16.901 – Lecture Notes: March 1, 2002

Topics:
* 1-D wave equation
* Domain of dependence & influence
* 1st order upwind scheme
* CFL condition

1-D Wave Equation

In 1-dimension, our model convection-diffusion equation reduces to:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial x^2}$$

When k is small, this can be further reduced to

$$\underbrace{\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = 0}_{\text{1-D (one-way) wave equation}} \qquad (1)$$

For $u(x,t) = \text{const}$, the solutions to this eqn are:

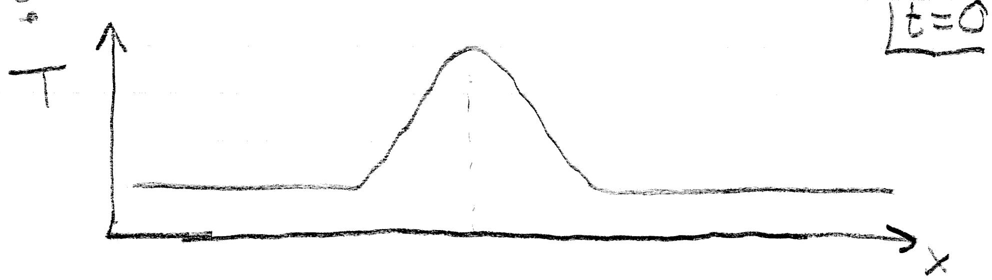
$$T(x,t) = T(x-ut) = T(\eta) \qquad \eta = x - ut \qquad (2)$$

To verify this, substitute (2) into (1):

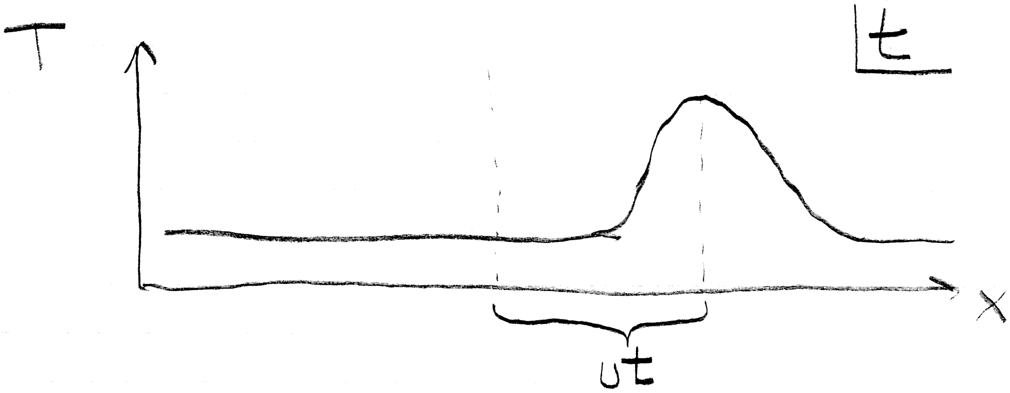
$$\frac{\partial T}{\partial t} = -u\frac{dT}{d\eta} \qquad \frac{\partial T}{\partial x} = \frac{dT}{d\eta}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = -u\frac{dT}{d\eta} + u\frac{dT}{d\eta} = 0!$$

These solutions are wavelike. Consider the following distribution of T at time $t = 0$:



Then, for some $t > 0$, the solution is:

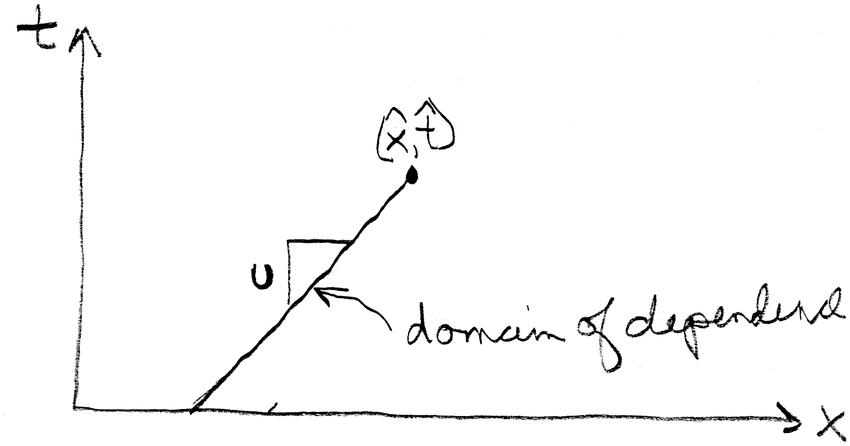


That is, the distribution of T shifts (i.e. propagat or convects) a distance ut to the right.

Domain of Dependence

Consider a point in space-time $(\hat{x}, \hat{t})$. The solution at this point, for a general partial differentia eqn depends on the solution at other points.

For the 1-D wave equation the domain of dependence for the point $(\hat{x},\hat{t})$ is the ray extending backwards in time with slope u:



Domain of dependence of $(\hat{x},\hat{t})$ = set of points satisfying $x-\hat{x} = u(t-\hat{t})$ for $t<\hat{t}$.

Definition: Domain of dependence

The domain of dependence for a point (x,t) is the set of points that the solution at (x,t) depends on.

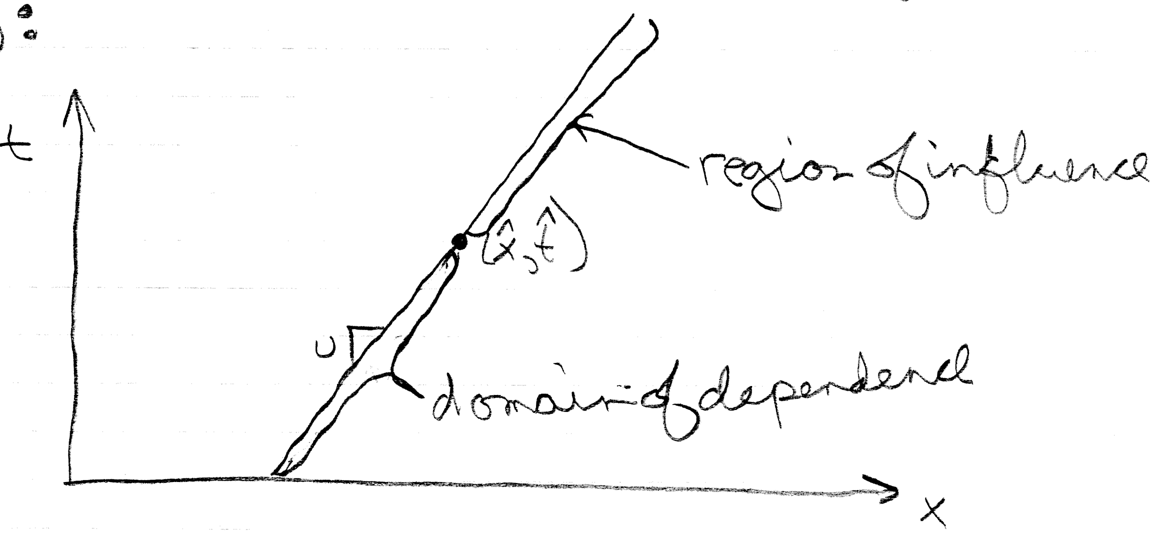
The analogous definition holds in multiple spatial dimensions.

The "inverse" of the <u>domain of dependence</u> is the <u>region of influence</u>.

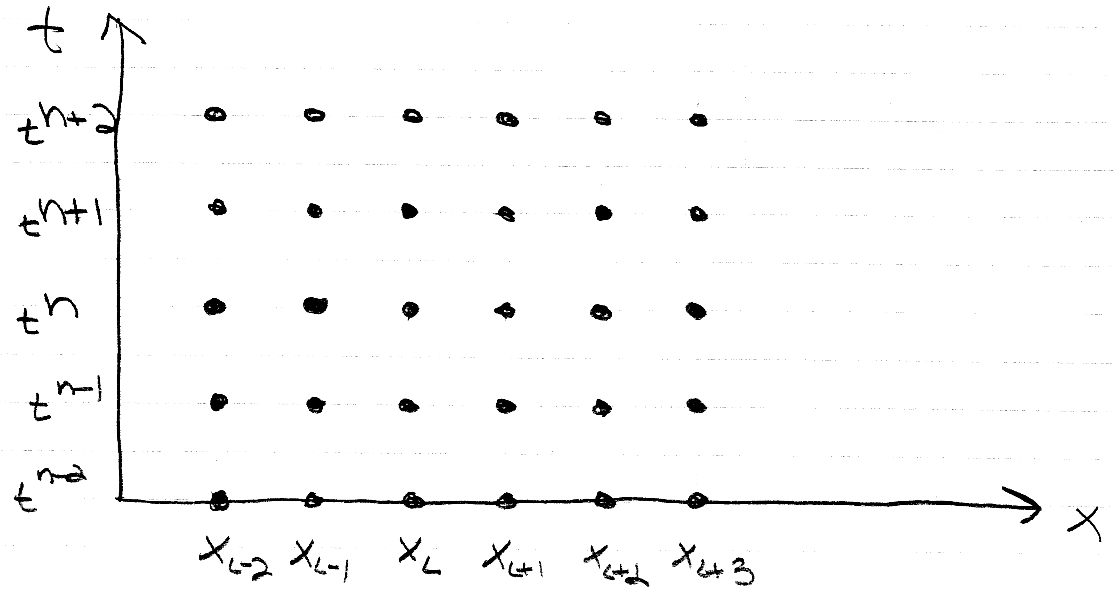
<u>Definition: Region of Influence</u>

The region of influence of a point (x,t) is the set of points whose domain of dependence includes (x,t).

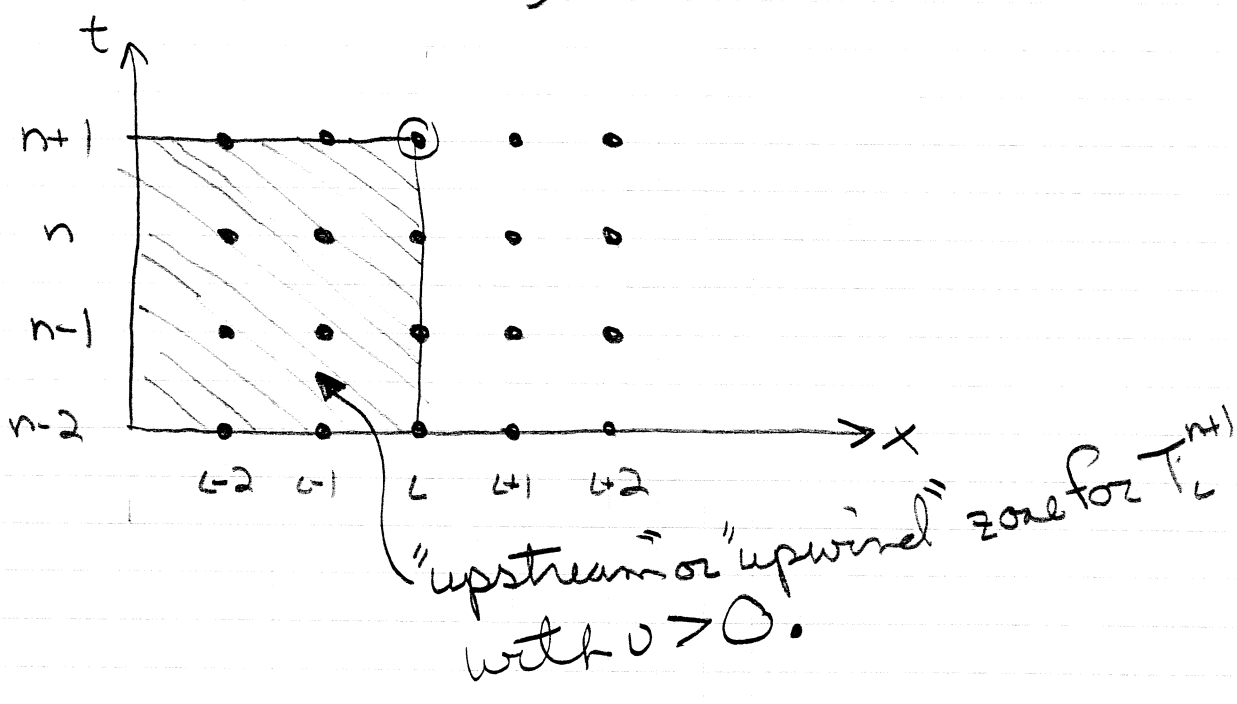
Returning to our 1-D wave eqn example, the region of influence at $(\hat{x},\hat{t})$ is the continuation of the line which was the domain of dependence for $(\hat{x},\hat{t})$:



Let's consider a discretization of (x,t) with constant Δx & Δt. This place a grid of points on the (x,t) plane:



Now, suppose we are solving $T_t + uT_x = 0$ with $u > 0$. In this case, the solution at T_i^{n+1} should only depend on information which is "upstream" (i.e. to the left for $u>0$). This is because we know the solutions behave as $T(x-ut)$.



So, let's consider the simplest scheme

which satisfies this constraint.

Specifically,

* explicit scheme $\Rightarrow T_i^{n+1}$ is only value at n+1 time level in the final approximation

* smallest number of unknowns at time n: select T_i^n and T_{i-1}^n

In other words,

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} + u \underbrace{\frac{T_i^n - T_{i-1}^n}{\Delta x}}_{\text{backwards or upwind difference}} = 0$$

Recall that $\frac{T_i^n - T_{i-1}^n}{\Delta x} = \frac{\partial T}{\partial x}\Big|_i^n + O(\Delta x)$.

Now let's test this algorithm. In the following results, Δx has been fixed and Δt is varied. Specifically:

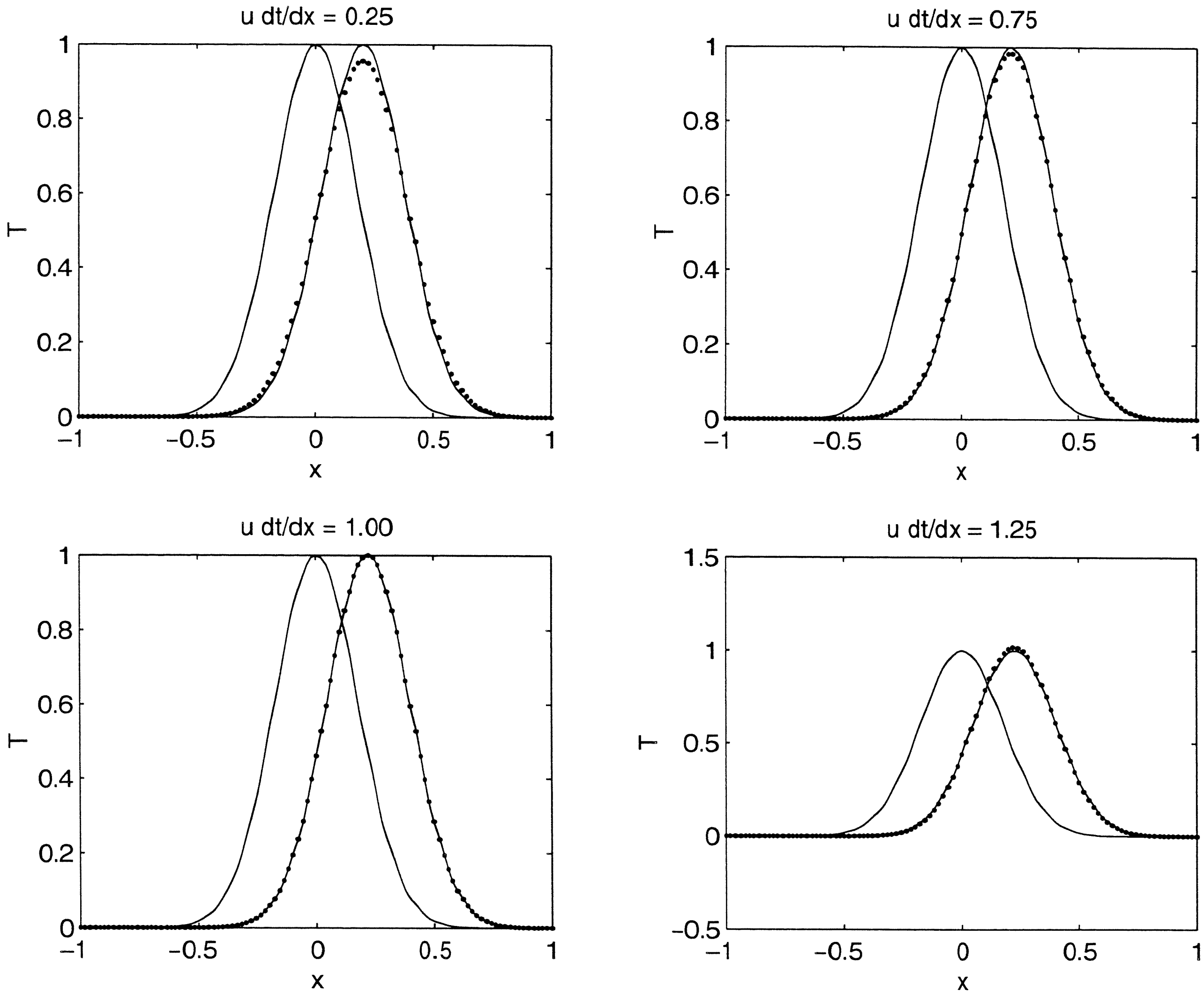
* Assume $u = 1$

* $\frac{u\Delta t}{\Delta x}$ is varied 0.25, 0.75, 1.0 & 1.25 holding Δx & u fixed.

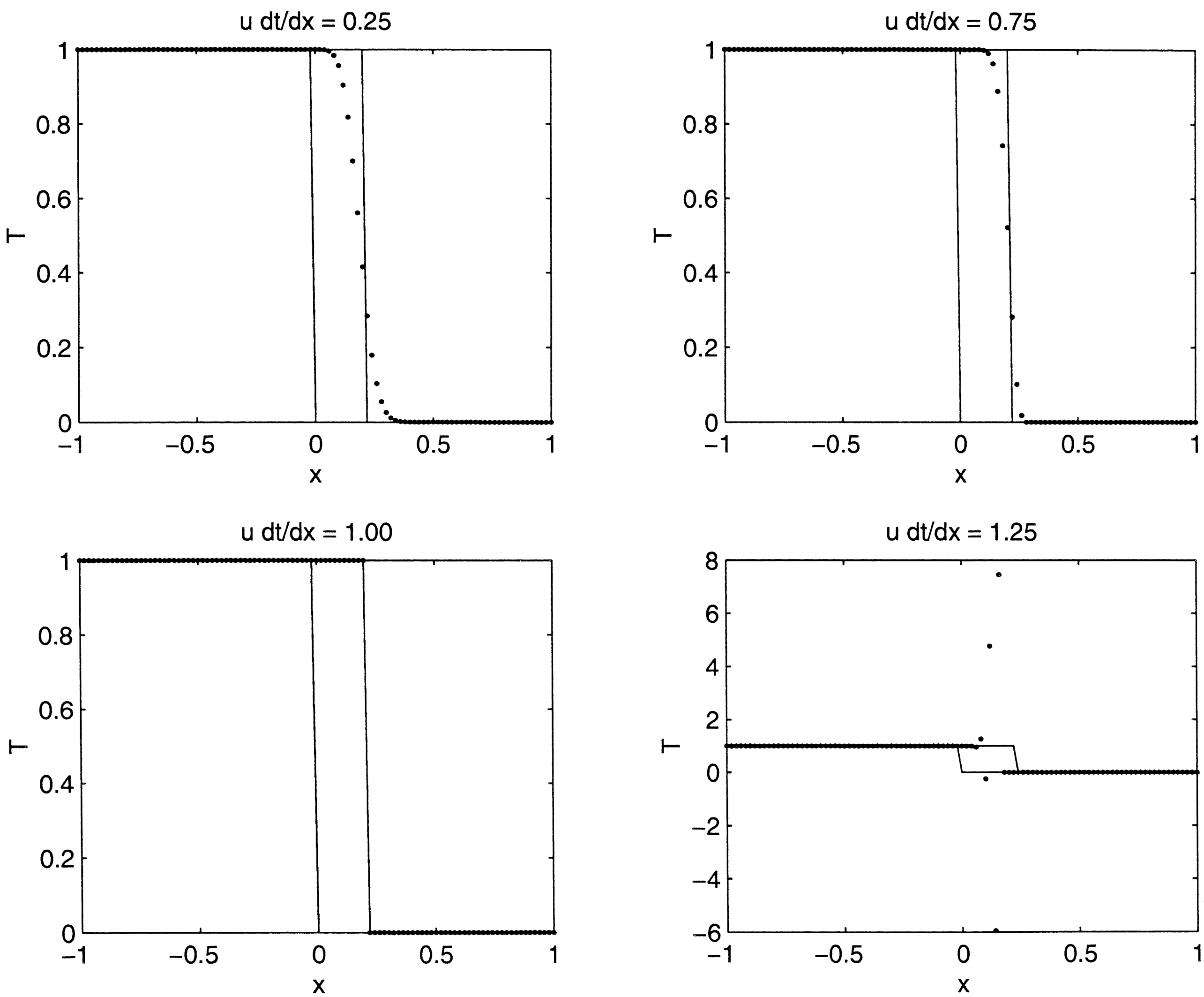
* Two initial conditions are tried: smooth and shocked.

```
CFL = 1.25;

u = 1;
L = 1;

tmax = 0.2;
imax = 100;

dx = 2*L/imax;

x = linspace(-L,L,imax+1);

dt = CFL*dx/u;

for ii = 1:imax+1,
  T(ii) = Texact(x(ii),0.0,u);
end

Tinit = T;
T0 = T;

t = 0;
while (t < tmax),

  t = t + dt;

  T0 = T;

  T(1) = Texact(x(1),t,u);

  for ii = 2:imax+1,
    T(ii) = T0(ii) - u*dt/dx*(T0(ii)-T0(ii-1));
  end


end

for ii = 1:imax+1,
  Tex(ii) = Texact(x(ii),t,u);
end

plot(x,T,'.');hold on;
plot(x,Tinit);
plot(x,Tex);
[mytitle] = sprintf('u dt/dx = %4.2f',CFL);
title(mytitle);
xlabel('x');
ylabel('T');
```

```
function [Tex] = Texact(x,t,u)

eta = x - u*t;

%Tex = exp(-16*eta^2);

if (eta < 0),
      Tex = 1;
else,
      Tex = 0;
end
```



u dt/dx = 0.25
u dt/dx = 0.75
u dt/dx = 1.00
u dt/dx = 1.25
T
x


u dt/dx = 0.25
u dt/dx = 0.75
u dt/dx = 1.00
u dt/dx = 1.25
T
x
-1
-0.5
0
0.5
1
0.2
0.4
0.6
0.8
8
6
4
2
-2
-4
-6

What is observed is:

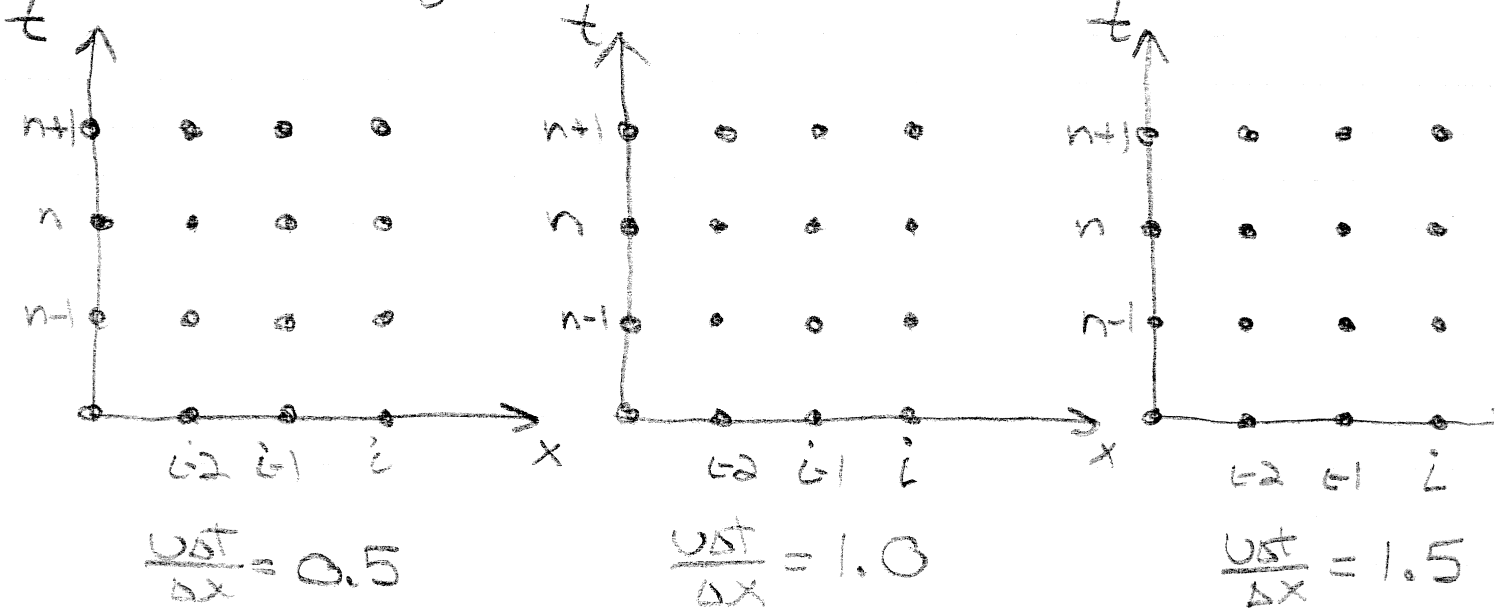
* For $\frac{U\Delta t}{\Delta x} > 1$, the smooth solution is increasing in amplitude. For $\frac{U\Delta t}{\Delta x} < 1$, the smooth solution decreases in amplitude. But $\frac{U\Delta t}{\Delta x} = 1$ looks to be exact.

* For shocked case, $\frac{U\Delta t}{\Delta x} > 1$ exhibits dramatic instabilities. For $\frac{U\Delta t}{\Delta x} = 1$, the solution again looks exact.

* For $\frac{U\Delta t}{\Delta x} < 1$, the shock spreads from the initial condition. The spreading is worse for lower $\frac{U\Delta t}{\Delta x}$ (i.e. lower Δt!)

The instability observed in this 1st order upwind method can be explained from a consideration of the numerical & physical domain of dependence.

Question: draw the physical domain of dependence for the point T_i^{n+1} if $\frac{u\Delta t}{\Delta x} = 0.5, 1.0 \text{ \& } 1.5$



Question: Now draw the numerical domain of dependence for the 1st order upwind scheme for the point T_i^{n+1} on the same (x,t) planes above.

Courant-Friedrichs-Lewy Condition:

A convergent numerical approximation to a PDE must contain the physical domain of dependence within the numerical domain of dependence.