16.901: ODE Integration Project
The Use of Swirl for Cleaning Nuclear Rocket Plumes
Due Date: February 25, 2pm

1 Background

Nuclear thermal rockets offer the combination of high thrust and high specific impulse making them ideal candidates for the transportation of large payloads in short times. The basic concept behind nuclear thermal propulsion is to use a critical mass of Uranium to heat hydrogen gas which is then accelerated out a nozzle. Unfortunately, a severe limitation is placed on nuclear thermal propulsion due to the potential for radioactive material to be dispersed. Thus, a critical need exists to develop efficient methods for extracting radioactive material from the plume prior to exhausting into the atmosphere.

In this project, we will consider the use of swirl in the rocket plume as a means to separate heavy Uranium particles from the hydrogen gas. The basic idea is that the heavy particles will be accelerated circumferentially due to the swirling gas flow. However, due to their large density, these particles cannot maintain their radial location and move to the outside of the plume where they can be efficiently skimmed from the main hydrogen gas path. In this project, numerical methods will be developed to trace the particle trajectories and determine the potential for particle separation using swirling flow.

NOTE: This project is based on the Master's Thesis of David Oh, M.I.T., 1993.

1.1 Rocket Configuration and Flow

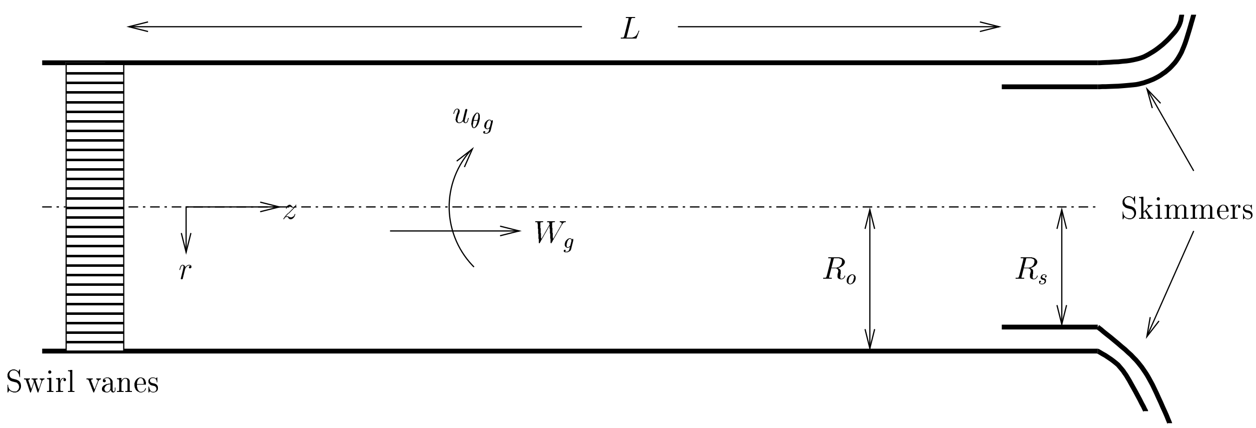

Figure 1: Rocket configuration and flow

We will study a simplified rocket configuration with a constant radius R_o and axial length L as shown in Figure 1. The flow through the chamber is assumed to have a constant axial velocity W_g. The swirl vanes have been designed to create a solid body rotating flow such that the circumferential velocity of the gas is given by,

$$u_{\theta g} = \Omega r.$$

At the exit of the rocket chamber, skimmers have been located at a radial location R_s. The parameters for this problem are defined in Table 1.

1.2 Particle Trajectory Equations

The location of a particle is denoted as $\vec{x}_p$. The particle motion is governed by,

$$m_p \ddot{\vec{x}}_p = \vec{F},$$

Parameter	Definition	Value
ρ_g	gas density	0.3 kg/m^3
ρ_p	particle density	12700 kg/m^3
a	particle radius	1 μm, 10 μm, 100 μm
R_o	radius of rocket	0.15 m
R_s	radius of skimmer	0.14 m
L	length of rocket	1 m
W_g	axial velocity of gas	1100 m/s
μ_g	gas viscosity	3×10^{-5} kg/(m sec)
Ω	rotation rate of gas	5000 rad/sec, 10000 rad/sec

Table 1: Parameter definitions and values

where m_p is the mass of the particle and $\vec{F}$ is force acting on the particle. The force is composed of a pressure and a drag force,

$$\begin{aligned}
\vec{F} &= \vec{F}_p + \vec{F}_d, \\
\vec{F}_p &= -\frac{4}{3}\pi a^3 \nabla p, \\
\vec{F}_d &= \frac{1}{2}\rho_g \pi a^2 |\vec{u}_{rel}|\vec{u}_{rel} C_D(Re), \\
\vec{u}_{rel} &= \vec{u}_g - \vec{u}_p, \\
CD(Re) &= \frac{24}{Re} + \frac{6}{1+\sqrt{Re}} + 0.4, \\
Re &= \frac{\rho_g |\vec{u}_{rel}| 2a}{\mu_g}.
\end{aligned}$$

We will assume that the gas is in radial equilibrium, thus the only pressure gradient is in the radial direction and given by,

$$\frac{dp}{dr} = \rho_g \frac{{u_\theta}_g^2}{r}.$$

Finally, if during the integration of the particle, its trajectory reaches the outer radius of the chamber, the particle can be assumed to remain near the wall (in its boundary layer) and slowly moves into the skimmer.

1.3 Particle Inlet Distribution and Cleaning Efficiency

For this work, we will assume that the particles are uniformly distributed at the inlet. Also, we will assume that the initial velocity of the particles is equal to the gas axial velocity, W_g. Thus, the particles enter the swirling flow region with only axial velocity. The cleaning efficiency, η, is defined as the fraction of particles which are cleaned,

$$\eta = \frac{n_p}{N_p}$$

where N_p is the total number of particles, and n_p is the number of particles removed by the skimmers. For the uniform distribution case with constant initial conditions on the particle velocity, this can be calculated from,

$$\eta = 1 - \left(\frac{R_\eta}{R_o}\right)^2,$$

where R_η is the initial radius of the inner-most particles which can be skimmed off the flow.

2 Tasks

2.1 Dimensional Analysis

Conduct a dimensional analysis of this problem. Specifically, for the cleaning efficiency, η, determine the number of independent non-dimensional parameters which govern the behavior of η. Then, determine the specific non-dimensional parameters which you will use in the remainder of the project.

Note: while the number of independent non-dimensional parameters is unique, the specific choice of which non-dimensional parameters to use is not. Thus, it is helpful to use some engineering judgement in finalizing on your choice of parameters. This may take some iteration.

2.2 Reduction to a First-Order System

Reduce the order of the system to first-order by introducing additional states to produce a set of coupled, nonlinear, first-order ODE's. Specifically, choose the states to be the Cartesian components of particle's velocity and position, i.e. u_p, v_p, w_p and x_p, y_p, and z_p. Thus, the final system will have the form,

$$\begin{bmatrix} \dot{u_p} \\ \dot{v_p} \\ \dot{w_p} \\ \dot{x_p} \\ \dot{y_p} \\ \dot{z_p} \end{bmatrix} = \begin{bmatrix} f_u(u_p, v_p, w_p, x_p, y_p, z_p, t) \\ f_v(u_p, v_p, w_p, x_p, y_p, z_p, t) \\ f_w(u_p, v_p, w_p, x_p, y_p, z_p, t) \\ f_x(u_p, v_p, w_p, x_p, y_p, z_p, t) \\ f_y(u_p, v_p, w_p, x_p, y_p, z_p, t) \\ f_z(u_p, v_p, w_p, x_p, y_p, z_p, t) \end{bmatrix}$$

Please leave the equations in symbolic form! Do not substitute the specific values for the input parameters given in Table 1.

2.3 Linearization and Eigenvalues

Linearize the governing equations about time $t = \tau$ to produce a set of linear equations,

$$\begin{bmatrix} \dot{\tilde{u}}_p \\ \dot{\tilde{v}}_p \\ \dot{\tilde{w}}_p \\ \dot{\tilde{x}}_p \\ \dot{\tilde{y}}_p \\ \dot{\tilde{z}}_p \end{bmatrix} = \mathbf{A} \begin{bmatrix} \tilde{u}_p \\ \tilde{v}_p \\ \tilde{w}_p \\ \tilde{x}_p \\ \tilde{y}_p \\ \tilde{z}_p \end{bmatrix} + \begin{bmatrix} \tilde{f}_u(t) \\ \tilde{f}_v(t) \\ \tilde{f}_w(t) \\ \tilde{f}_x(t) \\ \tilde{f}_y(t) \\ \tilde{f}_z(t) \end{bmatrix}$$

where $\mathbf{A}$ is a 6×6 matrix and $\tilde{u}_p(t) = u_p(t) - u_p(\tau)$ and similarly for the other perturbation states. Note: the forcing terms $\tilde{f}$ may depend on time and on the particle velocity and position at $t = \tau$, but they cannot depend on the perturbation states $\tilde{u}_p$, etc. To linearize the original equations, we will assume that all of the perturbation states are small. Again, leave the linearized equations in a symbolic form.

Assuming that the particle's axial velocity was initialized to W_g at $t = 0$, calculate the eigenvalues of this linearized set of equations in a symbolic form. Note: the linearization we have done to arrive at these equations probably will require that $t - \tau$ is small to remain valid. So, the eigenvalues which correspond to long time behavior are likely to be less meaningful. Fortunately, for numerical stability, it is the short time scales (i.e. the large magnitude eigenvalues) which are important.

Compare the timescales implied by these eigenvalues with each other and with the approximate time for the particle to traverse the length of the rocket chamber. Under what conditions will this problem be stiff? Using the specific values of the input parameters given in Table 1, is this problem stiff for the particular cases being studied in this project? If yes, describe the conditions under which this stiffness occurs.

2.4 Implementation of ODE Integration Methods

Implement the following numerical methods to integrate the nonlinear particle ODE's:

- First- and second-order Adams-Bashforth (note: First-order = Forward Euler)
- First- and second-order Adams-Moulton (note: First-order = Backward Euler, Second-order = Trapezoidal Rule)
- Second- and fourth-order Runge-Kutta (note: choose any of the popular variations for these two)

Use constant timesteps for all methods. In the write-up, briefly describe these methods. In particular, for the Adams-Moulton, which are implicit methods, discuss how you solved the nonlinear coupled equations at each timestep.

2.5 Study of Numerical Accuracy and Computational Work

In this section, you will study the accuracy and the computational work for each of the 6 methods described above.

1. In order to compare the work involved in these methods, we need a single 'currency' which is relatively independent of the particular computer, network, version of software or compiler, etc. We will use the computational time required to run a single iteration of the forward Euler method as our Work Unit (WU). To find the WU's for an iteration of the other methods, run each of them for a few hundred iterations and calculate the average time required for a single iteration. MAKE SURE TO DO THIS ON THE SAME COMPUTER, IMMEDIATELY AFTER EACH OTHER to reduce the possibility of varying conditions affecting the timings. Then, normalize these timings by the Forward Euler timings to calculate the WU for an iteration of each method. Report the WU for an iteration of each method in a table.

2. Using each of the methods, calculate the cleaning efficiency, η, to a precision of 0.01 over the entire range of conditions (note: make sure to do use the results of your dimensional anaylsis when possible). To do this, you will first need to calculate R_η (defined above) by running the particle integration for a series of initial radial positions with some fixed time step. From this, you can then calculate η. Then, decrease the timestep by some amount (say 50%), and re-run your procedure to find R_η and η. If the answer remains the same to within 0.01, then you have achieved your accuracy limit. Try to find the maximum timestep size for each method which would achieve your accuracy limit. Then, for each method and under each combination of operating conditions required, report the data in a table similar to Table 2.

Method	$\Delta t_{\max}$	WU/trajectory	Estimated η at $\Delta t_{\max}$
AB1			
AB2			
AM1			
AM2			
RK2			
RK4			

Table 2: Work requirements and η estimates for each method under the operating conditions X

3. Thoroughly discuss the results of the accuracy and work study of the different integration methods. Try to explain the WU results based on your understanding of the methods and their accuracy and stability.

4. For each of the methods, plot the trajectory of a particle using different timesteps. To do this, plot the trajectory in the (x, y) plane as the z coordinate only sets the terminal time. Do you observe any stability problems using larger timesteps for any of the methods? If so, can these stability limits be explained using the eigenvalues from the linearized analysis you performed above and your knowledge of the stability limits of the different methods?