16.901: Finite Element Method Project
Heat Transfer in a Turbine Blade
Sample Solution

1 Background

In the first stages of a turbine, the blades are subjected to a high temperature flow due to the hot gas produced in the combustor. As a result, turbine blades are often internally cooled by pumping low temperature air through the blades in passages. In this project, we will simulate the heat transfer in an internally cooled turbine blade using a finite element discretization. The blade with the four internal cooling passages is shown in Figure 1 including the coarse grid elements (which are triangular). The blade and the problem we are considering is actually representative of a large turbine used in power generation.

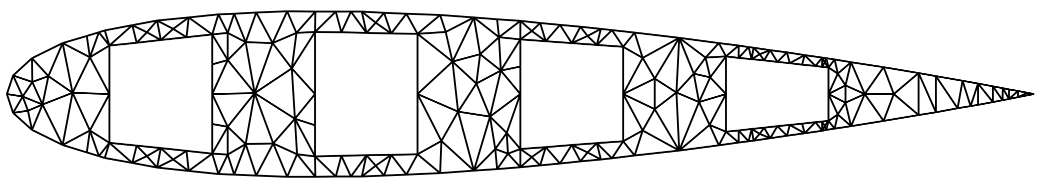


Figure 1: Blade geometry with cooling passages including coarse grid

The heat transfer equation is a diffusion equation for the temperature,

$$\nabla^2 T = 0, \tag{1}$$

Recall that the heat transfer rate is given by,

$$\vec{q} = -k\nabla T.$$

where k is the thermal conductivity of the blade material. The boundary conditions will all be convective heat transfer conditions. For the blade surface, the heat flux out of the blade will be given by,

$$\vec{q} \cdot \vec{n} = h_{ext} \left(T - T_{ext}\right),$$

where h_{ext} is the convective heat transfer coefficient and T_{ext} is the temperature outside the blade. Note, $\vec{n}$ is a normal pointing out of the blade so that $\vec{q} \cdot \vec{n}$ represents the heat flux out of the blade. Similarly for the heat flux at a cooling passage,

$$\vec{q} \cdot \vec{n} = h_{int} \left(T - T_{int}\right).$$

Again, $\vec{n}$ is out of the blade (into the cooling passage) so that $\vec{q} \cdot \vec{n}$ represents the heat flux out of the blade.

In the grids, the dimensions of the blade have been non-dimensionalized by the chord length, L. So, the coordinate values are actually x/L and y/L. The temperatures are,

$$T_{ext} = 1300°C \qquad T_{int} = 200°C$$

Also, the heat transfer coefficients in non-dimensional form are,

$$\frac{h_{ext}L}{k} = 14.0 \qquad \frac{h_{int}L}{k} = 4.7$$

2 Tasks

2.1 Finite Element Method Implementation

The first task was to develop an FEM solver for this problem. Three grids of triangular elements were provided for the blade stored in the MATLAB® datafiles **g0012coarse.mat**, **g0012medium.mat**, and **g0012fine.mat**

The convective heat transfer boundary condition implementation was based on the standard approach for implementing Robin-type conditions. Specifically, the weighted residual statement for Equation (1) is,

$$\int_\Gamma w \nabla T \cdot \vec{n}\, ds - \iint_\Omega \nabla w \cdot \nabla T\, dA = 0,$$

where w is the weight functions. On the boundaries Γ, the normal gradient of the temperature can be related to the temperature using the convective heat transfer boundary condition. For example, on the interior surface,

$$\begin{aligned} \nabla T \cdot \vec{n} &= -\frac{1}{k}\vec{q} \cdot \vec{n} \\ &= -\frac{h_{int}}{k}\left(T - T_{int}\right). \end{aligned}$$

An analogous expression exists for the external surface. Thus, the weighted residual statement will be (after multiplication by negative one),

$$\int_\Gamma w \frac{h_{bc}}{k}\left(T - T_{bc}\right) ds + \iint_\Omega \nabla w \cdot \nabla T\, dA = 0,$$

where T_{bc} and h_{bc} are the specified gas temperatures and heat transfer coefficients at the boundary.

The numerical implementation of this boundary integral term is as follows. First, we note that the integral is zero except for weight functions at nodes on the boundary. Given a boundary edge with nodes i and j, we have two integrals to construct for that edge, specifically,

$$\int_\Gamma w_i \frac{h_{bc}}{k}\left(T - T_{bc}\right) ds, \text{ and } \int_\Gamma w_j \frac{h_{bc}}{k}\left(T - T_{bc}\right) ds.$$

For a given edge, the values of T_{bc} and h_{bc} are constant. However, since we are using a linear element, the temperature T will vary linearly from node i to node j along the edge. Thus, for a given distance, s, along the edge from node i, the temperature is,

$$T(s) = T_i + \left(T_j - T_i\right)\frac{s}{l_e}$$

where l_e is the length of the edge. Furthermore, since we are using a Galerkin approximation, the weight functions also vary linearly taking a value of 1 at the node and dropping to zero at the other node,

$$\begin{aligned} w_i(s) &= 1 - \frac{s}{l_e} \\ w_j(s) &= \frac{s}{l_e} \end{aligned}$$

Substituting these expressions for the temperature and weight functions into the boundary integrals and performing the integration gives,

$$\begin{aligned} \int_\Gamma w_i \frac{h_{bc}}{k}\left(T - T_{bc}\right) ds &= \frac{h_{bc} l_e}{k}\left(\frac{1}{3}T_i + \frac{1}{6}T_j - \frac{1}{2}T_{bc}\right) \\ \int_\Gamma w_j \frac{h_{bc}}{k}\left(T - T_{bc}\right) ds &= \frac{h_{bc} l_e}{k}\left(\frac{1}{6}T_i + \frac{1}{3}T_j - \frac{1}{2}T_{bc}\right) \end{aligned}$$

Grid	Nodes	Elements	$h_{\max}/L$	$T_{\max}$ (C)	$T_{\min}$ (C)	CPU Time (sec)
Coarse	230	299	0.024	1299.8	911.4	0.1
Medium	424	574	0.016	1299.8	910.1	0.2
Fine	1425	2296	0.008	1299.8	908.3	2.1

Table 1: Comparison of number of nodes, elements, cell size, minimum and maximum blade temperatures, and CPU time for three grids.

Since these boundary integral terms include dependences on T_i and T_j they will alter the stiffness matrix, in addition to setting the right-hand side through the T_{bc} term. For more details of the computational implementation, a thoroughly commented MATLAB® source code has been included in the archive file distributed with this solution.

Plots of the temperature distributions on all three grids are shown in Figure 2. The temperature is largest at the trailing edge, and to a lesser extent, the leading edge. At these locations, the temperature is almost exactly the external temperature of $1300°C$. This occurs because the blade is thin in these regions and far from the cooling flow passages. Consider the trailing edge: since the temperature above and below is $1300\,°C$ and no cooling flow is near, the temperature must be nearly uniform and set by the external temperature.

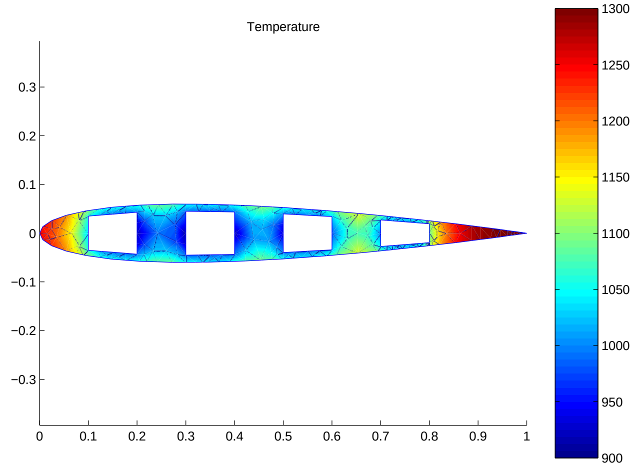
2.2 Study of Accuracy and Work

The results of the simulations, including the CPU time required, are given in Table 1. The grid size $h_{\max}$ was defined as the maximum circumradius of all elements, where the circumradius for an element is the radius of the circle through the nodes of the element. This can be calculated by intersecting the median segments from any two of the elements three edges. Note: the median segment of an edge is the line perpendicular to the edge passing through its midpoint. These median segments are equidistant to the nodes of the edge. Thus, interesting two median segments produces the point at which all three nodes are equidistant, i.e. the center of the circumcircle.

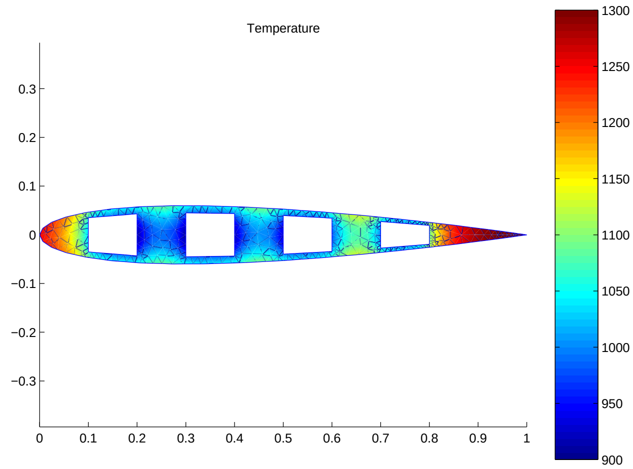
A shown in Table 1, the maximum cell size is a factor of two and three times larger for the medium and coarse meshes relative to the fine mesh. Of particular interest is the behavior of $T_{\max}$. $T_{\max}$ is constant and independent of the mesh resolution. As discussed above, this is due to the temperature in the leading and trailing edge regions being dominated by the external temperature. As a result, grid resolution does not impact its behavior. In contrast, $T_{\min}$ is changing and continues to decrease with increasing grid resolution. Since the change in $T_{\min}$ is larger from the medium-to-fine mesh ($1.8°C$) than the coarse-to-medium mesh ($1.3°C$), it appears that the results are not sufficiently resolved. If the results were approach the asymptotic result, we would tend to expect them to decrease more slowly as the mesh resolution were increased.

For the CPU times, we would expect two significant effects. First, in assembling the stiffness matrix, the work is proportional to the number of elements, which will be roughly proportional to the number of nodes. Second, in solving the linear system for the temperatures, we are using Matlab's® matrix division routine which is Gaussian elimination. Since Gaussian elimination is $O(N^3)$ work where N is the number of unknowns, we expect a portion of the CPU time to scale with the number of nodes cubed. Thus, for larger grids (i.e. more nodes), we expect the CPU time for scale with N^3 since the $O(N)$ effect will eventually be small by comparison. For the two smallest grids, we see the CPU time doubles. Since the number of nodes approximately doubled as well, this indicates the linear work term is dominating over the Gaussian elimination time. However, in moving to the largest mesh, the work jumps a factor of 10 to 2.1 seconds. Since the number of nodes increased by 3.4, this indicates that the CPU time has increased faster than $O(N)$. In short, the Gaussian elimination time is starting to be the dominant factor in the solution time.

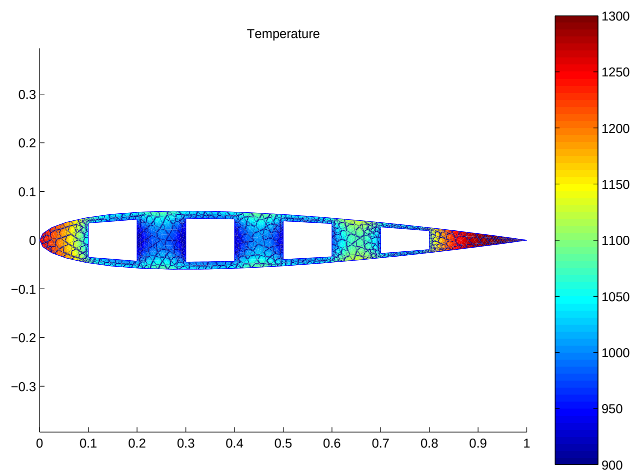
Finally, if we wish to have accuracy within $10°C$ of the exact answer, we can probably using any of the existing grids since the entire change of temperature is about 3 degrees. However, to be certain, additional meshes should be run to see if the temperature is approaching an asymptotic answer.



(a) Coarse grid



(b) Medium grid



(c) Fine grid

Figure 2: Temperature distribution for finite element simulations

2.3 Design Recommendations

At the leading and trailing edges, we basically have two choices. One would be to somehow lower the external temperatures. This can be done by blowing cool air from inside the blade onto the surface of the blade (this is known as film cooling). The result is that the temperature observed from the blade's view is lower since it will be some mix of the high gas path temperature and the lower cooling air temperature. The other option is to find a way to have cooling air reach the internal portions of the blade near the leading and trailing edge. At the leading edge, this is done by so-called 'showerhead' cooling in which cool air is blown on the internal leading surface. At the trailing edge, a series of thin slots is often manufactured in the chordwise direction through which the cooling air can travel. This cooling air then leaves the airfoil at the trailing edge. The result is that the thin trailing edge region does have some cooling air which will help to lower the temperatures locally.