Some Introductory Thoughts

Suppose you had a simple problem, such as,

$$\frac{du}{dt} = \lambda u \quad \text{with } u(0) = u_0 \qquad (1)$$

And, for your application, you were interested in the behavior of u for $t=0$ until $t=T$. Now, for the problem, the solution is well-known:

$$u(t) = u_0 e^{\lambda t} \qquad (2)$$

But, for most problems of interest, the exact solution is unknown. It is for these problems that numerical methods are critical. We will be studying these tougher problems throughout this class, but often we can learn a lot about the behavior of numerical methods from simple model problems such as (1).

So, suppose we developed a simple numerical method to solve (1):

$$\frac{v^{n+1} - v^n}{\Delta t} = \lambda v^n \quad \text{where } v^n = v(n\Delta t) \quad (3)$$
$$v^0 = u_0$$

This approach is known as a forward Euler method. It is based on approximating

$$\left.\frac{dv}{dt}\right|_{t^n} \approx \frac{v(t^n+\Delta t) - v(t^n)}{\Delta t}$$

To understand how accurate and efficient our Forward Euler method is, we can calculate the error behavior. Specifically, we want

$$v^n \approx u(t^n) \equiv u^n \text{ from } 0 \le t \le T$$

So, a good measure of our error might be:

$$E(T) = \int_0^T (v-u)^2 dt$$

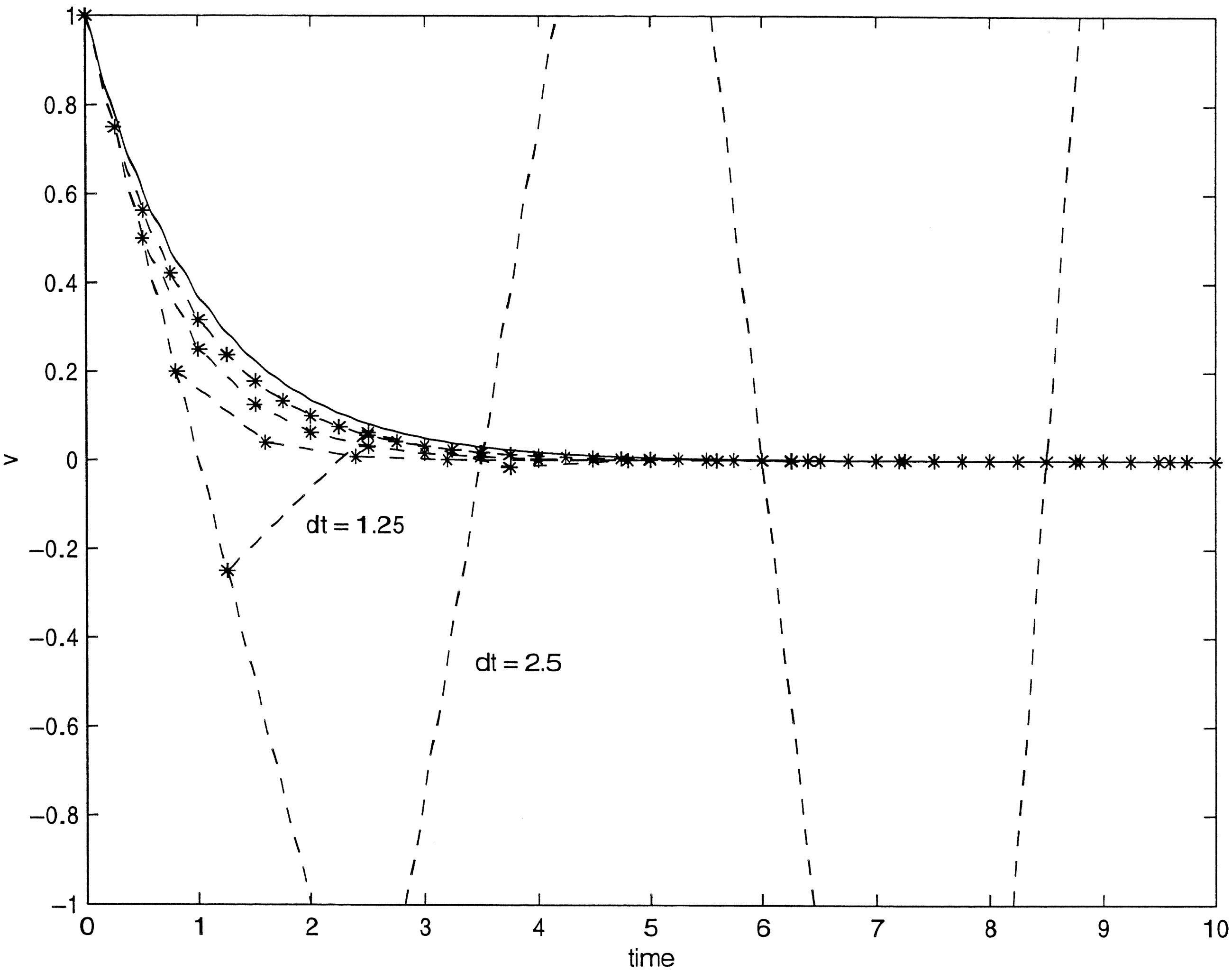
Discretely $\Rightarrow E(T) \equiv \sqrt{\Delta t \sum_{n=0}^{N} (v^n - u^n)^2}$ where $T = N\Delta t$

For general problem, we can't calculate u (the exact solution) and therefore we can't know for sure the behavior of $E(T)$. But, for this problem we can. So, let's first try a specific example and plot the results.

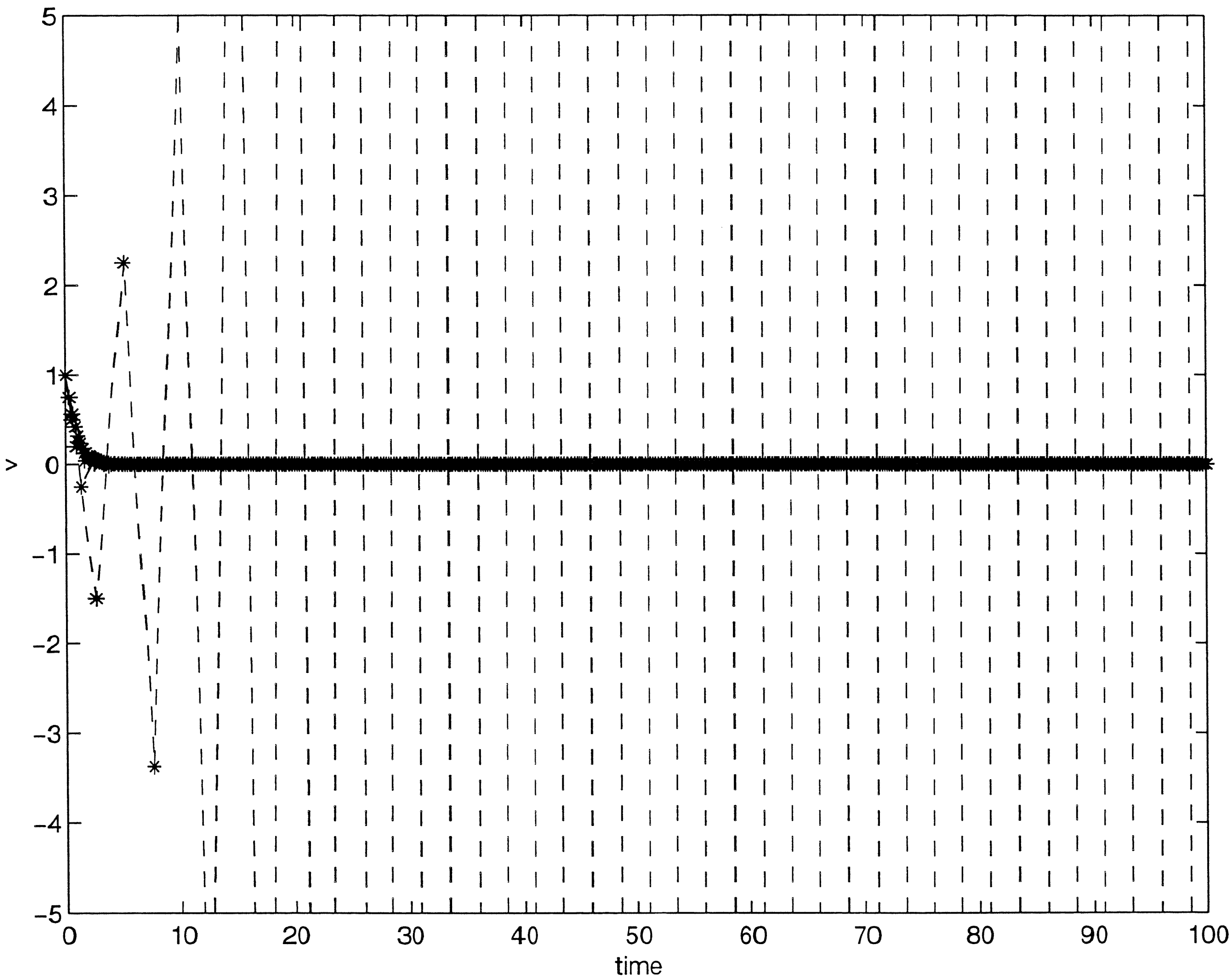
<u>Example:</u> $\lambda = -1$ $T = 1000$ $u_0 = 1$

$$\Rightarrow \boxed{u(t) = e^{-t}}$$

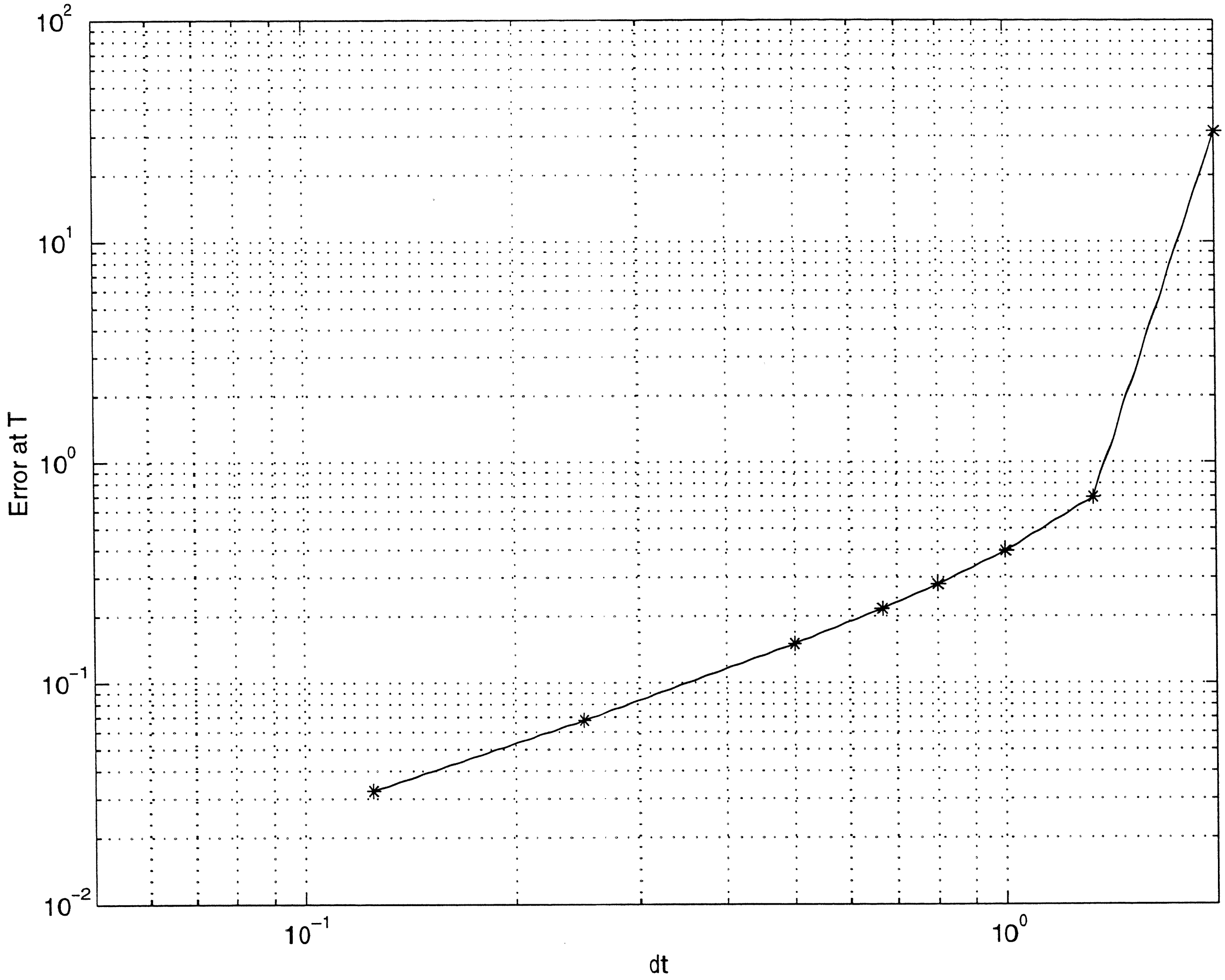
The Forward Euler method gives:



1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
v
0
1
2
3
4
5
6
7
8
9
10
time
dt = 1.25
dt = 2.5



5
4
3
2
1
0
-1
-2
-3
-4
-5
v
0
10
20
30
40
50
60
70
80
90
100
time



Error at T
dt
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-1}
10^0

Let's try to look at the Forward Euler method analytically:

$$\frac{v^{n+1}-v^n}{\Delta t} = \lambda v^n \Rightarrow v^{n+1} = (1+\lambda\Delta t)v^n$$

$$\Rightarrow \boxed{v^n = v^0(1+\lambda\Delta t)^n}$$

So, for this example:

$$\boxed{v^n = (1-\Delta t)^n} \qquad (4)$$

Then, for the error we have:

$$E(T) = \sqrt{\Delta t \sum_{n=0}^{N}\left[(1-\Delta t)^n - e^{-n\Delta t}\right]^2}$$

Simplifying a little more, we note that:

$$T = 1000, \quad \underbrace{N = \frac{T}{\Delta t} = \frac{1000}{\Delta t}}_{\text{we assume } \Delta t \text{ is chosen to give integer values of } N}$$

$$\Rightarrow \boxed{E(T) = E(1000) = \sqrt{\Delta t \sum_{n=0}^{1000/\Delta t}\left[(1-\Delta t)^n - e^{-n\Delta t}\right]^2}} \qquad (5)$$

Comments

* Notice that v^n actually will grow in time (with increasing n) if Δt is large enough! Specifically, if

$$\Delta t > 2 \text{ then } |v^{n+1}| > |v^n| !$$

An interesting aspect of this is that if we let $\Delta t = 2$, this would give $N = 500$ timesteps to reach $t = T = 1000$, which would seem reasonable. But, in fact, what really matters is $\lambda \Delta t$! If $\lambda < 0$, then in order for $|v^{n+1}| < |v^n|$, we require $|\lambda \Delta t| < 2$

This type of behavior in which $|v^n|$ grows with n is known as the stability of the numerical method.

* Closely related to the growth of $|v^n|$ if $\Delta t > 2$, we notice that v^n will be oscillatory if $\Delta t > 1$! From (4):

$$v^{n+1} = (1-\Delta t)^n = (1-\Delta t)v^n$$

So, if $\Delta t > 1$ then v^{n+1} & v^n will be of opposite sign (i.e. $v^{n+1} * v^n < 0$).