16.901: Finite-difference PDE Project
The Impact of Film Cooling on Combustor Liner Temperatures
Due Date: March 18, 2pm

1 Background

The temperatures within the primary zone of a combustor are significantly higher than the temperatures which most materials can withstand without significant deterioration. Thus, a critical aspect of the design of a combustor is the development of a method to cool the liner walls of a combustor such that the temperatures which the liner temperatures are well below the limit of the material. A typical method to cool a combustor liner is through film cooling. Film cooling consists of diverting air from the main flow path prior to combustion and then re-introducing this air along the liner surface to provide a film of cooler air to protect the liner.

In this project, you will simulate the air flow and the liner wall to estimate the effectiveness of a film cooling strategy. The specific model problem is shown in Figure 1 and the values of the specific parameters are given in Table 1. We will assume that the flow enters the computational domain of interest at $x = 0$ with the following conditions:

For $x = 0$, $y > h$: $U = U_{hot}, T = T_{hot}$.
For $x = 0$, $0 < y \leq h$: $U = U_{cool}, T = T_{cool}$.
For $x = 0$, $-t_w \leq y \leq 0$: $U = 0, T = T_{cool}$. (Note: no convection in liner wall!)
For $x = 0$, $y < -t_w$: $U = U_{cool}, T = T_{cool}$.

The velocity throughout the domain is only in the x-direction and is given by the inlet value, i.e.,

$$U(x, y) = U(0, y), \qquad V(x, y) = 0.$$

The governing equation for our model problem will be the convection-diffusion equation and we will include a variable thermal conductivity to handle the change in conductivity between the air flow and the liner wall. Thus, the governing equation throughout the entire domain of interest is,

$$U\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right).$$

At the outlet of the computational domain, we will use a 'parabolized' version of this equation in which we will assume the second derivative in x is small compared to the second derivative in y,

$$\text{At } x = L, \qquad U\frac{\partial T}{\partial x} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right).$$

2 Tasks

2.1 Finite Difference Approximations for Non-uniform Grid Spacing

To efficiently study the film cooling problem, a grid with small y-spacing is desired in the region around the liner but large y-spacing away from the liner. Thus, the y-spacing will be a function of the j-index,

$$\Delta y_{j+\frac{1}{2}} \equiv y_{j+1} - y_j.$$

In the x-direction, the spacing will be kept constant and equal to Δx. Using a Taylor series analysis, find the values of a and b such that,

$$\frac{a\left(k_{j+\frac{1}{2}}\frac{T_{j+1}-T_j}{\Delta y_{j+\frac{1}{2}}}\right) - b\left(k_{j-\frac{1}{2}}\frac{T_j-T_{j-1}}{\Delta y_{j-\frac{1}{2}}}\right)}{\frac{1}{2}\left(\Delta y_{j+\frac{1}{2}} + \Delta y_{j-\frac{1}{2}}\right)}$$

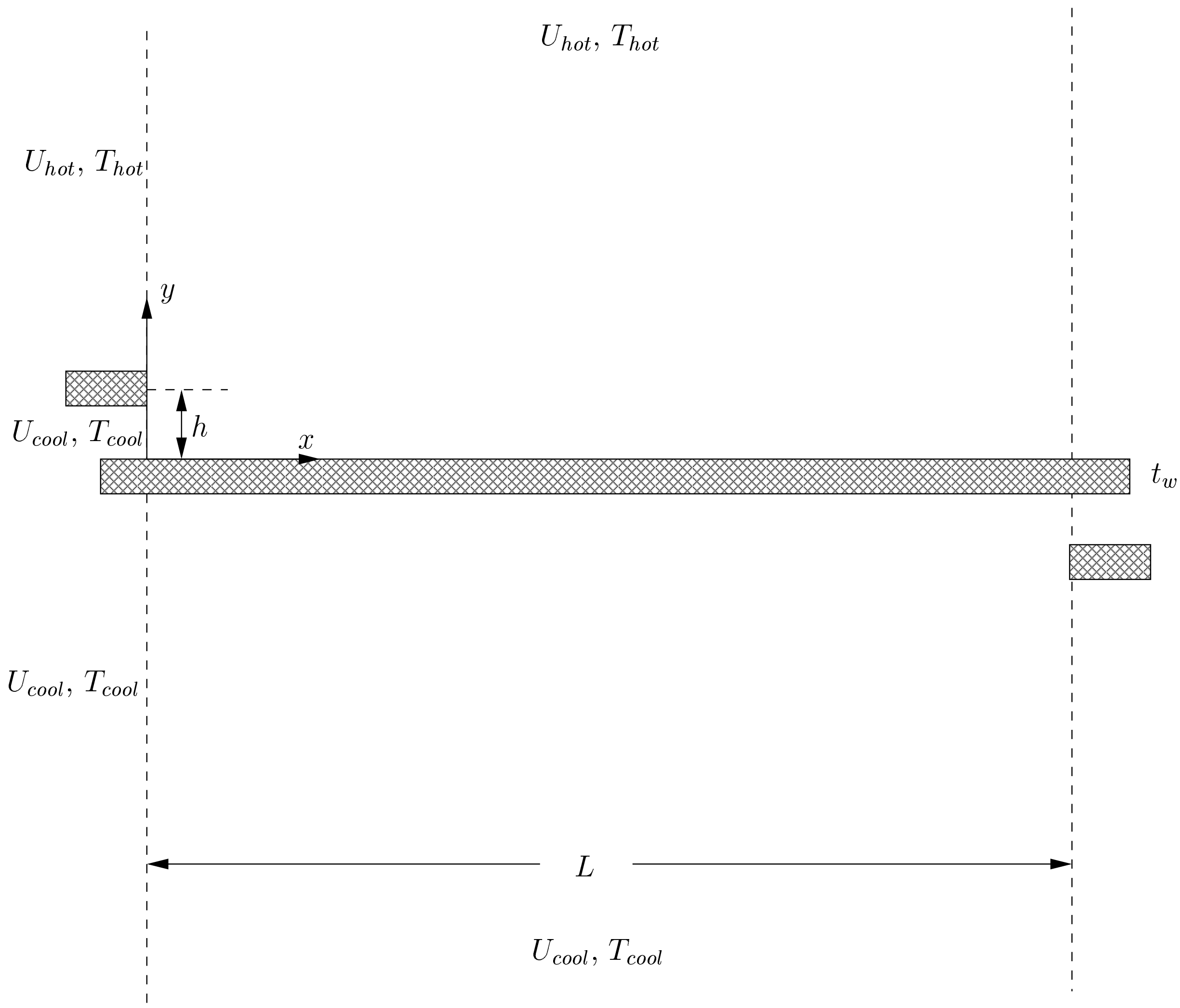

Figure 1: Combustor liner with film cooling

Parameter	Definition	Value
k_g	air conductivity	0.1 $W/(m\,K)$
k_w	wall conductivity	26.0 $W/(m\,K)$
h	height of cooling passage	0.003 m
L	axial length between cooling passages	0.3 m
U_{hot}	velocity of hot flow	100
U_{cool}	velocity of cool flow	150
T_{hot}	temperature of hot flow	2200 K
T_{cool}	temperature of cool flow	800 K
t_w	thickness of liner wall	0.0015 m

Table 1: Parameter definitions and values

is a second-order accurate approximation to $\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)$ at the j-th location. Note, the notation $k_{j+\frac{1}{2}}$ is defined as,

$$k_{j+\frac{1}{2}} = k\left(\frac{1}{2}\left(y_j + y_{j+1}\right)\right).$$

2.2 Grid Generation

The first step in the implementing the finite difference method for this problem is to develop a grid generator for the problem. To simplify the implementation of the finite difference method, we will use constant spacing in x as mentioned above (i.e. $\Delta x =$ constant). You need to develop your own approach for the y-spacing, but here are the requirements you should follow:

- To simplify the implementation, we will require that the surfaces of the liner wall at $y = -t_w$ and $y = 0$ are both grid lines.
- Use constant y-spacing within the liner wall.
- In the flow, the y-spacing will need to be smaller near both walls and around $y = h$ than away from the walls to correctly simulate the thermal layers which will be evolving from these locations. As $|y|$ increases, the y-spacing should increase since nothing is happening in these regions.
- Your method to generate points in y should be somewhat flexible so that you can easily add more points, increase stretching, etc. This will be useful when you study the accuracy of the method.

Describe your grid generation algorithm and provide a few plots to show samples of the grids which are generated.

2.3 Implementation of Finite Difference Method for Analysis of Liner

Using a second-order accurate upwind difference for $\frac{\partial T}{\partial x}$, a standard second-order accurate central difference for $\frac{\partial^2 T}{\partial x^2}$, and the approximation derived in Section 2.1, implement a finite difference algorithm for the problem. Solve the linear system of equations directly using Matlab's capability to solve matrix equations (as shown in the sample convection-diffusion equation solver available on-line). Here are some specific suggestions/comments:

- Within the liner wall, we are dealing with a solid material for which no convection occurs, so the governing equation reduces to,
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = 0.$$
- On the grid lines which are on the surface of the material, we will also use the liner equation, but in this case it is important to remember that $k_{j\pm\frac{1}{2}}$ will be equal to k_w or k_g depending on whether the $j \pm \frac{1}{2}$ location is in the liner or in the flow, respectively. For the values of k_j needed for the x-derivative approximation, use $k_j = k_w$ at these surface points.
- You have some freedom in setting up how far out to place the upper and lower boundaries for large $|y|$. The only constraint is that the boundaries should be placed far enough away from the wall so that the wall temperature distributions are not affected.

Implement the method and describe your algorithm. Include the source code with the report. Note: you don't have to provide an exhaustive description of the discretization. But, the description should provide enough information for an expert in the area could implement your discretization and boundary conditions without requiring the source code.

2.4 Accuracy Study of Temperature Predictions

In order for your model to be useful in the design of a liner film cooling approach, the temperature predictions for the liner wall must be accurate. At the same time, to allow rapid design studies to be conducted, the run-time of the model should be as small as possible. Using the grid generation strategy you have develop, vary the size (both in x and y)and the y-stretching of the grid to develop a baseline grid which provide wall temperatures that are accurate to within 10 K for the current set of parameter values. Here are some suggestions when performing this grid study:

- You might start with a grid and then study the accuracy as you continually double the number of points (a) in x only, (b) in y only, and (c) in both x and y simultaneously. From this, you can to establish an 'exact' answer for the temperature distribution in the wall as well as about how much resolution is needed in x and y. When you do this study, you should try to hold fixed the manner in which you do the stretching (i.e. you can use stretching, but do not change the manner in which you are doing it so that you are only studying grid size and not stretching effects).
- Once establishing a reasonably accurate grid in the first suggestion, you should also assess how far away you can place the farfield boundaries.
- Having done the previous suggestions, you could make some adjustments to how your stretching is done and see how much you can decrease the total number of points in the farfield regions without sacrificing accuracy.

In the report, provide documentation on your accuracy study to support the baseline grid which you have developed.

2.5 Von Neumann Analysis

Suppose that the model were to be used for an unsteady problem, so that the governing equation became,

$$\frac{\partial T}{\partial t} + U\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right).$$

You wish to determine if a Forward Euler method can be used to integrate the equation in time given the spatial discretization you used above. Use a Von Neumann analysis to assess the stability of the method and determine which region in the problem (i.e. in the wall, in the flow but near the wall, or in the flow but away from the surface) would most constrain the time-step. Use the grid sizes for your baseline grid and the parameter values studied above for the final result. Note: only substitute specific values for the grid and parameter values after the analytic expression for the von Neumann analysis has been derived.