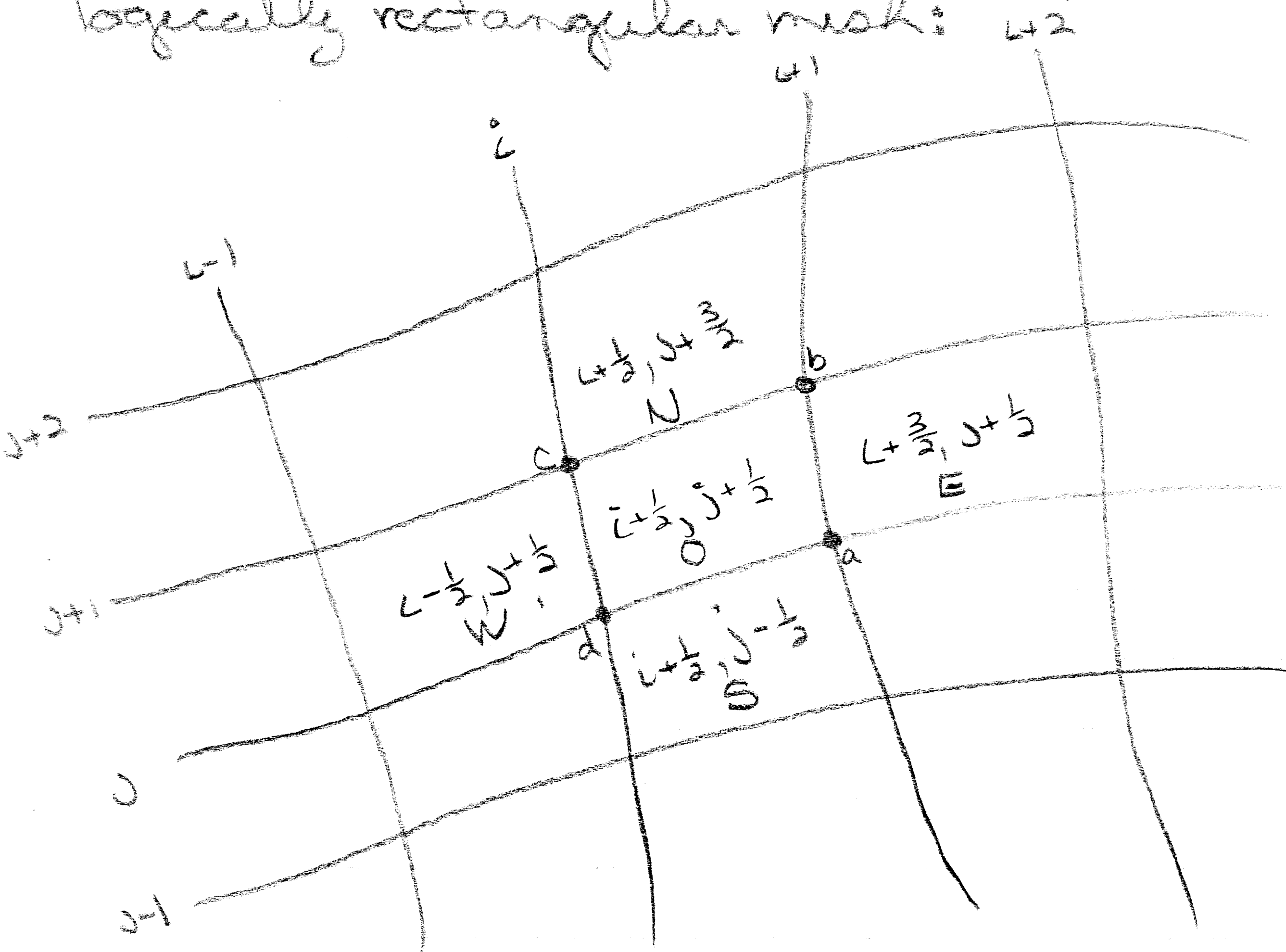
16.901 Notes for March 15, 2002

* Finite Volume schemes in multiple dimensions
* Unstructured vs. structured grids

Let's return to the energy equation without diffusion in multiple dimensions:

$$\iiint_V \frac{\partial E}{\partial t}\, dV + \oiint_S E\vec{u}\cdot\vec{n}\, dS = 0$$

Suppose we have a non-uniform though logically rectangular mesh:



As in 1-D, define a cell-average state:

$$E_{i+\frac{1}{2},j+\frac{1}{2}} \equiv \frac{1}{A_{i+\frac{1}{2},j+\frac{1}{2}}} \iint_{A_{i+\frac{1}{2},j+\frac{1}{2}}} E \, dA$$

where $A_{i+\frac{1}{2},j+\frac{1}{2}}$ = cell area of $i+\frac{1}{2},j+\frac{1}{2}$

The conservation law applied to $A_{i+\frac{1}{2},j+\frac{1}{2}}$ is:

$$A_{i+\frac{1}{2},j+\frac{1}{2}} \frac{dE_{i+\frac{1}{2},j+\frac{1}{2}}}{dt} + \int_a^b E\vec{u}\cdot\vec{n}\,ds + \int_b^c E\vec{u}\cdot\vec{n}\,ds + \int_c^d E\vec{u}\cdot\vec{n}\,ds + \int_d^a E\vec{u}\cdot\vec{n}\,ds = 0$$

The next issue is to evaluate the line integrals of the fluxes on each face. Some options:

ab face length

$$\text{Average:} \quad \int_a^b E\vec{u}\cdot\vec{n}\,ds \cong \frac{1}{2}\left[\left(E\vec{u}\right)_E + \left(E\vec{u}\right)_D\right]\cdot\vec{n}_{ab}\,\Delta S_{ab}$$

ab face unit normal

This produce a central difference approximat

$$A_{i+\frac{1}{2},j+\frac{1}{2}}\frac{dE_{i+\frac{1}{2},j+\frac{1}{2}}}{dt} + \frac{1}{2}(E\vec{v})_O \cdot \overbrace{\left[\vec{n}_{ab}\Delta S_{ab} + \vec{n}_{bc}\Delta S_{bc} + \vec{n}_{cd}\Delta S_{cd} + \vec{n}_{d}\right.}^{=0 \text{ for closed cell}}$$

$$+ \frac{1}{2}(E\vec{v})_E \cdot \vec{n}_{ab}\Delta S_{ab} + \frac{1}{2}(E\vec{v})_N \cdot \vec{n}_{bc}\Delta S_{bc}$$

$$+ \frac{1}{2}(E\vec{v})_W \cdot \vec{n}_{cd}\Delta S_{cd} + \frac{1}{2}(E\vec{v})_S \cdot \vec{n}_{da}\Delta S_{da} = 0$$

$\Rightarrow$ No dependence on $E_O = E_{i+\frac{1}{2},j+\frac{1}{2}}$ throug flux term! $\Rightarrow$ Central difference!

We can also implement an upwind scheme as follows:

For edge ab: if $\vec{v}_{AB} \cdot \vec{n}_{AB} > 0$ then

$$\int_a^b E\vec{v}\cdot\vec{n}\,ds \hat{=} E_O \vec{v}_{AB}\cdot\vec{n}_{AB}\,\Delta S_{AB}$$

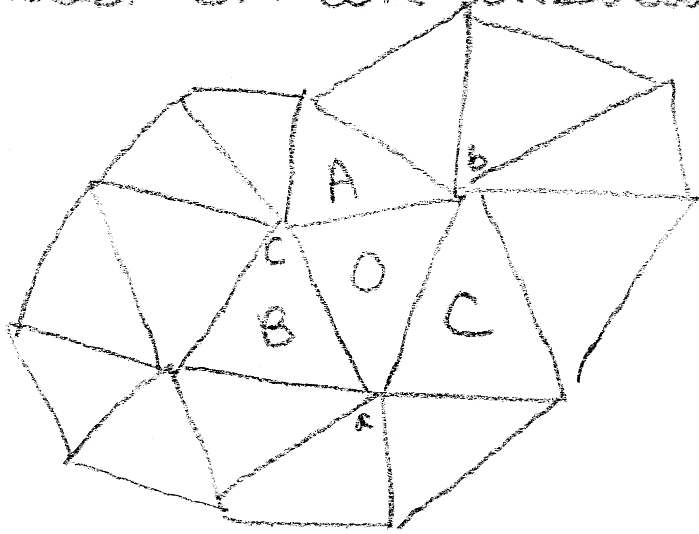
if $\vec{v}_{AB} \cdot \vec{n}_{AB} < 0$ then

$$\int_a^b E\vec{v}\cdot\vec{n}\,ds \hat{=} E_E \vec{v}_{AB}\cdot\vec{n}_{AB}\,\Delta S_{AB}$$

Or, more compactly:

$$\int_a^b E\vec{v}\cdot\vec{n}\,ds = \left\{\frac{1}{2}(E_E + E_O)\vec{v}_{AB}\cdot\vec{n}_{AB} - \frac{1}{2}(E_E - E_O)\left|\vec{v}_{AB}\cdot\vec{n}_{AB}\right|\right\}\Delta S_{A}$$

This finite volume scheme can also be implemented on an "unstructured" mesh:



In this case, the finite volume method for cell O is:

$$A_0 \frac{dE_0}{dt} + \int_a^b E\vec{u}\cdot\vec{n}\,ds + \int_b^c E\vec{u}\cdot\vec{n}\,ds + \int_c^a E\vec{u}\cdot\vec{n}\,ds = 0$$

As before, we need to evaluate the flux integrals & discretize in time.

Why would unstructured meshes be used

* better suited for complex geometry
* can adapt to local features

Why would structured meshes be used?

* require less computer memory than unstructured
  (unstructured mesh must store connectivity info)

* structured equations can general be solved with techniques that take advantage of structure
  $\Rightarrow$ faster solutions