16.901 Notes - March 4, 2002

Topics {
* 2^{nd} order upwind & CFL condition
* Fourier analysis of wave eqn
* Von Neumann analysis of finite differenced wave equation

Last time, we investigated the first order upwind approximation to the wave eqn:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \Rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} + u\frac{T_i^n - T_{i-1}^n}{\Delta x} = 0$$

assumed $u>0$

Suppose we wish to replace the 1^{st} order upwind derivative with a 2nd order, upwind approximation:

$$\left.\frac{\partial T}{\partial x}\right|_i \cong \frac{1}{\Delta x}\left\{aT_i + bT_{i-1} + cT_{i-2}\right\} \qquad (1)$$

where a, b & c are unknown constants which we will set such that the approximation is at least 2^{nd} order. As usual, use a Taylor series:

$$T_{i-1} = T_i - \Delta x\left.\frac{\partial T}{\partial x}\right|_i + \frac{1}{2}\Delta x^2\left.\frac{\partial^2 T}{\partial x^2}\right|_i - \frac{1}{6}\Delta x^3\left.\frac{\partial^3 T}{\partial x^3}\right|_i + O(\Delta$$

$$T_{i-2} = T_i - 2\Delta x\left.\frac{\partial T}{\partial x}\right|_i + 2\Delta x^2\left.\frac{\partial^2 T}{\partial x^2}\right|_i - \frac{4}{3}\Delta x^3\left.\frac{\partial^3 T}{\partial x^3}\right|_i + O(\Delta x$$

Then substituting these into (1) and collecting terms

$$\frac{1}{\Delta x}\left\{aT_i + bT_{i-1} + cT_{i-2}\right\} = \frac{T_i}{\Delta x}(a+b+c) \rightarrow = 0$$

$$+ \left.\frac{\partial T}{\partial x}\right|_i (-b-2c) \rightarrow = 1 \text{ for } \left.\frac{\partial T}{\partial x}\right|_i \text{ approxim}$$

$$+ \Delta x \left.\frac{\partial^2 T}{\partial x^2}\right|_i \left(\tfrac{1}{2}b + 2c\right) \rightarrow = 0 \text{ for higher ord}$$

$$+ \Delta x^2 \left.\frac{\partial^3 T}{\partial x^3}\right|_i \left(-\tfrac{1}{6}b - \tfrac{4}{3}c\right)$$

So,

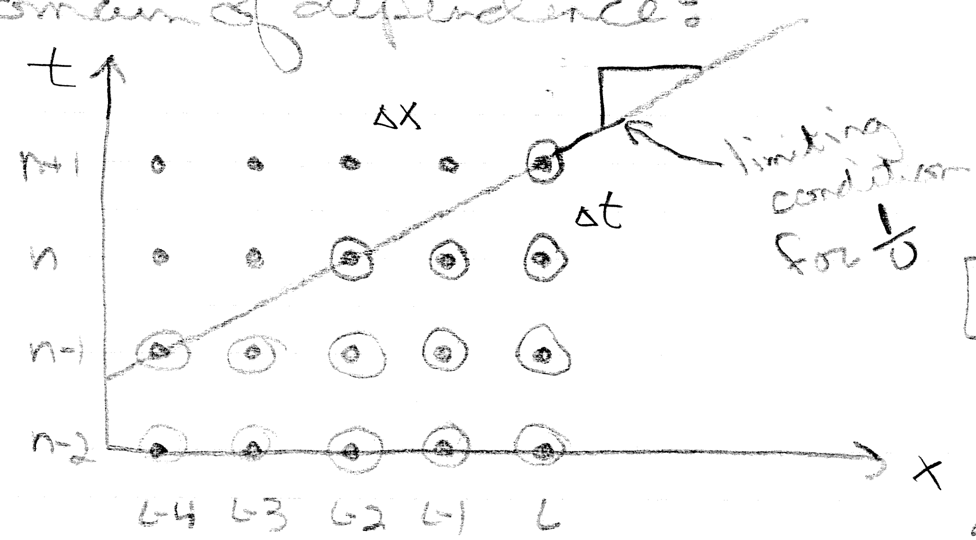
$$\left.\begin{aligned} a+b+c &= 0 \\ -b-2c &= 1 \\ \tfrac{1}{2}b + 2c &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= \tfrac{3}{2} \\ b &= -2 \\ c &= \tfrac{1}{2} \end{aligned}$$

$$\Rightarrow \left.\frac{\partial T}{\partial x}\right|_i = \boxed{\underbrace{\frac{1}{\Delta x}\left(\frac{3}{2}T_i - 2T_{i-1} + \frac{1}{2}T_{i-2}\right)}_{\substack{\text{2nd order upwind} \\ \text{discretization} \\ \text{of } \left.\frac{\partial T}{\partial x}\right|_i \text{ for } u>0}}} - \underbrace{\frac{1}{3}\Delta x^2 \frac{\partial^3 T}{\partial x^3}}_{\text{2nd order error}} + O(\Delta x^3)$$

Combining this with the Forward Euler time-discretization gives:

$$\boxed{\frac{T_i^{n+1} - T_i^n}{\Delta t} + u\frac{1}{\Delta x}\left(\frac{3}{2}T_i^n - 2T_{i-1}^n + \frac{1}{2}T_{i-2}^n\right) = 0}$$

Next, let's consider what the CFL condition requires in order for this scheme to be convergent. To do this, we need to determine the conditions on Δt & Δx for the physical domain of dependence to lie within the numerical domain of dependence:

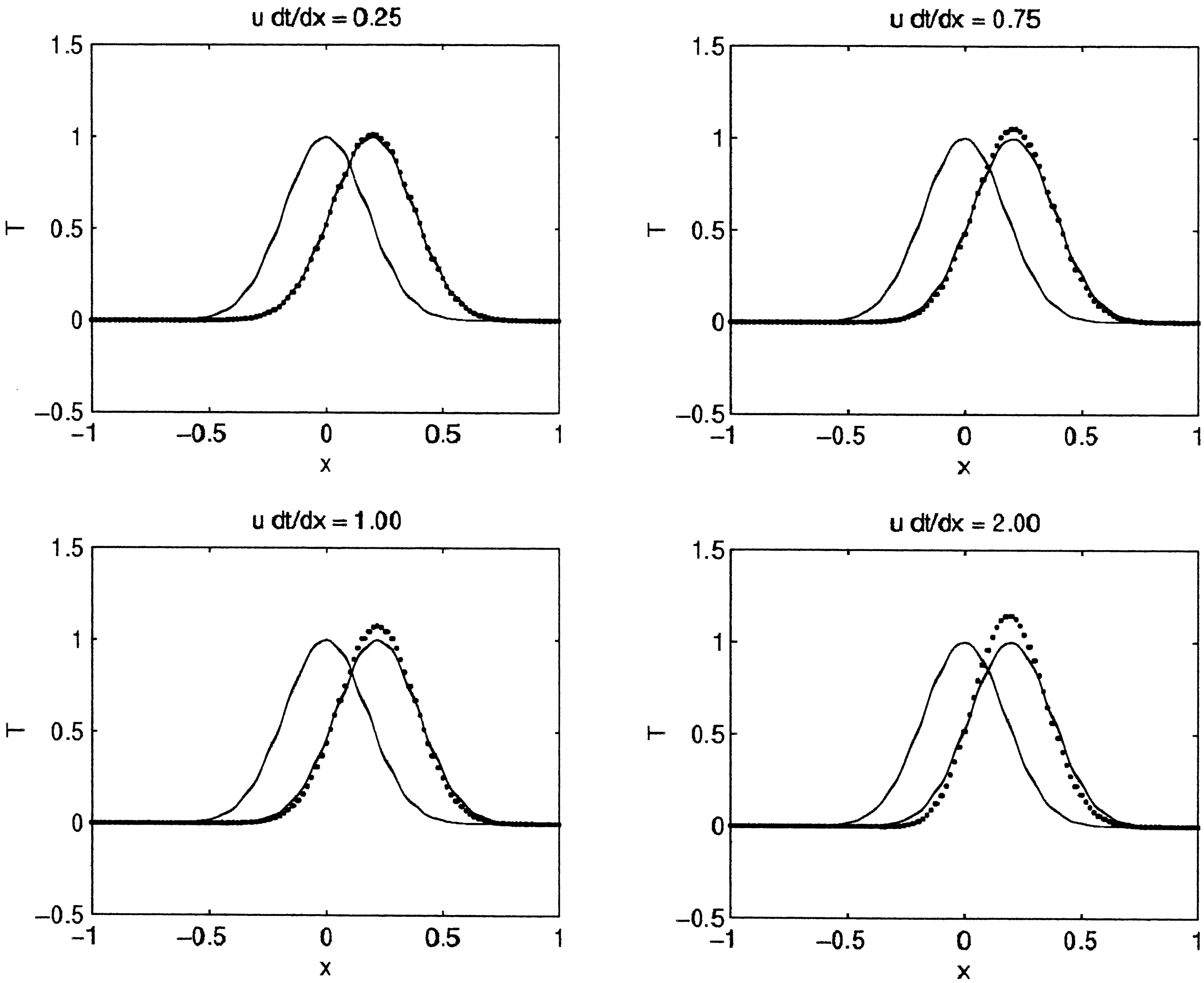


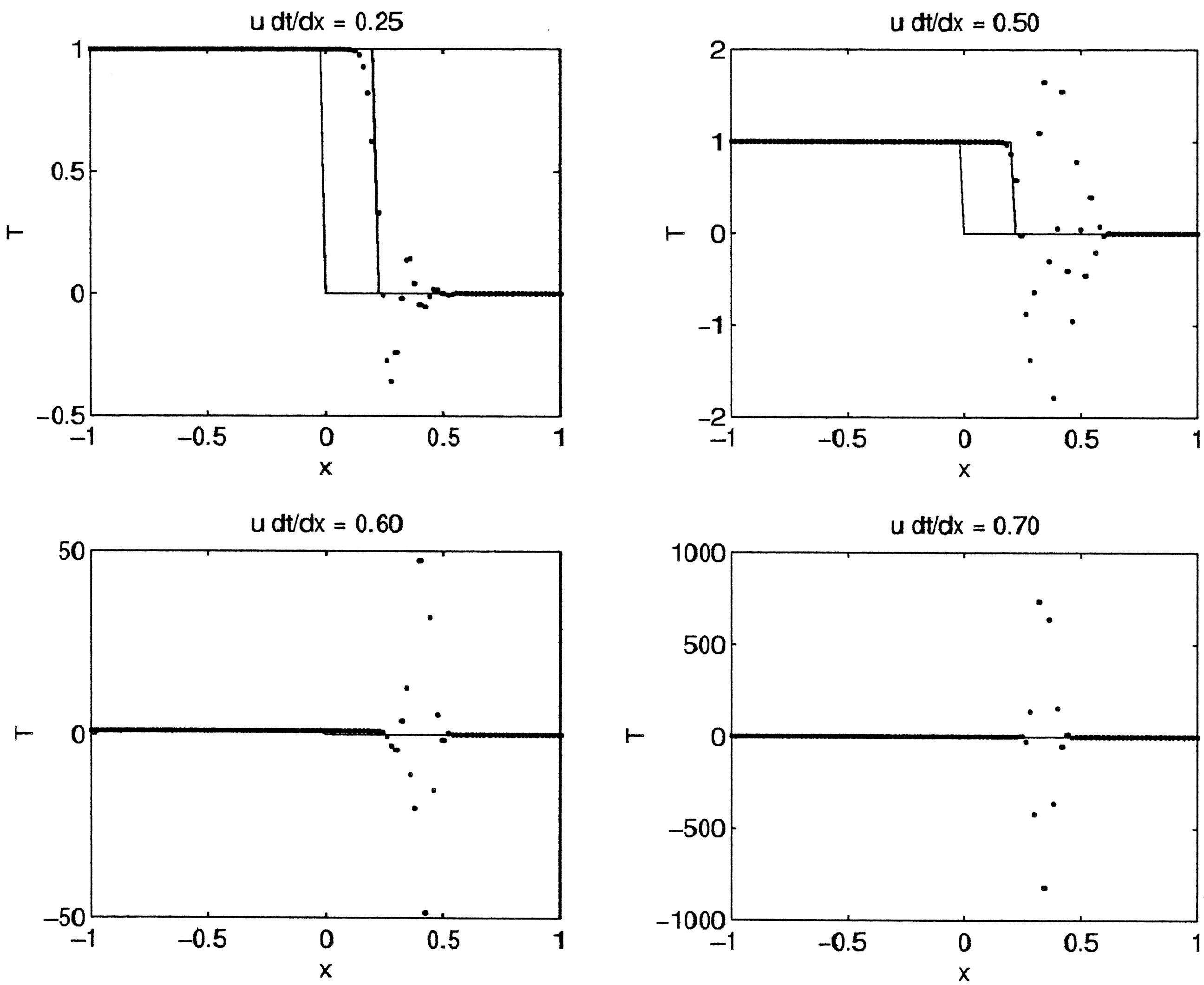
Recall: the physical domain of dependence is a line of slope $\frac{1}{U}$.

$$\Rightarrow \frac{1}{U} \geq \frac{\Delta t}{2\Delta x}$$

$$\Rightarrow \boxed{\frac{U\Delta t}{\Delta x} \leq 2}$$

Now let's look at the results of applying this scheme on the next page.


u dt/dx = 0.25
u dt/dx = 0.75
u dt/dx = 1.00
u dt/dx = 2.00
T
x
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1



u dt/dx = 0.25
u dt/dx = 0.50
u dt/dx = 0.60
u dt/dx = 0.70
T
x

The results show that

* for the smooth, Gaussian initial conditions, the solution grows for all $\frac{u \Delta t}{\Delta x}$ tested (0.25, 0.75, 1.0 & 2.0)

* For discontinuous solutions, the growth from the numerical instability is explosive.

$$\boxed{\text{Conclusion: CFL condition is a necessary but not sufficient condition for convergence!}}$$

$\Uparrow$

<u>Very Important!</u>

Thus, we will need a more precise tool to help analyze the stability and convergence properties of finite difference methods for PDE's.