Accuracy & Consistency for Finite Difference Methods

We begin with defining a <u>convergent method</u>:

Definition: A finite difference method for solving

Initial Value Problem $\Longrightarrow$ $$\frac{d\vec{u}}{dt} = \vec{f}(\vec{u}, t) \qquad \vec{u}(0) = \vec{u}_0$$ We will call this our IVP from now on

from $t=0$ to T is convergent if

$$\max_{n=\left[0, \frac{T}{\Delta t}\right]} \left\| \vec{v}^n - \vec{u}(n\Delta t) \right\| \to 0 \quad \text{as } \Delta t \to 0$$

The global order of accuracy measures how quickly the finite difference method converges:

Definition: The global order of accuracy for a finite difference method for IVP is p when:

$$\max_{n=\left[0, \frac{T}{\Delta t}\right]} \left\| \vec{v}^n - \vec{u}(n\Delta t) \right\| \leq C(\Delta t)^p \quad \text{as } \Delta t \to 0$$

Assumes $\vec{f}$ has continuous derivatives up to and including $\frac{\partial^p \vec{f}}{\partial t^p}$ and $\frac{\partial^p \vec{f}}{\partial \vec{u}^p}$.

Another definition we require is consistency

Definition: A finite difference method is consistent if

$$\| \vec{v}^{\,1} - \vec{u}(\Delta t) \| \leq C \Delta t \qquad \text{as } \Delta t \to 0$$

for any initial condition.

Consistency is a bare minimum condition required of any numerical algorithm. If a method were not consistent, then $\vec{v}^{\,1}$ can never approach $\vec{u}(\Delta t)$ no matter how small Δt is chosen.

Consistency is checked through a Taylor series analysis. Here's an example for Forward Euler:

Forward Euler: $\vec{v}^{\,1} = \vec{v}^{\,0} + \Delta t \vec{f}(\vec{v}^{\,0}, 0)$

$$\vec{v}^{\,1} - \vec{u}(\Delta t) = \vec{v}^{\,0} + \Delta t \vec{f}(\vec{v}^{\,0}, 0) - \vec{u}(\Delta t)$$

But, we assume $\vec{v}^{\,0} = \vec{u}(0)$

$$\vec{v}^{\,1} - \vec{u}(\Delta t) = \vec{u}(0) + \Delta t \vec{f}(\vec{u}(0), 0) - \vec{u}(\Delta t)$$

and $\vec{f}(\vec{u}(0), 0) = \dot{\vec{u}}(0)$

$$\vec{v}' - \vec{u}(\Delta t) = \vec{u}(0) + \Delta t\,\dot{\vec{u}}(0) - \vec{u}(\Delta t)$$

But, a Taylor series expansion of $\vec{u}(\Delta t)$ gives:

$$\vec{u}(\Delta t) = \vec{u}(0) + \Delta t\,\dot{\vec{u}}(0) + \tfrac{1}{2}\Delta t^2\,\ddot{\vec{u}}(0) + O(\Delta t^3)$$

Thus, substituting this Taylor series gives:

$$\vec{v}' - \vec{u}(\Delta t) = -\tfrac{1}{2}\Delta t^2\,\ddot{\vec{u}}(0) + O(\Delta t^3)$$

Clearly, this is faster than Δt as required by consistency.

$$\Longrightarrow \boxed{\text{F.E. is consistent}}$$

The global error can be viewed as a sum of local errors at every iteration.

Define: $\vec{E}^n \equiv$ Global error at $t^n \equiv \vec{v}^n - \vec{u}(n\Delta t)$

The difference between the global error $\vec{E}^{n+1} - \vec{E}^n$ is the local error:

$$\Delta\vec{E}^n \equiv \vec{E}^{n+1} - \vec{E}^n = \vec{v}^{n+1} - \vec{v}^n - \vec{u}^{n+1} + \vec{u}^n$$

We will write $\vec{v}^{n+1} = \mathbf{N}(\vec{v}^n)$ where $N(\vec{v}^n)$ is the numerical integration method:

$$\Delta\vec{E}^n(\vec{v}^n) = N(\vec{v}^n) - \vec{v}^n - \vec{u}^{n+1} + \vec{u}^n$$

$$\Rightarrow \vec{E}^n = \sum_{i=0}^{n-1} \Delta\vec{E}^i$$

Now, suppose that $\Delta\vec{E}^i = O(\Delta t^{p+1})$

$$\Rightarrow \vec{E}^n = O(n\,\Delta t^{p+1})$$

The largest error occurs for the largest n which is $n = N = \frac{T_{max}}{\Delta t}$

$$\Rightarrow \vec{E}^N = O\left(\frac{T_{max}}{\Delta t}\Delta t^{p+1}\right)$$

$\Rightarrow$ Global error $= O(\Delta t^p)$ if local error is $O(\Delta t^{p+1})$

Calculation of the order of $\Delta\vec{E}^n$ is difficult. Instead, we define the closely related local truncation error (LTE) as the difference of $N(\vec{U}^n)$ and $\vec{U}^{n+1}$:

$$\boxed{\vec{e} \equiv N(\vec{U}^n) - \vec{U}^{n+1}} \leftarrow \text{LTE}$$

Note: $\vec{e} = \Delta\vec{E}^n(\vec{U}^n)$.

Definition: The <u>local order of accuracy</u> of the numerical integration method is p where:

$$\|N(\vec{U}^n) - \vec{U}^{n+1}\| \leq C\Delta t^{p+1} \quad \text{as } \Delta t \to 0$$

and $\vec{U}$ = exact solution of ODE's.

We note that the local order of accuracy is closely tied to the schemes consistency.

Local order of accuracy of Forward Euler

$$\vec{v}^{\,n+1} = \vec{v}^{\,n} + \Delta t\, \vec{f}(\vec{v}^{\,n}, t^n) = N(\vec{v}^{\,n})$$

$$\Rightarrow N(\vec{u}^{\,n}) = \vec{u}^{\,n} + \Delta t\, \vec{f}(\vec{u}^{\,n}, t^n)$$

$$\Rightarrow N(\vec{u}^{\,n}) - \vec{u}^{\,n+1} = \vec{u}^{\,n} + \Delta t\, \vec{f}(\vec{u}^{\,n}, t^n) - \vec{u}^{\,n+1}$$

$$= \vec{u}^{\,n} + \Delta t\, \dot{\vec{u}}(t^n) - \vec{u}^{\,n+1}$$

$$= \vec{u}^{\,n} + \Delta t\, \dot{\vec{u}}(t^n) - \left[\vec{u}^{\,n} + \Delta t\, \dot{\vec{u}}(t^n) + \tfrac{1}{2}\Delta t^2 \ddot{\vec{u}}(t^n) + O(\Delta t^3)\right]$$

$$= -\tfrac{1}{2}\Delta t^2 \ddot{\vec{u}}(t^n) + O(\Delta t^3)$$

$$\Rightarrow \left\| N(\vec{u}^{\,n}) - \vec{u}^{\,n+1} \right\| \le C \Delta t^2 \quad \text{as } \Delta t \to 0$$

$\Rightarrow p = 1$ for Forward Euler.

$\Rightarrow$ local accuracy of F.E. is 1st ($p = 1$) order

$\Rightarrow$ Expect global order of F.E. is 1st order.

A scheme which we have not yet studied is trapezoidal integration:

$$\frac{\vec{v}^{\,n+1} - \vec{v}^{\,n}}{\Delta t} = \frac{1}{2}\left[\vec{f}(\vec{v}^{\,n}, t^n) + \vec{f}(\vec{v}^{\,n+1}, t^{n+1})\right]$$

This is an implicit scheme since $\vec{f}^{\,n+1}$ is involved. Let's look at its local accuracy:

Trapezoidal Local Accuracy

$$N(\vec{v}^n) = \vec{v}^n + \frac{\Delta t}{2}\left[\vec{f}(\vec{v}^n, t^n) + \vec{f}(\vec{v}^{n+1}, t^{n+1})\right]$$

$$\Rightarrow N(\vec{U}^n) - \vec{U}^{n+1} = \vec{U}^n + \frac{\Delta t}{2}\left[\vec{f}(\vec{U}^n, t^n) + \vec{f}(\vec{U}^{n+1}, t^{n+1})\right] - \vec{U}^{n+1}$$

$$= \vec{U}^n - \vec{U}^{n+1} + \frac{\Delta t}{2}\left[\dot{\vec{U}}(t^n) + \dot{\vec{U}}(t^{n+1})\right]$$

Now, Taylor series expand $\vec{U}^{n+1}$ & $\dot{\vec{U}}(t^{n+1})$:

$$\vec{U}^{n+1} = \vec{U}^n + \Delta t\,\dot{\vec{U}}(t^n) + \tfrac{1}{2}\Delta t^2\,\ddot{\vec{U}}(t^n) + \tfrac{1}{6}\Delta t^3\,\dddot{\vec{U}}(t^n) + O(\Delta t^4)$$

$$\dot{\vec{U}}^{n+1} = \dot{\vec{U}}^n + \Delta t\,\ddot{\vec{U}}^n + \tfrac{1}{2}\Delta t^2\,\dddot{\vec{U}}^n + \tfrac{1}{6}\Delta t^3\,\ddddot{\vec{U}} + O(\Delta t^4)$$

$$\Rightarrow N(\vec{U}^n) - \vec{U}^{n+1} = -\Delta t\,\dot{\vec{U}}^n - \tfrac{1}{2}\Delta t^2\,\ddot{\vec{U}}^n - \tfrac{1}{6}\Delta t^3\,\dddot{\vec{U}}^n + O(\Delta t^4) + \frac{\Delta t}{2}\left[\dot{\vec{U}}^n + \dot{\vec{U}}^n + \Delta t\,\ddot{\vec{U}}^n + \tfrac{1}{2}\Delta t^2\,\dddot{\vec{U}}^n + O(\Delta t^3)\right]$$

$$\Downarrow \; 0 \qquad\qquad \Downarrow \; 0$$

$$= \frac{1}{12}\Delta t^3\,\dddot{\vec{U}}^n + O(\Delta t^4)$$

$$\Rightarrow \boxed{\|N(\vec{U}^n) - \vec{U}^{n+1}\| \leq C\Delta t^3 \Rightarrow p = 2!}$$

Trapezoidal has 2nd order accuracy

Example: Derivation of 2nd order Explicit Scheme

The Adams-Bashforth family of integrators are explicit and approximate the time derivatives as:

$$\left.\frac{d\vec{U}}{dt}\right|_{t^n} = \frac{\vec{U}^{n+1} - \vec{U}^n}{\Delta t}$$

1st A-B (=Forward Euler): $\frac{\vec{V}^{n+1} - \vec{V}^n}{\Delta t} = \vec{f}(\vec{V}^n, t^n)$

The 2nd A-B scheme has the form:

$$\frac{\vec{V}^{n+1} - \vec{V}^n}{\Delta t} = \beta_1 \vec{f}(\vec{V}^n, t^n) + \beta_2 \vec{f}(\vec{V}^{n-1}, t^{n-1})$$

where β_1, β_2 are (currently) unknown constants. We can use our local error analysis to determine $\beta_{1,2}$.

$$\vec{V}^{n+1} = \vec{V}_n + \Delta t \beta_1 \vec{f}^n + \Delta t \beta_2 \vec{f}^{n-1}$$

$$N(\vec{U}^n) - \vec{U}^{n+1} = \vec{U}_n + \Delta t \beta_1 \vec{f}(\vec{U}^n, t^n) + \Delta t \beta_2 \vec{f}(\vec{U}^{n-1}, t^{n-1}) - \vec{U}^{n+1}$$

Then, substitute and Taylor expand:

$$\vec{f}(\vec{U}^n, t^n) = \dot{\vec{U}}^n$$

$$\vec{f}(\vec{U}^{n-1}, t^{n-1}) = \dot{\vec{U}}^{n-1} = \dot{\vec{U}}^n - \Delta t \ddot{\vec{U}}^n + \frac{1}{2}\Delta t^2 \dddot{\vec{U}}^n - \frac{1}{6}\Delta t^3 \ddddot{\vec{U}}^n + O(\Delta$$

$$\vec{U}^{n+1} = \vec{U}^n + \Delta t \dot{\vec{U}}^n + \frac{1}{2}\Delta t^2 \ddot{\vec{U}}^n + \frac{1}{6}\Delta t^3 \dddot{\vec{U}}^n + O(\Delta t^4)$$ 8

Group terms in powers of Δt:

$$N(\vec{U}^n) - \vec{U}^{n+1} = \Delta t \dot{\vec{U}}^n \left(\beta_1 + \beta_2 - 1\right) + \Delta t^2 \ddot{\vec{U}}^n \left(-\beta_2 - \frac{1}{2}\right)$$
$$+ \Delta t^3 \dddot{\vec{U}}^n \left(\frac{1}{2}\beta_2 - \frac{1}{6}\right) + O(\Delta t^4)$$

Then, by setting: $\left\{\begin{array}{l}\beta_2 = -\frac{1}{2} \\ \beta_1 = \frac{3}{2}\end{array}\right\}$ the leading

error term is $O(\Delta t^3)$. Thus, the 2nd A-B scheme is:

$$\boxed{\begin{array}{c}\dfrac{\vec{V}^{n+1} - \vec{V}^n}{\Delta t} = \dfrac{3}{2}\vec{f}\left(\vec{V}^n, t^n\right) - \dfrac{1}{2}\vec{f}\left(\vec{V}^{n-1}, t^{n-1}\right) \\ \text{2nd order A-B scheme } (p=2)\end{array}}$$

Note: what happens on first iteration when $\vec{V}^{n-1}$ is not defined?

** Usually use Forward Euler on first iteration.

Summarizing Multi-Step Methods (SO FAR)

Adams-Bashforth:

These are explicit schemes.

$$p=1: \quad \frac{\vec{V}^{n+1}-\vec{V}^{n}}{\Delta t} = \vec{f}(\vec{V}^{n}, t^{n}) \qquad \text{(Forward Euler)}$$

$$p=2: \quad \frac{\vec{V}^{n+1}-\vec{V}^{n}}{\Delta t} = \frac{3}{2}\vec{f}(\vec{V}^{n}, t^{n}) - \frac{1}{2}\vec{f}(\vec{V}^{n-1}, t^{n-1})$$

Adams-Moulton:

These are implicit schemes.

$$p=1: \quad \frac{\vec{V}^{n+1}-\vec{V}^{n}}{\Delta t} = \vec{f}(\vec{V}^{n+1}, t^{n+1}) \qquad \text{(Backward Euler)}$$

$$p=2: \quad \frac{\vec{V}^{n+1}-\vec{V}^{n}}{\Delta t} = \frac{1}{2}\left[\vec{f}(\vec{V}^{n+1}, t^{n+1}) + \vec{f}(\vec{V}^{n}, t^{n})\right] \qquad \text{(Trapezoidal)}$$

Variations of both of these schemes exists for $p > 2$ also.