Probabilistic Methods in Engineering
Course Notes for 16.901
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David Darmofal

1 Overview of Probabilistic Concepts

The purpose of this section is to provide a quick review of some basic concepts in probability.

1.1 Outcomes and Events

Consider an experiment or activitiy which will be performed several times. Each time the experiment is performed, the outcome ζ can be recorded. An event A is set of outcomes for which certain conditions have been met. An elementary event consists of only a single outcome.

Example: Consider the inspection of rotor blades in the turbine of a jet engine. Suppose that the total number of rotor blades in the engine is N. The outcome of a single inspection is recorded as the number of blades that must be replaced due to damage. Thus, the outcomes are the set of non-negative integers, $\{0, 1, 2, ..., N\}$. If the number of blades replaced in a single engine is greater than some number, say 5, than this may indicate more significant damage has occurred and the engine will need to be inspected more thoroughly. In this situation, we would be interested in the event where the number of replaced blades is $\{6, 7, 8, ..., N\}$. This is not an elementary event since it consists of more than one outcome. However, we would also be interested in the situation where no blades are replaced. In this case, the event consists of a single outcome (i.e. 0) and therefore is an elementary event.

1.2 The Meaning of Probability

Given an event, A, the probability of the event is $P\{A\}$. Probabilities are assumed to satisfy the following properties:

- $P\{A\} \geq 0$.
- If and only if the event is certain to occur, then $P\{A\} = 1$.
- Given two mutually exclusive events, A and B, then $P\{A + B\} = P\{A\} + P\{B\}$.

1.3 Random Variables

The utility of probability theory is in describing the behavior of random variables. In the simplest terms, a random variable can be defined as a variable or parameter whose values depend on the particular outcome of the experimental trial. Thus, random variables will be a function of the outcome. We will use boldface letters to denote a random variable, for example, $\mathbf{x}$. It is understood that the value of $\mathbf{x}$ depends on the outcome which has occurred, i.e. $\mathbf{x} = \mathbf{x}(\zeta)$.

Example: We will continue with the rotor blade inspection example. A very simple example of a random variable would be the number of blades which are replaced for a particular inspection. In this case, the random variable is just the outcome itself! Specifically,

$$\mathbf{N}(\zeta) = \zeta.$$

Now, let's try something a little more complicated. The airline is concerned about the cost of the inspection and repair of its engines. Not including the cost of replacing any blades, simply performing the inspection costs the airlines C_I dollars due to employee salaries (labor). The cost to replace a single blade is C_B (including the cost of the new blade and the labor). Also, if the number of replacements is greater than 5, the cost rises dramatically since a more thorough inspection must be performed. We will model this as an increment C_D. Since the cost of the inspection and repair depends on the outcome of the inspection (and the outcome is random), clearly the cost of the inspection is a random variable. Specifically,

$$\mathbf{C}(\zeta) = \begin{cases} C_I + C_B\zeta, & \text{for } 0 \le \zeta \le 5, \\ C_I + C_B\zeta + C_D, & \text{for } 6 \le \zeta \le N. \end{cases}$$

1.4 Probability density functions (PDF)

We are often concerned with probabilities of parameters which are real numbers (i.e. which have infinitely many values). In this case, a probability density function (PDF) is used to describe the probability of the parameter being in some range. In particular, given a random variable $\mathbf{x}$, the probability that $a \le \mathbf{x} \le b$ is,

$$P\left\{a \le x \le b\right\} = \int_a^b f(x)dx,$$

where $f(x)$ is the PDF of $\mathbf{x}$.

A common (and probably the simplest) distribution is the uniform distribution. In this case, we assume the probability density is constant within some range and zero outside of this range,

$$\textbf{Uniform:}\ f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Other distribution types are described in Section 1.9

1.5 Expected value and mean

Given the PDF, $f(x)$ of a random variable $\mathbf{x}$, the expectated value is defined as,

$$E\left\{\mathbf{x}\right\} \equiv \int_{-\infty}^{+\infty} x f(x)\, dx.$$

The expected value of $\mathbf{x}$ is also known as the mean value. We will also use the symbol μ_x for the expected value of $\mathbf{x}$.

1.6 Variance and standard deviation

The variance of $\mathbf{x}$ is defined as,

$$\sigma_x^2 \equiv \int_{-\infty}^{+\infty} (x - \mu_x)^2 f(x)\, dx.$$

The value σ_x is known as the standard deviation of $\mathbf{x}$. The variance is a measure of the variability in $\mathbf{x}$ about its mean value. A frequently used relationship exists between the mean and variance,

$$\sigma_x^2 = E\left\{\mathbf{x}^2\right\} - \mu_x^2.$$

Try proving this!

1.7 Cumulative distribution functions (CDF)

Cumulative distribution function (CDF) of $\mathbf{x}$ is defined as the probability that $\mathbf{x} \le x$. Specifically,

$$F(x) \equiv P\{\mathbf{x} \le x\}.$$

From the basic assumptions of probability, it can be shown that $F(-\infty) = 0$ (i.e. the probability of $\mathbf{x}$ becoming infinite is zero) and $F(+\infty) = 1$ (i.e. the probability of $\mathbf{x}$ being less than infinity is one). The CDF and PDF of $\mathbf{x}$ are related as follows,

$$F(a) = \int_{-\infty}^{a} f(x)\, dx$$

Thus, we can show,

$$F(b) - F(a) = \int_{a}^{b} f(x)\, dx.$$

Furthermore, this implies that

$$f = \frac{dF}{dx}.$$

1.8 Percentiles

The u percentile of $\mathbf{x}$ is the smallest number x_u such that,

$$u = P\{\mathbf{x} \le x_u\} = F(x_u).$$

Note, since u is a probability, its range is $0 \le u \le 1$.

1.9 Common distribution types

1.9.1 Normal distribution

The normal (or Gaussian) distribution is,

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-(x-\mu_x)^2/2\sigma_x^2}.$$

We will use the common notation $\mathbf{x} = N(\mu; \sigma)$ to indicate that $\mathbf{x}$ is a normally-distributed random variable with mean μ and standard deviation σ.

1.9.2 Triangle distribution

This distribution is often used when the minimum (x_{min}), most likely (x_{mpp}), and maximum values (x_{max}) of a random variable can be estimated, but not the actual probability density. The triangle distribution assumes that the probability density is maximum at x_{mpp} and varies linearly to zero at x_{min} and x_{max}. Let's derive the PDF for this case.