Lax-Wendroff Algorithms

Another very popular approach for introducing numerical stabilization for convection problems is through a Lax-Wendroff discretization. Here's how these algorithms are derived.

Suppose we start with a 1-D convection problem,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0 \quad , \; u = u(x) \leftarrow \text{not a function of } t$$

Next, let's introduce a second-order Taylor series in time for T^{n+1} about T^n:

$$T^{n+1} \cong T^n + \Delta t \left.\frac{\partial T}{\partial t}\right|_{t=t^n} + \frac{1}{2}\Delta t^2 \left.\frac{\partial^2 T}{\partial t^2}\right|_{t=t^n}$$

From the governing equation, we note that:

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$$

$$\frac{\partial^2 T}{\partial t^2} = \frac{\partial}{\partial t}\left(-u\frac{\partial T}{\partial x}\right) = -u\frac{\partial^2 T}{\partial t \partial x} = -u\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) = -u\frac{\partial}{\partial x}\left(-u\right.$$

$$\Rightarrow \frac{\partial^2 T}{\partial t^2} = u^2\frac{\partial^2 T}{\partial x^2}$$

Plugging back into Taylor series gives:

$$T^{n+1} = T^n + \Delta t \left(-u\frac{\partial T}{\partial x}\right)^n + \frac{1}{2}\Delta t^2 \left(u^2 \frac{\partial^2 T}{\partial x^2}\right)^n$$

The next step is to discretize $\frac{\partial T}{\partial x}$ & $\frac{\partial^2 T}{\partial x^2}$. Let's use standard central differences:

$$\left.\frac{\partial T}{\partial x}\right|_j^n = \frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x} \qquad \left.\frac{\partial^2 T}{\partial x^2}\right|_j^n = \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}$$

$$\boxed{\Rightarrow T_j^{n+1} = T_j^n - \Delta t\, u_j \left(\frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x}\right) + \frac{1}{2}\Delta t^2 u_j^2 \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}}$$

We can re-arrange this to look more like the convection equation:

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u_j \frac{T_{j+1}^n - T_{j-1}^n}{2\Delta x} = \underbrace{\frac{1}{2}\Delta t\, u^2 \frac{T_{j+1}^n - 2T_j^n + T_{j-1}^n}{\Delta x^2}}_{\text{numerical stabilization}}$$

$$\approx \underbrace{\frac{1}{2}\Delta t\, u^2}_{\text{this is a positive coefficient!}} \frac{\partial^2 T}{\partial x^2}$$

Fourier Analysis of Wave Equation

Before analyzing the finite-differenced approximations of the wave equation, we first look at the wave equation. We will investigate Fourier solutions on a periodic domain. Specifically,

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0$$

Assuming periodicity over length L:

$$T(x+mL, t) = T(x,t) \text{ for any integer } m$$

Why assume periodicity? Simplest analysis but can include other boundary conditions if desired.

Next, we assume a Fourier series solution form which has the correct periodicity:

$$T(x,t) = \sum_{m=-\infty}^{+\infty} \hat{g}_m(t) e^{ik_m x}$$

$$k_m \equiv 2\pi m/L \longleftarrow \text{satisfies periodicity}$$

Plug this into wave eqn:

$$\frac{\partial}{\partial t}\left\{\sum_m \hat{g}_m(t)e^{ik_m x}\right\} + u\frac{\partial}{\partial x}\left\{\sum_m \hat{g}_m(t)e^{ik_m x}\right\} = 0$$

$$\sum_m\left\{\left(\frac{d\hat{g}_m}{dt} + iuk_m\hat{g}_m\right)e^{ik_m x}\right\} = 0$$

Multiply by $e^{-ik_j x}$ and integrate from $x = 0 \to L$:

$$\sum_m\left\{\left(\frac{d\hat{g}_m}{dt} + iuk_m\hat{g}_m\right)\int_0^L e^{-ik_j x}e^{ik_m x}\,dx\right\} = 0$$

But $$\int_0^L e^{-ik_j x}e^{ik_m x}\,dx = \begin{cases} 0 & j \neq m \\ L & j = m \end{cases}$$

Thus, the governing equations (ODE's) for $\hat{g}_m(t)$ are decoupled by the multiplication and integration:

$$\frac{d\hat{g}_m}{dt} + iuk_m\hat{g}_m = 0$$

$$\Rightarrow \hat{g}_m(t) = \hat{g}_m(0)e^{-iuk_m t} \qquad (2)$$

comes from initial cond.

$$\Rightarrow \boxed{T(x,t) = \sum_{m=-\infty}^{+\infty} \hat{g}_m(0)\, e^{i k_m (x-vt)}}$$

note appearance of $\eta = x - vt$ dependence

Since each Fourier mode behaves independently we could have, and in the future we will, just consider a single Fourier mode.
i.e. we will assume

$$T(x,t) = \hat{g}_m(t) e^{i k_m x} \quad , k_m \equiv \frac{2\pi m}{L}$$

no summation over m

In terms of stability of the wave eqn, we see from (2) that the amplitude of the m^{th} wave (i.e. $\hat{g}_m(t)$) is constant in time:

$$\hat{g}_m(t) = \hat{g}_m(0) e^{-i v k_m t}$$

$$\Rightarrow \quad |\hat{g}_m(t)| = \left|\hat{g}_m(0) e^{-i v k_m t}\right|$$

$$= |\hat{g}_m(0)| \underbrace{\left|e^{-i v k_m t}\right|}_{=1} \quad \Rightarrow \boxed{|\hat{g}_m(t)| = |\hat{g}_m(0)|}$$

* * *

* * (neutrally) stable!

So, for the wave equation, the Fourier modes maintain their amplitude neither growing nor decaying.

Von Neumann Analysis

Von Neumann analysis is the discrete analog of the Fourier analysis performed above:

$$\boxed{\text{PDE's and Fourier analysis}} \Longleftrightarrow \boxed{\text{Finite Differenced PDE's and Von Neumann Analys}}$$

We start with a governing finite difference algorithm (we'll use 1st order upwin

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} + u\,\frac{T_j^n - T_{j-1}^n}{\Delta x} = 0 \quad , u > 0.$$

Then, assume a periodic function:

$$T_j^n = \overset{\text{raised to power } n}{\hat{g}_m^{\,n}}\, e^{i k_m x_j} \quad , \qquad x_j = j\Delta x$$
$$k_m = \frac{2\pi}{L} m$$

Plug this into the governing finite difference equation:

$$0 = \frac{\hat{g}_m^{n+1} e^{ik_m j\Delta x} - \hat{g}_m^{n} e^{ik_m j\Delta x}}{\Delta t} + u\,\frac{\hat{g}_m^{n} e^{ik_m j\Delta x} - \hat{g}_m^{n} e^{ik_m (j-1)\Delta x}}{\Delta x}$$

Separate out a factor $\hat{g}_m^n e^{ik_m j\Delta x}$ to give:

$$\hat{g}_m^n e^{ik_m x_j} \left\{ \frac{\hat{g}_m - 1}{\Delta t} + u\,\frac{1 - e^{-ik_m\Delta x}}{\Delta x} \right\} = 0$$

This gives n roots of $\hat{g}_m = 0$

This gives the other roots which are the key for stability.

Re-arrange expression in $\{\}$ to give:

$$\hat{g}_m = 1 - \frac{u\Delta t}{\Delta x}\left(1 - e^{-ik_m\Delta x}\right)$$

For long time stability (i.e. $n \to \infty$ for fixed $\Delta t, \Delta x$), we require $|\hat{g}_m| \le 1$. Let's check this:

$$|\hat{g}_m|^2 = \left| 1 - \frac{u\Delta t}{\Delta x}\left(1 - e^{-ik_m\Delta x}\right)\right|^2$$

$$= \left| 1 - \frac{u\Delta t}{\Delta x}\left(1 - \cos k_m\Delta x + i \sin k_m \Delta x\right)\right|^2$$

$$= \left| 1 - \frac{u\Delta t}{\Delta x}\left(1 - \cos k_m \Delta x\right) - i \frac{u\Delta t}{\Delta x}\sin k_m \Delta x \right|^2$$

$$= \left[1 - \frac{u\Delta t}{\Delta x}\left(1 - \cos k_m\Delta x\right)\right]^2 + \left(\frac{u\Delta t}{\Delta x}\right)^2 \sin^2 k_m \Delta x$$

$$= 1 - 2\frac{u\Delta t}{\Delta x}\left(1-\cos k_m\Delta x\right) + \left(\frac{u\Delta t}{\Delta x}\right)^2\left(1 - \cos k_m \Delta x\right)^2 + \left(\frac{u\Delta t}{\Delta x}\right)^2 \sin^2 k$$

$$= 1 - 2\frac{u\Delta t}{\Delta x}\left(1-\cos k_m\Delta x\right) + \left(\frac{u\Delta t}{\Delta x}\right)^2\left(2 - 2\cos k_m \Delta x\right)$$

$$= 1 + \underbrace{2\frac{u\Delta t}{\Delta x}}_{\geq 0}\underbrace{\left(1 - \cos k_m \Delta x\right)}_{0 \to 2}\underbrace{\left(\frac{u\Delta t}{\Delta x} - 1\right)}_{\text{this must be negative to assure } |\hat{g}_m| \leq 1}$$

$$\Longrightarrow \boxed{\frac{u\Delta t}{\Delta x} \leq 1}$$

Note: * we already saw this from the CFL condition requirements

* $\frac{u\Delta t}{\Delta x}$ is a special, very important non-dimensional number (as we have seen).

It is generally referred to as the CFL number:

$$\boxed{\text{CFL number} \equiv \frac{|u|\Delta t}{\Delta x}}$$

Physically, $|u|\Delta t$ represents the distance travelled by a wave in a single timestep.

Thus,

$$\boxed{\text{CFL \#} = \frac{\text{distance travelled by a wave}}{\text{cell length}}}$$

* $k_m \Delta x$ is also a very important non-dimensional number. Let's expand it:

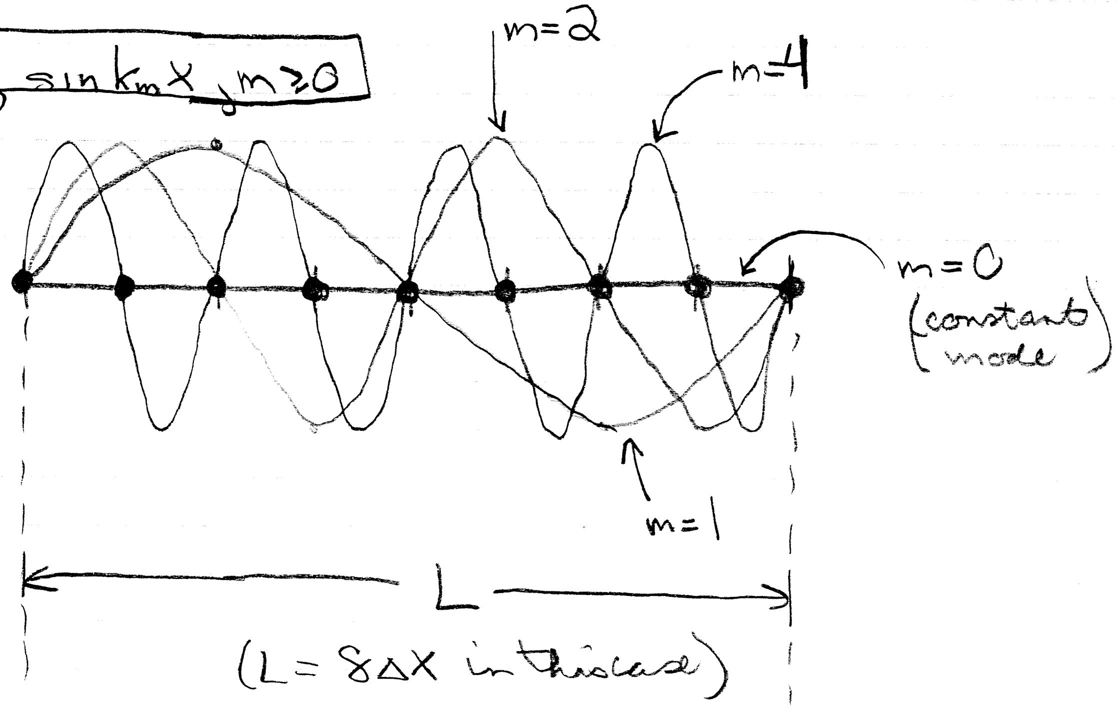
$$k_m \Delta x = 2\pi m \frac{\Delta x}{L}$$

$$k_m \Delta x = 2\pi \frac{\Delta x}{\underbrace{L/m}_{\text{This is the wavelength of the modes}}}$$

Recall, for the previous Fourier analysis, m could be any integer $-\infty \to 0 \to +\infty$. For the discrete problem, that's not quite true.

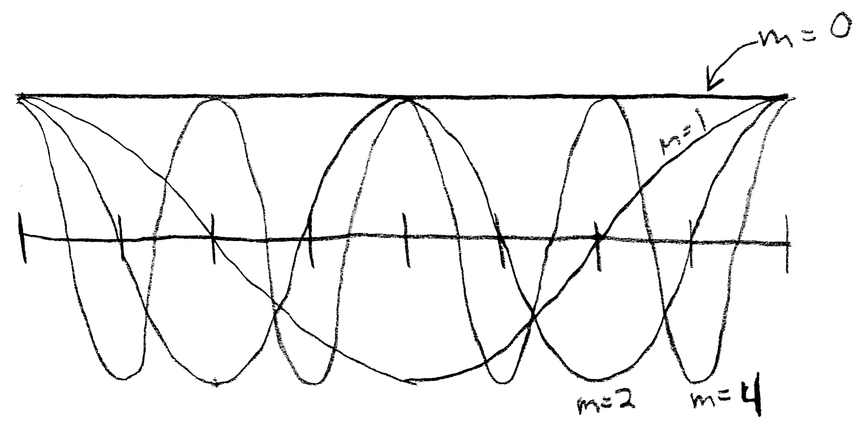
In particular, there is a limit on the high frequency (i.e. short wave length) size due to the grid:

$\boxed{\text{Plots of } \sin k_m x,\ m \geq 0}$



Aliasing: the $m=4$ mode is aliased with the $m=0$ mode!

$\boxed{\text{Plots of } \cos k_m x,\ m \geq 0}$



Aliasing does not occur until $m \geq 5$ for cosine modes

In summary, the smallest wavelength mode which is not aliased will have wavelength $2\Delta x$!

$$\max\left(k_m \Delta x\right) = 2\pi \frac{\Delta x}{(L/m)_{min}}$$

$$= 2\pi \frac{\Delta x}{2\Delta x} = \pi$$

Allowing for positive and negative k_m's this gives:

$$\boxed{-\pi \leq k_m \Delta x \leq \pi}$$

Since $k_m \Delta x$ appears so frequently in our analyses, we assign it a symbol:

$$\boxed{\beta_x \equiv k_m \Delta x \quad \text{and} \quad |\beta_x| \leq \pi}$$

For the first-order upwind scheme,

$$\boxed{\hat{g}(\beta_x) = 1 - \frac{u\Delta t}{\Delta x}\left(1 - e^{-i\beta_x}\right)}$$