

12.215 Modern Navigation

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Review of last class

- Estimation methods
 - Restrict to basically linear estimation problems (also non-linear problems that are nearly linear)
 - Restrict to parametric, over determined estimation
 - Concepts in estimation:
 - Mathematical models
 - Statistical models
 - Least squares and Maximum likelihood estimation
 - Covariance matrix of estimated parameters
 - Statistical properties of post-fit residuals

Today's class

- Finish up some aspects of estimation
 - Propagation of variances for derived quantities
 - Sequential estimation
 - Error ellipses
- Discuss correlations: Basic technique used to make GPS measurements.
 - Correlation of random signals with lag and noise added (varying amounts of noise)
 - Effects of length of series correlated
 - Effects of clipping (ex. 1-bit clipping)

Covariance of derived quantities

- Propagation of covariances can be used to determine the covariance of derived quantities. Example latitude, longitude and radius. θ is co-latitude, λ is longitude, R is radius. ΔN , ΔE and ΔU are north, east and radial changes (all in distance units).

Geocentric Case :

$$\begin{bmatrix} \Delta N \\ \Delta E \\ \Delta U \end{bmatrix} = \begin{bmatrix} -\cos(\theta)\cos(\lambda) & -\cos(\theta)\sin(\lambda) & \sin(\theta) \\ -\sin(\lambda) & \cos(\lambda) & 0 \\ X/R & Y/R & Z/R \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

A matrix for use in propagation from V_{xx}

Covariance of derived quantities

- Using the matrix on the previous page to find a linear relationship (matrix A) between changes in XYZ coordinates and changes in the North ($\delta\phi R$), East ($\delta\lambda R \cos\phi$) and height (Up), we can find the covariance matrix of NE and U from the XYZ covariance matrix using propagation of variances
- This is commonly done in GPS, and one thing which stands out is that height is more determined than the horizontal position (any thoughts why?).
- This fact is common to all applications of GPS no matter the accuracy.

Estimation in parts/Sequential estimation

- A very powerful method for handling large data sets, takes advantage of the structure of the data covariance matrix if parts of it are uncorrelated (or assumed to be uncorrelated).

$$\begin{bmatrix} \mathbf{V}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_3 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{V}_1^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_3^{-1} \end{bmatrix}$$

Sequential estimation

- Since the blocks of the data covariance matrix can be separately inverted, the blocks of the estimation ($\mathbf{A}^T \mathbf{V}^{-1} \mathbf{A}$) can be formed separately and combined later.
- Also since the parameters to be estimated can be often divided into those that effect all data (such as station coordinates) and those that effect data a one time or over a limited period of time (clocks and atmospheric delays) it is possible to separate these estimations (shown next page).

Sequential estimation

- Sequential estimation with division of global and local parameters. \mathbf{V} is covariance matrix of new data (uncorrelated with priori parameter estimates), \mathbf{V}_{xg} is covariance matrix of prior parameter estimates with estimates \mathbf{x}_g and \mathbf{x}_l are local parameter estimates, \mathbf{x}_g^+ are new global parameter estimates.

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{x}_g \end{bmatrix} = \begin{bmatrix} \mathbf{A}_g & \mathbf{A}_l \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_g \\ \mathbf{x}_l \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}_g^+ \\ \mathbf{x}_l \end{bmatrix} = \begin{bmatrix} (\mathbf{A}_g^T \mathbf{V}^{-1} \mathbf{A}_g + \mathbf{V}_{xg}^{-1}) & \mathbf{A}_g^T \mathbf{V}^{-1} \mathbf{A}_l \\ \mathbf{A}_l^T \mathbf{V}^{-1} \mathbf{A}_g & \mathbf{A}_l^T \mathbf{V}^{-1} \mathbf{A}_l \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_g^T \mathbf{V}^{-1} \mathbf{y} + \mathbf{V}_{xg}^{-1} \mathbf{x}_g \\ \mathbf{A}_l^T \mathbf{V}^{-1} \mathbf{y} \end{bmatrix}$$

Sequential estimation

- As each block of data is processed, the local parameters, \mathbf{x}_l , can be dropped and the covariance matrix of the global parameters \mathbf{x}_g passed to the next estimation stage.
- Total size of adjustment is at maximum the number of global parameters plus local parameters needed for the data being processed at the moment, rather than all of the local parameters.

Eigenvectors and Eigenvalues

- The eigenvectors and values of a square matrix satisfy the equation $\mathbf{Ax}=\lambda\mathbf{x}$
- If A is symmetric and positive definite (covariance matrix) then all the eigenvectors are orthogonal and all the eigenvalues are positive.
- Any covariance matrix can be broken down into independent components made up of the eigenvectors and variances given by eigenvalues. One method of generating samples of any random process (ie., generate white noise samples with variances given by eigenvalues, and transform using a matrix made up of columns of eigenvectors.

Error ellipses

- One special case is error ellipses. Normally coordinates (say North and East) are correlated and we find a linear combinations of North and East that are uncorrelated. Given their covariance matrix we have:

$$\begin{bmatrix} \sigma_n^2 & \sigma_{ne} \\ \sigma_{ne} & \sigma_e^2 \end{bmatrix} \quad \text{Covariance matrix;}$$

$$\text{Eigenvalues satisfy } \lambda^2 - (\sigma_n^2 + \sigma_e^2)\lambda + (\sigma_n^2\sigma_e^2 - \sigma_{ne}^2) = 0$$

$$\text{Eigenvectors } \begin{bmatrix} \sigma_{ne} \\ \lambda_1 - \sigma_n^2 \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_2 - \sigma_e^2 \\ \sigma_{ne} \end{bmatrix}$$

Error ellipses

- These equations are often written explicitly as:

$$\left. \begin{array}{l} \lambda_1 \\ \lambda_2 \end{array} \right\} = \frac{1}{2} \left(\sigma_n^2 + \sigma_e^2 \pm \sqrt{(\sigma_n^2 + \sigma_e^2)^2 - 4(\sigma_n^2 \sigma_e^2 - \sigma_{ne}^2)} \right)$$

$$\tan 2\phi = \frac{2\sigma_{ne}}{\sigma_n^2 - \sigma_e^2} \text{ angle ellipse make to N axis}$$

- The size of the ellipse such that there is P (0-1) probability of being inside is (area under 2-D Gaussian). ρ scales the eigenvalues

$$\rho = \sqrt{-2 \ln(1 - P)}$$

Error ellipses

- There is only 40% chance of being inside the 1-sigma error ellipse (compared to 68% of 1-sigma in one dimension)
- Commonly you will see 95% confidence ellipse which is 2.45-sigma (only 2-sigma in 1-D).
- Commonly used for GPS position and velocity results
- The specifications for GPS standard positioning accuracy are given in this form and its extension to a 3-D error ellipsoid (cigar shaped)

Example of error ellipse

Covariance

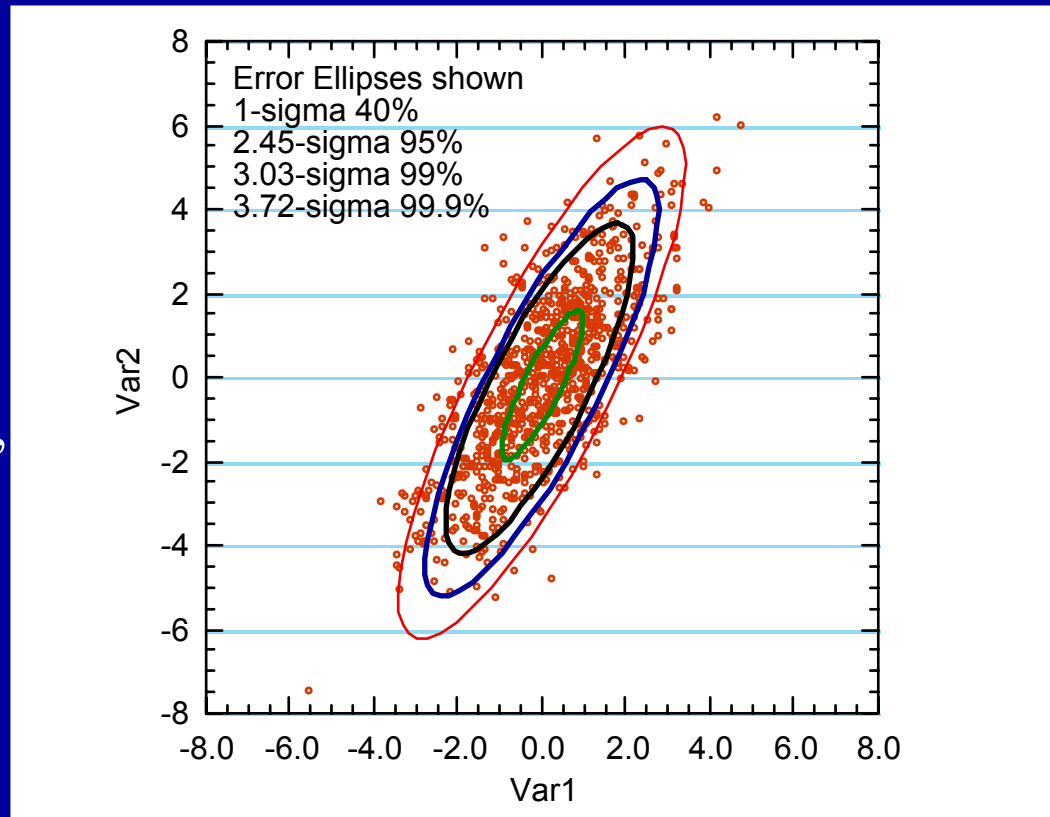
2 2

2 4

Eigenvalues

0.87 and 3.66,

Angle -63°



Correlations

- Stationary: Property that statistical properties do not depend on time
- Autocorrelation and Power Spectral Density (PSD)

$$\text{Autocorrelation } \varphi(t_1, t_2) = \iint_{x_1 x_2} x_1 x_2 f(x_1, t_1; x_2, t_2) dx_1 dx_2$$

For stationary process only depends of $\tau = t_1 - t_2$

$$\varphi_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int x(t)x(t + \tau) dt$$

$$\text{PSD } \Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \varphi_{xx}(\tau) e^{-i\omega\tau} d\tau$$

Cross Correlation

- Cross correlation is similar except that it is being two different time series rather than the same time series.
- If two time series have the same imbedded signal with difference noise on the two series then cross-correlation can be used to measure the time offset between the two series (example in next few slides)

Cross - correlation $\varphi_{xy}(t_1, t_2) = \iint_{xy} xy f(x, t_1; y, t_2) dx dy$

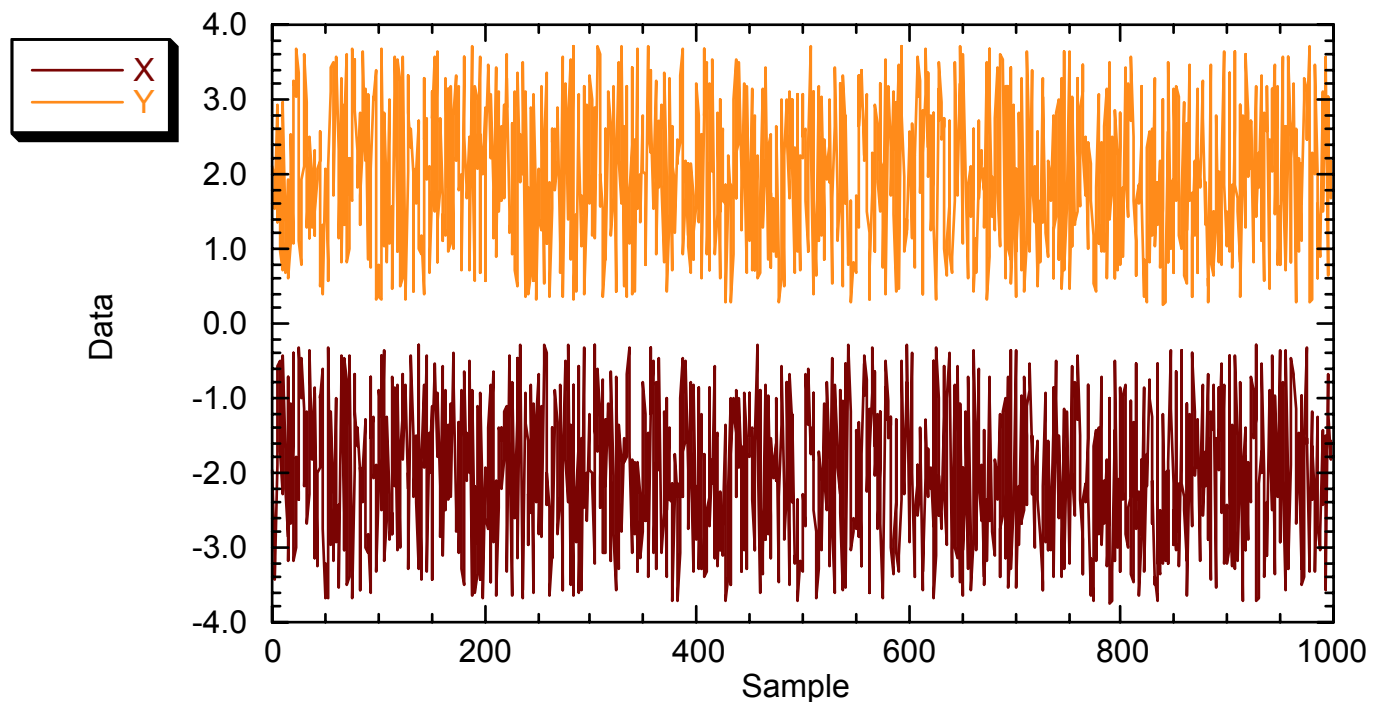
$$\varphi_{xy}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int x(t)y(t + \tau) dt \text{ for stationary process}$$

Example

- We will now show some time series of random time series and examine the correlations and cross correlations.
- We will present normalized cross and auto correlation function in that they have been divided by the standard deviations of the time series.
- (In reality, these are uniform probability density function values between $-0.5\sqrt{12}$ and $0.5\sqrt{12}$).
- Why multiply by $\sqrt{12}$? Series have also been offset so that they can be seen.

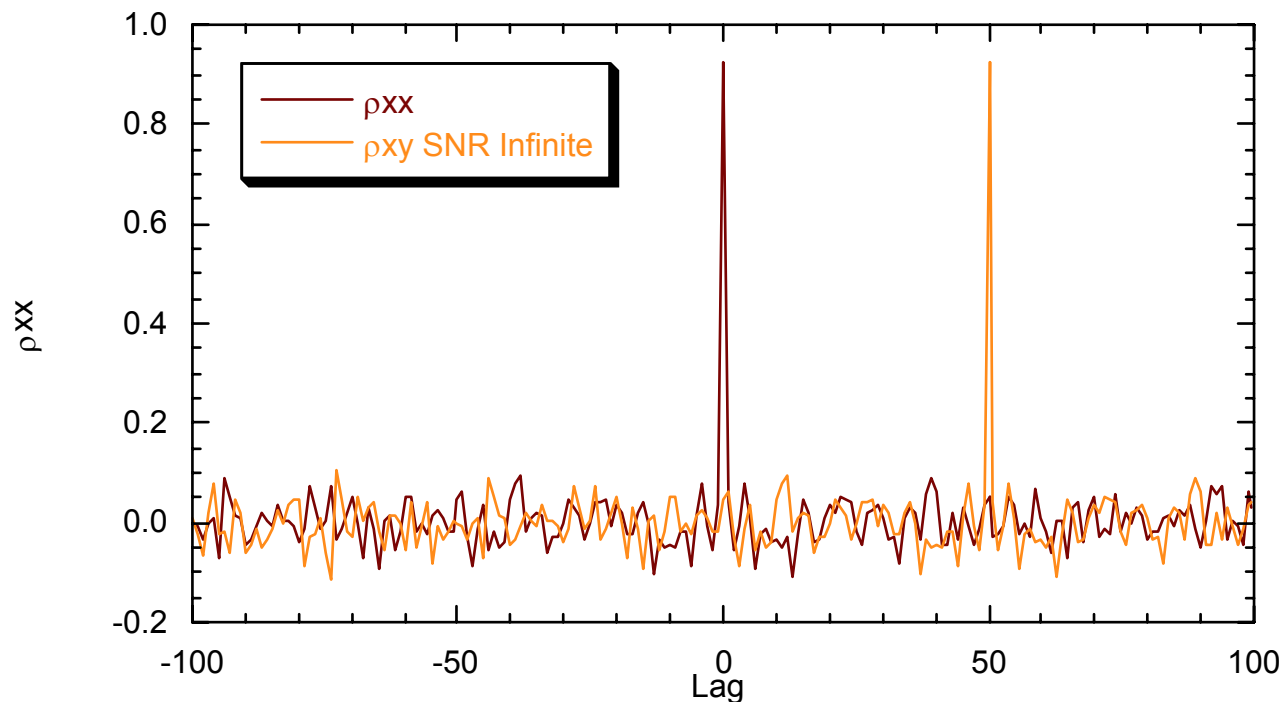
Time series (infinite signal to noise)

- Here the two time series are the same, one is simply displaced in time.



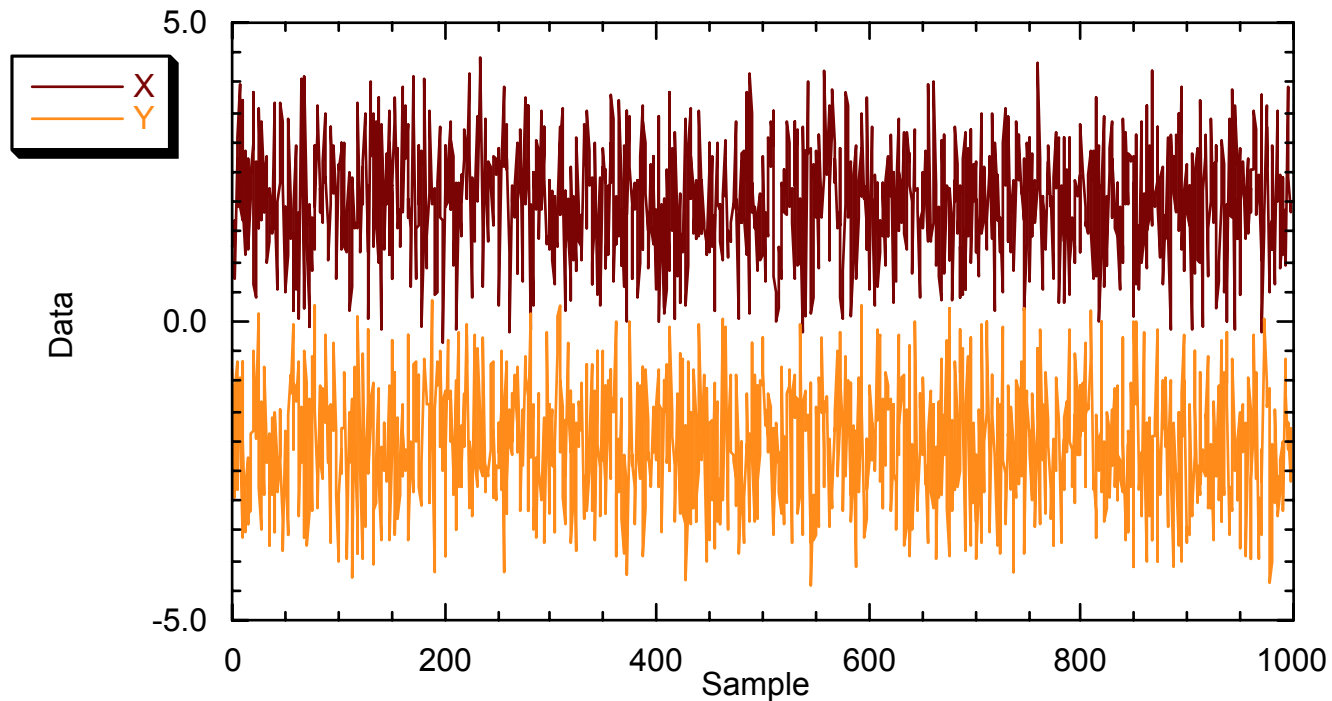
Auto and cross correlations

- Auto and cross correlations by summing over samples (discrete version of integral). Notice autocorrelation function is symmetric.



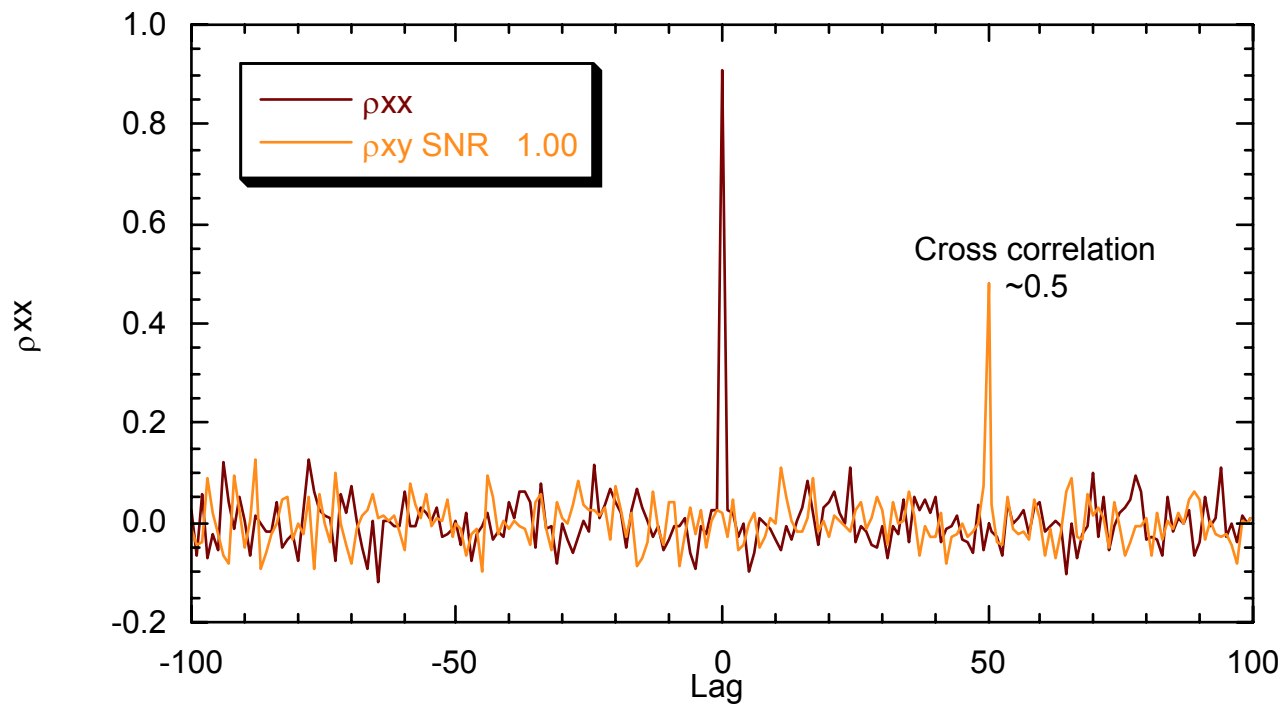
Signal plus noise

- We now add noise (independent) to the x and y time series. In this case equal noise and signal



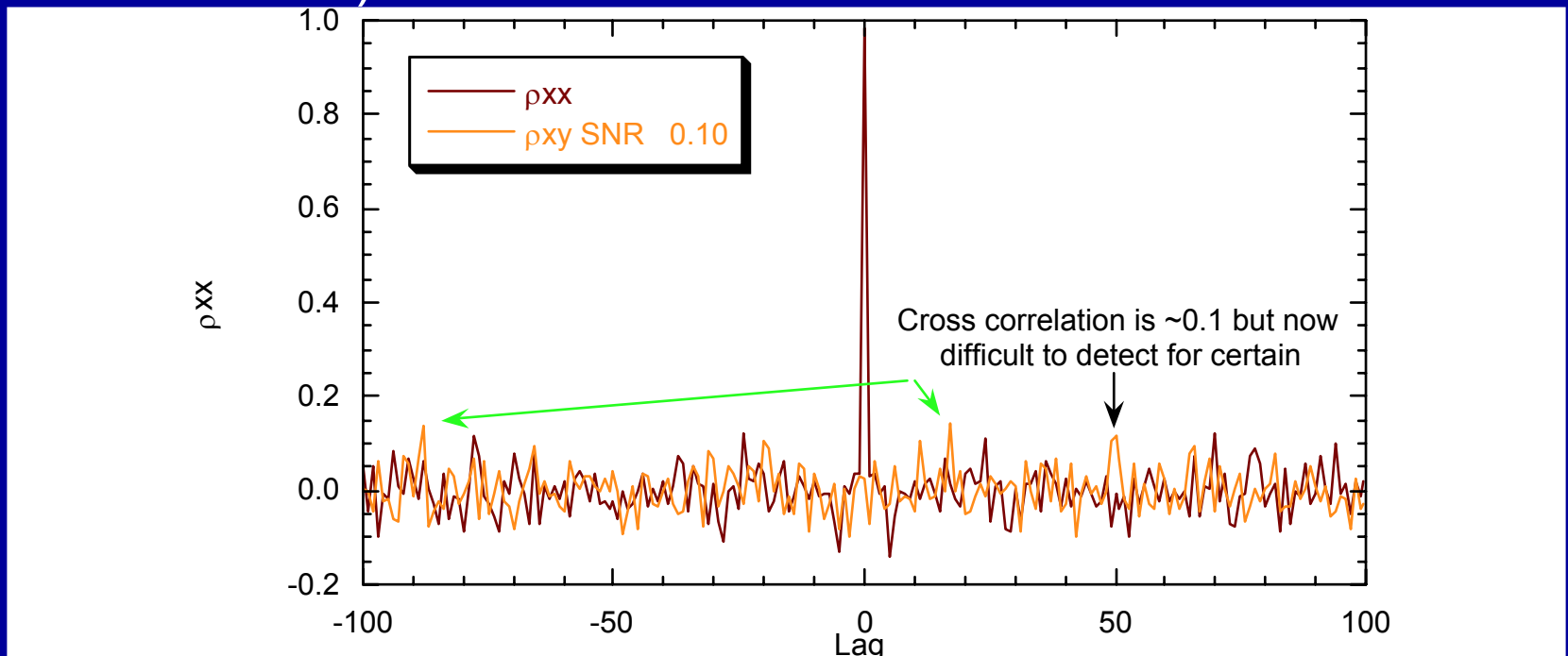
Auto and cross correlations

- Same lag in signal but now equal noise and signal in time series.



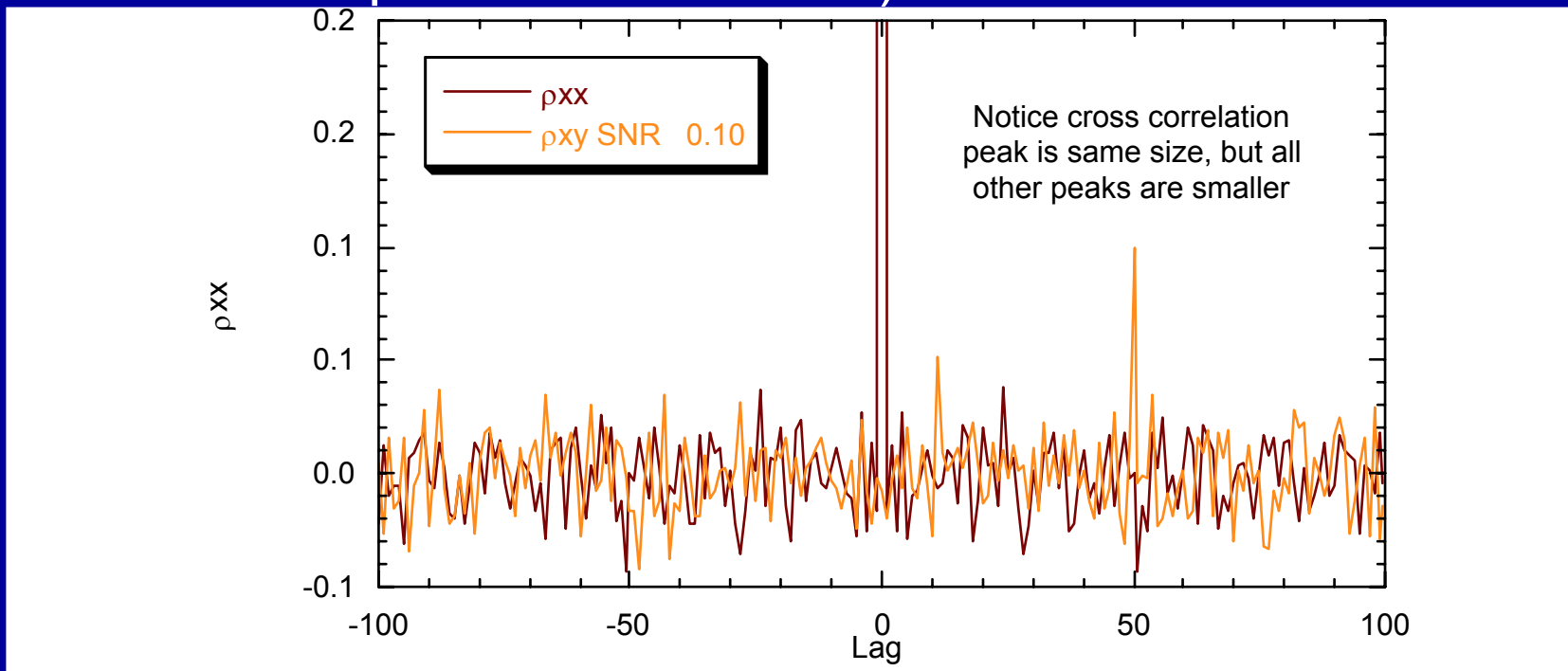
Low SNR case

- Here the SNR is 0.1 (that is ten times more noise than signal). Now we can not see clear peak in cross-correlation)



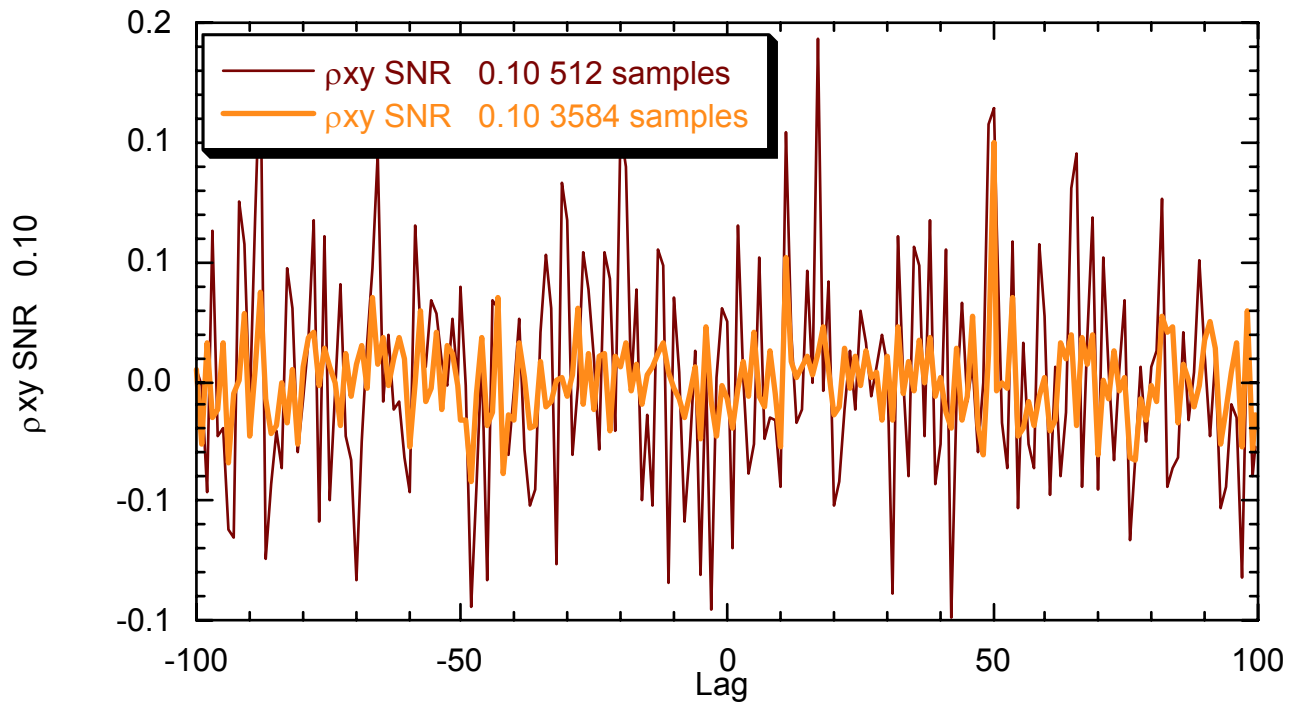
Low SNR, longer time series

- Example of improving detection by increasing the length of the time series correlated (in this case to 3600 samples instead of 500)



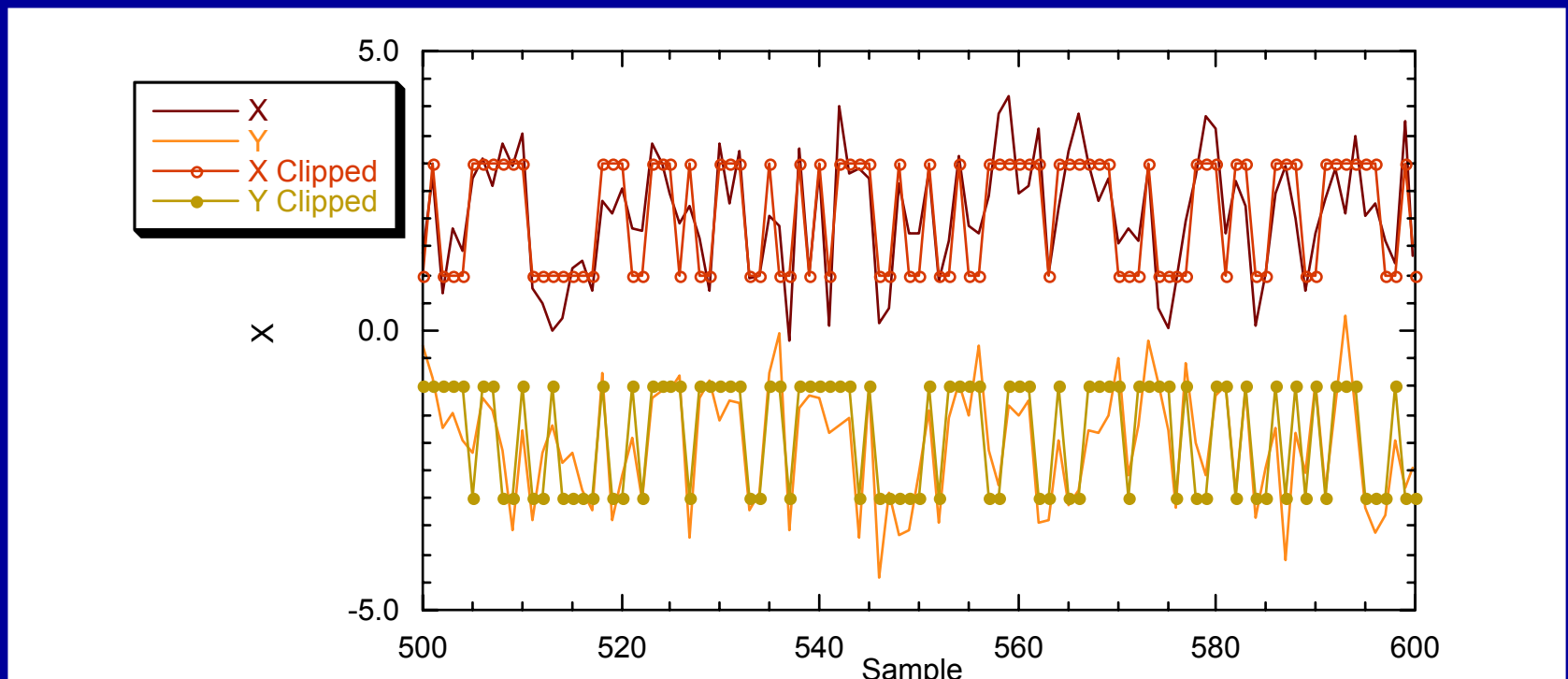
Cross Correlations comparison

- Comparison of the two cross correlations with different sample lengths



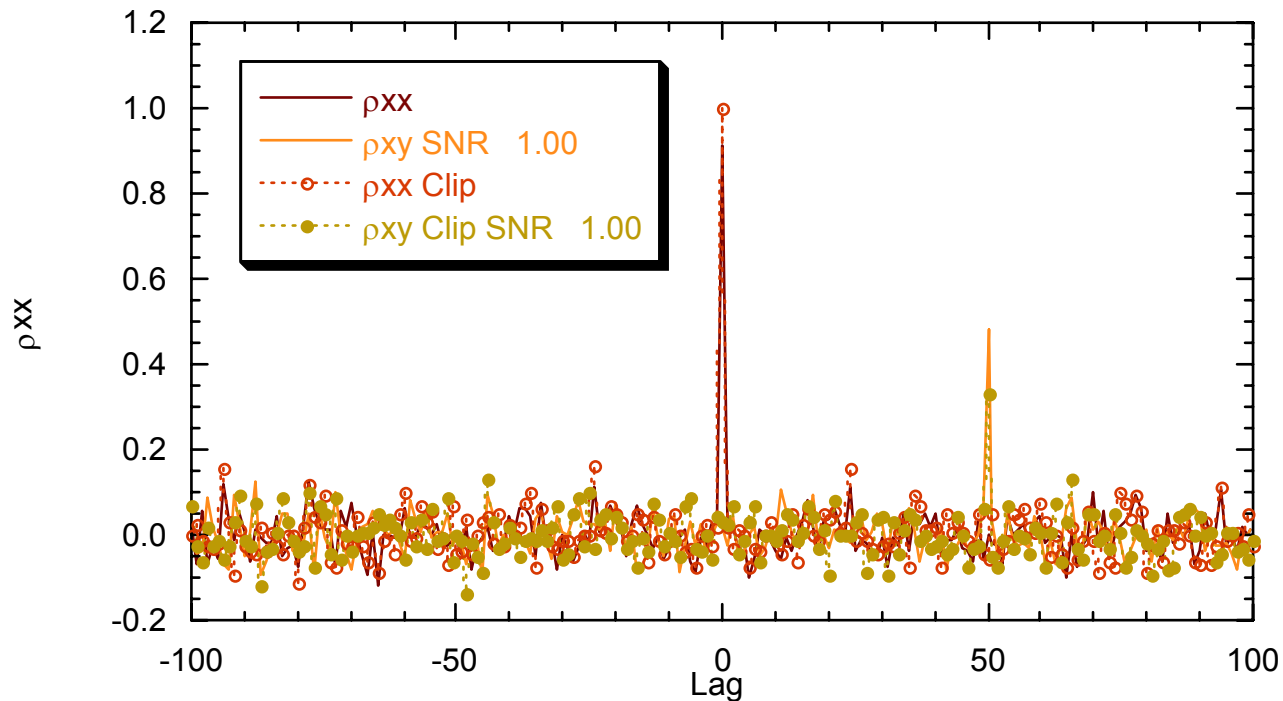
Effects of clipping

- Clipping when a signal is sampled digitally. 1-bit sampling detects only if the signal is positive or negative. SNR=1 example shown below (zoomed)



Auto and cross correlations

- Below are the auto and cross correlation functions for the original and clipped signals. Notice there is small loss of correlation but not much.



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