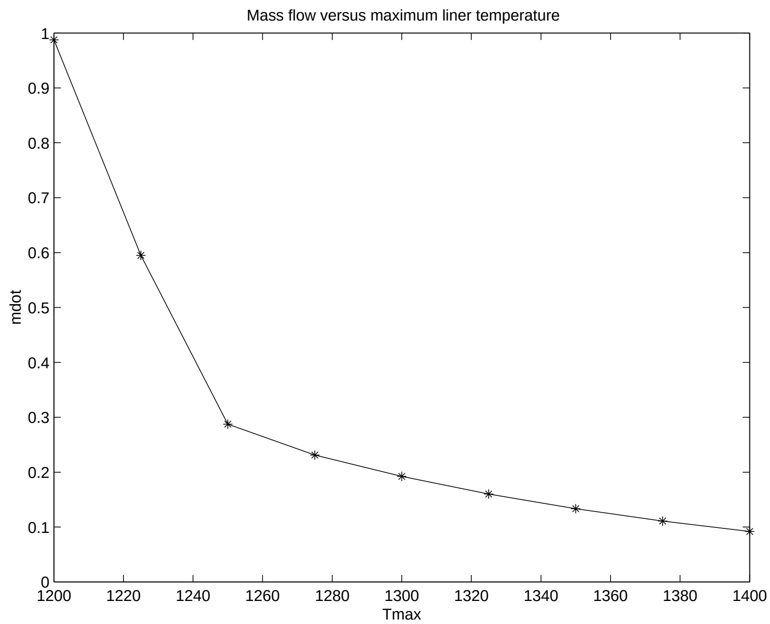
Comparing the work with finite-difference sensitivity derivatives to the adjoint-based analytic derivatives, clearly the finite-differenced sensitivity derivative case required 77 function evaluations compared to only 22 evaluations for the analytic derivatives. We note, however, that the analytic derivatives requires the inversion of $\frac{\partial \vec{R}}{\partial \vec{T}}^T$. Thus, a single evaluation of the function and derivatives is about twice as expensive than a single function evaluation in the finite-differenced version.

2.4 Parametric study of cooling mass flow minimization (20%)

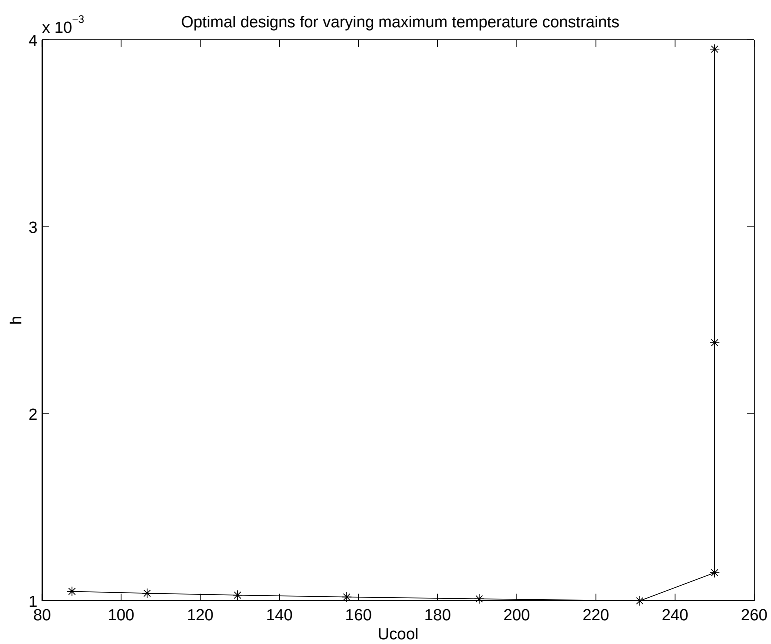
Using the constrained optimization with analytic derivatives developed in the previous section, a trade study was performed by varying T_{lim} from $1200\,K$ to $1400\,K$. The results are shown in Figure 1.

The variation of $\min \dot{m}$ versus T_{lim} shows that $\dot{m}$ increases with decreasing T_{lim}. This trend is expected as the lower temperature will require larger liner cooling, and therefore higher mass flows.

The location of the designs in the two-dimensional design space show that at the most stringent temperature constraint (i.e. lowest $T_{\rm lim}$), the largest mass flow is used with U_{cool} at its maximum value and h nearly at its largest value. As T_{lim} is increased, the h value decreases while U_{cool} remains at the maximum limit of 250 m/sec. Eventually, at $T_{lim} = 1275K$, the value of U_{cool} decreases from the maximum boundary value. At this point, further increases in T_{lim} cause further descreases of U_{cool} while h is fixed at the minimum value of 0.001 m.



(a) Optimum $\dot{m}$ versus T_{lim}



(b) Optimum design variables versus T_{lim}

Figure 1: Trade study for varying maximum temperature limits, T_{lim}.