16.901 Project #1
Due Date: March 3, 2pm

1 Background

Modeling the dynamics of an airplane can be very complex. Twelve state variables are required to describe the motion: three states for position, three for velocity, three for angular orientation, and three angular rates. However, this system of twelve coupled differential equations can be simplified greatly by linearization and some assumptions. These simplifications yield two decoupled four-state systems, one describing longitudinal motion and the other describing lateral motion.

The longitudinal modes can be simplified even more to give fairly accurate decoupled models of the familiar pheugoid and short-period modes. The lateral system cannot be simplified as easily to solve for the three lateral modes: Dutch roll, spiral mode, and roll mode. The Dutch roll mode is a combination of yawing and rolling oscillations. In the roll mode, the roll rate reaches steady state quickly. The spiral mode can be slightly stable or unstable. An unstable spiral mode can lead to divergence from the flight path or result in a spiral dive. These lateral modes will be the topic of the first programming project and will be the basis for most of the questions in this homework assignment.

1.1 Definition of coordinate systems and variables

The motion of the aircraft is measured relative to a fixed frame; however, the properties of the aircraft are often known in a coordinate system relative to the aircraft body. Thus, the states to be determined are the the relative position, motion, angular orientation, and angular rate of the body frame. Figure 1 shows the forces (X, Y, Z), moments (L, M, N), angular rates of rotation (p, q, r) with respect to the body axes (x_b, y_b, z_b).

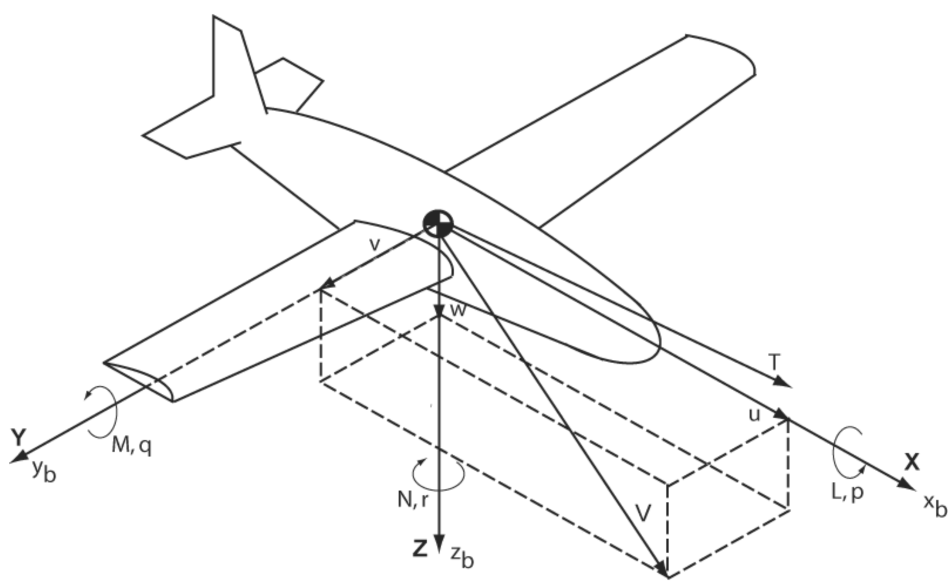

Figure 1: Definition of body coordinate system attached to the aircraft

The Table 1 defines the relevant variables. The linearized equations of motion are written in terms of stability derivatives, the first derivatives of the forces and moments with respect to the states. These are generally written in subscript notation. For example, $Y_\beta \equiv \frac{\partial Y}{\partial \beta}$. Often stability derivatives are given in a

	Symbol	Description
States	$\Delta\beta$	Side-slip angle perturbation
	Δp	Roll rate perturbation
	Δr	Yaw rate perturbation
	$\Delta\phi$	Roll angle perturbation
Forces and Moments	Y	Side force (Force in y direction)
	L	Rolling moment (Moment about x axis)
	N	Yawing moment (Moment about z axis)
Other quantities	Q	Dynamic pressure
	S	Planform area
	b	Wingspan
	W	Weight of aircraft
	m	Mass of aircraft
	I_x, I_z, I_{xz}	Momentw of inertia
	g	Gravitational acceleration
	M_0	Mach number
	h_0	altitude
	u_0	Initial velocity in x direction

Table 1: Explanation of Variables

$$\begin{array}{ccccccccc}
Y_\beta & = & QSC_{y_\beta}, & L_\beta & = & QSbC_{l_\beta}, & N_\beta & = & QSbC_{n_\beta} \\
Y_p & = & \frac{QSb}{2u_0}C_{y_p}, & L_p & = & \frac{QSb^2}{2u_0}C_{l_p}, & N_p & = & \frac{QSb^2}{2u_0}C_{n_p} \\
Y_r & = & \frac{QSb}{2u_0}C_{y_r}, & L_r & = & \frac{QSb^2}{2u_0}C_{l_r}, & N_r & = & \frac{QSb^2}{2u_0}C_{n_r}
\end{array}$$

Table 2: Stability derivative non-dimensional definitions

non-dimensional form, such as C_{y_β} as the non-dimensionalized form of Y_β. Specifically, the definitions in Table 2 are conventional and will be used in this homework and project. Values of the stability derivatives and dimensions for the airplanes we will study in this homework and project are given in Table 3.

1.2 Equations of motion

The linearized equations of lateral motion are given in Equation (1)-(4). Equation (1) is the conservation of y-momentum. Equation (2) is the conservation of x-angular momentum. Equation (3) is the conservation of z-angular momentum. Finally, Equation (4) is the relation between the roll angle and the roll rate. Specifically, the governing equations take the following form:

$$\left(mu_0\frac{d}{dt} - Y_\beta\right)\Delta\beta - Y_p\Delta p + (mu_0 - Y_r)\,\Delta r - mg\Delta\phi = 0 \tag{1}$$

$$-L_\beta\Delta\beta + \left(I_x\frac{d}{dt} - L_p\right)\Delta p - \left(I_{xz}\frac{d}{dt} + L_r\right)\Delta r = 0 \tag{2}$$

$$-N_\beta\Delta\beta - \left(I_{xz}\frac{d}{dt} + N_p\right)\Delta p + \left(I_z\frac{d}{dt} - N_r\right)\Delta r = 0 \tag{3}$$

$$\frac{d}{dt}\Delta\phi = \Delta p \tag{4}$$

Quantity	Values for 747	Values for F-104
C_{y_β}	-0.96	-1.17
C_{y_p}	0	0
C_{y_r}	0	0
C_{l_β}	-0.221	-0.175
C_{l_p}	-0.45	-0.285
C_{l_r}	0.101	0.265
C_{n_β}	0.150	0.50
C_{n_p}	-0.121	-0.14
C_{n_r}	-0.30	-0.75
S	5500 ft^2	196.1 ft^2
b	195.68 ft	21.94 ft
h	Sea level	Sea level
M_0	0.25	0.257
W	636600 lbs	16300 lbs
I_x	18.2×10^6 $slug\, ft^2$	3549 $slug\, ft^2$
I_z	49.7×10^6 $slug\, ft^2$	59669 $slug\, ft^2$
I_{xz}	0.97×10^6 $slug\, ft^2$	0

Table 3: Stability derivatives and dimensions for 747 and F-104 at sea level

2 Assignment

- Implement the following six integration methods:
  - 1st order Adams-Bashforth (AB1). More commonly known as Forward Euler.
  - 2nd order Adams-Bashforth (AB2).
  - 1st order Adams-Moulton (AM1). More commonly known as Backward Euler.
  - 2nd order Adams-Moulton (AM2). More commonly known as Trapezoidal Method.
  - 2-stage Runge-Kutta (RK2).
  - 4-stage Runge-Kutta (RK4).

  Turn in hard copies of your algorithms with your project write-up.

  We will study the accuracy and efficiency of these 6 methods applied to the following problem: Find $u(t)$ from $t = 0s$ to $t = 30s$ for the following initial condition:

$$\Delta\beta\,(0) = 0.1\,rad, \Delta p\,(0) = 0, \Delta r\,(0) = 0, \Delta\phi\,(0) = 0$$

  NOTE: the final time has been shortened compared to Homework #2.

- For this assignment, the measure of accuracy will be the maximum error in the sideslip angle perturbation (at any iteration), i.e.,

$$E \equiv \max_n \left| \Delta\hat{\beta}^n - \Delta\beta(n\Delta t) \right|$$

  where $\Delta\hat{\beta}^n$ is the value of the sideslip angle perturbation calculated by the integration method at the n-th iteration. For both airplanes and using all integration methods, determine the accuracy for timesteps of $\Delta t = 0.001$, 0.01, 0.1, and 1.0 seconds. Specifically, fill in the results in a table similar to Table 4. For both aircraft, please discuss which methods exhibit their predicted global order of accuracy as the timestep decreases? Explain your answer.

Δt	AB1	AB2	AM1	AM2	RK2	RK4
1.0						
0.1						
0.01						
0.001						

Table 4: Accuracy convergence study for Boeing 747

	AB1	AB2	AM1	AM2	RK2	RK4
WU	1.0					

Table 5: Work Units for a single iteration of each integration method.

- In order to compare the work involved in these methods, we need a single 'currency' which is relatively independent of the particular computer, network, version of software compiler, etc. We will use the computational time required to run a single iteration of the Forward Euler method (AB1) as our Work Unit (WU). To find the WU's for an iteration of the other methods, run each of them for a few hundred iterations and calculate the average time required for a single iteration. MAKE SURE TO DO THIS ON THE SAME COMPUTER, IMMEDIATELY AFTER EACH OTHER to reduce the possibility of varying conditions affecting the timings. In Matlab®, the command to use to find the current time is called **cputime** (look up this command in the Matlab® help to see its usage). Then, normalize these timings by the Forward Euler timings to calculate the WU for an iteration of each method. Report the WU for an iteration of each method in a table such as Table 5. Note: refer to the discussion in the sample solution to Homework #2 to see what the expected behavior should be. If you get something very different from the discussion in that sample solution, you probably have a bug or have not implemented your method efficiently.
- Based on the results in the tables for the accuracy study and the known computational work costs of each algorithm, estimate the amount of work required (i.e. in terms of WU) for each method to achieve an accuracy of 0.0001 radians for β for each airplane (you might use a table like Table 6. Try to explain the WU results based on your understanding of the methods and their accuracy and stability. Which method would you recommend for efficiently solving these aircraft lateral dynamics equations?

Aircraft	AB1	AB2	AM1	AM2	RK2	RK4
747						
F-104						

Table 6: Work Unit requirements to achieve an accuracy of 0.0001 rads for the sideslip angle.