A STUDY OF THE INTERFERENCE OF POLARIZED LIGHT

by

the Method of Coherency Matrices

by

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I. INTRODUCTION

In the harmonic analysis of a wave function Wiener[1] developed the coherency matrix. This method has applications in the field of optics and the quantum theory. He has applied this theory to a single beam of polarized light. Poincaré[2] has analyzed this problem by a different method evolving the Poincaré sphere. Tuckerman[3] has made an analysis of this problem by a third method. It is the purpose of this thesis to correlate the work of these three men, and to develop the theory for the case of two polarized beams of light.

This problem affords a physical picture in the field of optics. Since the method of coherency matrices is applicable to the quantum theory, which theory does not readily submit to a physical picture, it is felt that this study is merited.

II. INVARIANTS UNDER A ROTATION OF AXES

The determination of the invariants of these papers can be brought under the following theorem.

If the axes of
$$A_1 \xi + B_1 \eta + C_1 = 0$$
$$\bar{A}_1 \bar{\xi} + \bar{B}_1 \bar{\eta} + \bar{C}_1 = 0$$

are rotated through an angle θ, then $A_1 \bar{A}_1 + B_1 \bar{B}_1$ and $\bar{A}_1 B_1 - A_1 \bar{B}_1$ are invariant under this rotation.

Proof:

Let
$$\xi' = \xi \cos\theta + \eta \sin\theta$$
$$\eta' = -\xi \sin\theta + \eta \cos\theta$$
$$\bar{\xi}' = \bar{\xi} \cos\theta + \bar{\eta} \sin\theta$$
$$\bar{\eta}' = -\bar{\xi} \sin\theta + \bar{\eta} \cos\theta$$

If
$$A_2 \xi' + B_2 \eta' + C_2 = 0$$
$$\bar{A}_2 \bar{\xi}' + B_2 \bar{\eta}' + \bar{C}_2 = 0$$

Then
$$A_1 = A_2 \cos\theta - B_2 \sin\theta \qquad B_1 = A_2 \sin\theta + B_2 \cos\theta$$
$$\bar{A}_1 = \bar{A}_2 \cos\theta - \bar{B}_2 \sin\theta \qquad \bar{B}_1 = \bar{A}_2 \sin\theta + \bar{B}_2 \cos\theta$$
$$A_1\bar{A}_1 = A_2\bar{A}_2 \cos^2\theta - (\bar{A}_2 B_2 + A_2 \bar{B}_2)\cos\theta \sin\theta + B_2 \bar{B}_2 \sin^2\theta$$
$$B_1\bar{B}_1 = A_2\bar{A}_2 \sin^2\theta + (\bar{A}_2 B_2 + A_2 \bar{B}_2)\cos\theta \sin\theta + B_2 \bar{B}_2 \cos^2\theta$$
$$\therefore A_1\bar{A}_1 + B_1\bar{B}_1 = A_2\bar{A}_2 + B_2\bar{B}_2$$
$$\bar{A}_1 B_1 = (A_2\bar{A}_2 - B_2\bar{B}_2)\cos\theta \sin\theta + \bar{A}_2 B_2 \cos^2\theta - A_2\bar{B}_2 \sin^2\theta$$
$$A_1 \bar{B}_1 = (A_2\bar{A}_2 - B_2\bar{B}_2)\cos\theta \sin\theta - \bar{A}_2 B_2 \sin^2\theta + A_2\bar{B}_2 \cos^2\theta$$
$$\therefore \bar{A}_1 B_1 - A_1 \bar{B}_1 = \bar{A}_2 B_2 - A_2 \bar{B}_2$$

f we take for our parametric equations

$$\xi = A e^{i(pt+\psi)}$$
$$\eta = B e^{i(pt+\psi)}$$

re can bring these equations into the form of the equations of the invariant theorem as follows:

$$\xi B e^{i(pt+\psi)} - \eta A e^{i(pt+\varphi)} = 0$$
$$\bar{\xi} B e^{-i(pt+\psi)} - \bar{\eta} A e^{-i(pt+\varphi)} = 0$$

$\therefore A^2 + B^2$ is an invariant under a rotation

and $AB e^{i(\psi-\varphi)} - AB e^{-i(\psi-\varphi)} = 2i AB \sin(\psi-\varphi)$

$\therefore AB \sin(\psi - \varphi)$ is invariant.

III. SOME GENERAL CONSIDERATIONS OF POLARIZED LIGHT

Given two simple harmonic motions acting at right angles to each other.

Let $\xi = A \cos pt$

$\eta = B \cos (pt + \beta)$ where $\beta = (\psi - \varphi)$

Eliminating the time factor between these two equations gives

$$y - \frac{Bx}{A}\cos\beta = B\sqrt{1 - \frac{x^2}{A^2}}\sin\beta$$

or

$$\frac{x^2}{A^2} - \frac{2xy}{AB}\cos\beta + \frac{y^2}{B^2} = \sin^2\beta$$

which represents an ellipse except when $\sin\beta = 0$; when $\sin\beta = 0$ i. e., when $\beta = n\pi$, $\cos\beta = \pm 1$ and the equation then may be written

$$\left(\frac{x}{A} \mp \frac{y}{B}\right)^2 = 0$$ the equation of two straight lines.

If A = B, then the equation represents a circle, or two straight lines making an angle of 45° with the axes. Therefore, under certain conditions two simple harmonic motions at right angles to each other may be represented by an elliptic harmonic vibration. Similarly an elliptic harmonic vibration may be represented by two simple harmonic motions at right angles.

The ellipticity of an ellipse is given by the ratio of its axes such as $\frac{B}{A}$.

If θ equals the angle between the x axis and the major axis of the ellipse

$$\theta = \frac{1}{2} \tan^{-1} \frac{AB \cos \beta}{A^2 - B^2} .$$

A plane which is parallel to the optic axis of a crystal and perpendicular to the face through which the light enters is defined as the principal plane of that face. If a beam of light having vibrations equally in all directions falls upon a doubly refracting crystal, this crystal will resolve the light into two component beams, the vibrations of which are in one,parallel, and in the other, perpendicular, to the principal plane. Light restricted to a single plane of vibration is said to be polarized, or more specifically, plane polarized. In the direction of the optic axis of the crystal both waves travel with the same velocity, and double refraction fails. If the two components pass through a crystal slip cut in such a way that the optic axis is parallel to the surface, the two components will travel with different velocities, causing a difference in phase between them on emergence from the slip. This difference in phase is a function of the thickness of the crystal slip. When the difference in phase is $\frac{\pi}{2}$, it is called a quarter wave plate.

From the consideration of two simple harmonic motions acting at right angles to each other, it is readily seen that light may be elliptically polarized.

Two rays of light from the same source may be caused to interfere, while from different sources this phenomenon does not occur. In the first case, the light is called coherent, and in the latter, incoherent.

IV. INVARIANTS IN THE CASE OF A SINGLE BEAM OF POLARIZED LIGHT

1. Tuckerman

Tuckerman considers a plane wave of monochromatic elliptically polarized light falling normally on a series of plane parallel doubly refracting plates. The axes of reference chosen in each plate are the planes of polarization of the ordinary and extraordinary vibrations.

Let

$$\xi_1 = A_1 e^{i(pt+\varphi_1)}$$
$$\eta_1 = B_1 e^{i(pt+\psi_1)} \quad (1)$$

represent the beam of light referred to the first plate, where A_1 and B_1 represent the amplitudes of the ordinary and extraordinary vibrations, respectively, and $(\psi_1 - \varphi_1)$ the phase lag of the extraordinary over the ordinary.
If we submit ξ_1 and η_1 to a rotation through an $\angle \omega$, where ω represents the angle between the reference axes of the first and second plates, we obtain the displacements referred to the axes of the second plate. The passage of this light through a series of such plates rotates the components of the beam through an angle, $= \sum_{i=1}^{n} \omega_i$ for n plates.
Under a rotation, from the invariant theorem

$$A_1^2 + B_1^2 = \text{constant} \quad (2)$$

$A_1 B_1 \sin(\psi_1 - \varphi_1)$ = constant, that is, that the sum of the energies of the two wave components as well as the mutual energy between these two components are invariants under a rotation.

Following Tuckerman's notation, we will let

$$A^2 + B^2 = 2P \qquad AB\cos(\psi-\varphi) = K$$

$$A^2 - B^2 = 2Q \qquad AB\sin(\psi-\varphi) = S \qquad (3)$$

Therefore P and S are the invariants, but the value of Q and K vary under a rotation.

2. Poincaré

We have shown that by the elimination of the time factor from the parametric wave equations the resulting equation is that of an ellipse. Poincaré starts from this point of view.

He defines

$$\frac{\eta}{\xi} = u + iv = w$$

$$\frac{\bar{\eta}}{\bar{\xi}} = u - iv = \bar{w}$$

$$u^2 + v^2 = w\bar{w} = \frac{\eta\bar{\eta}}{\xi\bar{\xi}} = \frac{B^2}{A^2} = \varepsilon^2$$

$$u = \frac{w+\bar{w}}{2} = \frac{\eta\bar{\xi}+\xi\bar{\eta}}{2A^2} = \frac{AB\cos(\psi-\varphi)}{A^2} = \varepsilon\cos(\psi-\varphi)$$

$$v = \frac{w-\bar{w}}{2i} = \frac{\eta\bar{\xi}-\xi\bar{\eta}}{2iA^2} = \frac{AB\sin(\psi-\varphi)}{A^2} = \varepsilon\sin(\psi-\varphi)$$

Poincaré defines

$$w = u + iv = \frac{a+bt}{c+dt}$$

$$u^2 + v^2 = w\bar{w} = \frac{P_1}{P_4}$$

$$u = \frac{w+\bar{w}}{2} = \frac{P_2}{P_4}$$

$$v = \frac{w-\bar{w}}{2} = \frac{P_3}{P_4}$$

where P_1, P_2, P_3, P_4 are functions of a, b, c, d, and their conjugates and

Poincaré's $P_1 + P_2 = A^2 + B^2$ is invariant and corresponds to Tuckerman's $2P$

Poincaré's invariants are $P_1 + P_4$ and $P_3 = A\,B \sin(\psi - \varphi)$.

$$u + iv = \varepsilon\left[\cos(\psi-\varphi) + i\sin(\psi-\varphi)\right] = \varepsilon e^{i(\psi-\varphi)}$$

It can be clearly seen that

$$\varepsilon = f(u+iv) \quad \text{or} \quad u+iv = f(\varepsilon)$$

If $u = 0$, $\cos(\psi - \varphi) = 0$

$$\therefore \psi = \varphi \pm (2n+1)\frac{\pi}{2} \qquad v = \pm\frac{B}{A} = \pm\varepsilon$$

Thus when $u = 0$ the light is elliptically polarized.

If $\frac{\eta}{\xi} = \pm i$ i. e., when $\varepsilon = 1$ the points $v = \pm 1$ represent right or left circular vibrations.

When $v = 0$, $\sin(\psi - \varphi) = 0$

$$\therefore \psi = \varphi \pm n\pi \qquad u = \frac{\eta}{\xi}$$

This is the case of rectilinearly polarized light.

If the ray traverses a crystal slip where the principal sections are oriented as the coordinate axes; ξ and η are propagated with unequal velocities, their phases differ.

$$\frac{B}{A} = \frac{B_1}{A_1} e^{i\omega}$$ when ω is the phase difference introduced by the crystal slip.

Poincaré shows that since P_1, P_2, P_3, P_4 are not independent functions we have

$$C_1(u^2+v^2) + C_2 u + C_3 v + C_4 = 0$$

an equation of a circle as the locus of the point ellipses.

If the axes of the ellipse make an angle θ with the coordinate axes,

ξ and η are turned through an $\angle \theta$. From the foregoing considerations Poincaré shows that

$$\frac{\eta'}{\xi'} = \frac{-\xi \sin\theta + \eta\cos\theta}{\xi\cos\theta + \eta\sin\theta} = i\tau$$

when $\tau = u_1 + i v_1$

$$= \frac{u + iv - \tan\theta}{1 + (u+iv)\tan\theta}$$

If θ is kept constant while τ varies from $-\infty$ to $+\infty$ the point u, v describes a circle.

$$u_1^2 + v_1^2 = \frac{(u - \tan\theta)^2 + v^2}{(1+u\tan\theta)^2 + v^2\tan^2\theta}$$

$$u_1 = \frac{(u^2+v^2)\tan\theta + u(1-\tan^2\theta) - \tan\theta}{(1+u\tan\theta)^2 + v^2\tan^2\theta}$$

$$v_1 = \frac{v(1-\tan^2\theta)}{v^2\tan^2\theta + (1+u\tan\theta)^2}$$

if $u_1 = 0$ then

$$(u^2+v^2)+u\left(\frac{1-\tan^2\theta}{\tan\theta}\right)=1$$

the equation for a circle

When $\theta = \frac{\pi}{4}$ center of circle $(0,0)$.

If $v = 1$, then $u = 0$

If $v = 0$ $\quad u = \tan\theta$ or $-\frac{1}{\tan\theta}$

Similarly if γ is held constant and θ allowed to vary

$$\tan\theta = \frac{u+i(v-\gamma)}{(1-\gamma v)+i\gamma u} = u_2 + i v_2$$

$$u_2^2+v_2^2 = \frac{u^2+(v-\gamma)^2}{(1-\gamma v)^2+\gamma^2u^2}$$

$$u_2 = \frac{u(1-\gamma v)+\gamma u(v-\gamma)}{(1-\gamma v)^2+\gamma^2u^2}$$

$$v_2 = \frac{(v-\gamma)(1-\gamma v)-\gamma u^2}{(1-\gamma v)^2+\gamma^2u^2} = 0$$

$$\therefore (v-\gamma)(1-\gamma v) = \gamma u^2$$

$$v-\gamma-\gamma v^2+\gamma^2 v = \gamma u^2$$

$$u^2+v^2-v\left(\gamma+\frac{1}{\gamma}\right) = -1$$

equation for a circle

If $\gamma = 1$ then center of circle is $(0,1)$ a point circle

when $u = 0$, $v = \gamma$ or $\frac{1}{\gamma}$

i. e. when $\theta = 0$.

The two circles

$$\mu^2 + \upsilon^2 - \upsilon\left(\gamma + \frac{1}{\gamma}\right) = -1$$

$$\mu^2 + \upsilon^2 + \mu\left(\frac{1}{\tan\theta} - \tan\theta\right) = 1$$

intersect orthogonally.

The ellipticity is given by the value of υ and the angle θ by μ. The representation on a plane is then stereographically projected on a unit sphere, the origin O, being the point of contact of the sphere with the plane u, v. The μ axis is projected into a great circle called the equator, and the υ axis into a great circle orthogonal to the first called the first meridian.

Thus the two effects of double refraction and power of rotation when superposed may be represented by the rotation of the Poincaré sphere about some axis. Poincaré gives physical interpretations of groups of rotations on his sphere. Tuckerman has shown that the sphere defined by $Q^2 + K^2 + S^2 = P^2$ is the Poincaré sphere

or $$(\eta\bar{\eta} - \xi\bar{\xi})^2 + \left[\frac{\eta\bar{\xi} + \xi\bar{\eta}}{2}\right]^2 + \left[\frac{\eta\bar{\xi} - \xi\bar{\eta}}{2}\right]^2 = \left[\eta\bar{\eta} + \xi\bar{\xi}\right]^2$$

The point $S = P$, $Q = K = 0$ is chosen as the pole of the sphere, the plane $S = 0$ the equatorial plane.

Where $\sin l = \frac{S}{P} = \frac{2\varepsilon}{1+\varepsilon^2}$ $\qquad$ $\operatorname{tg} m = \frac{K}{Q} = \operatorname{tg} 2\theta$

or $\varepsilon = \operatorname{tg}\frac{1}{2} l$ $\qquad$ $m = 2\theta$

where l and m represent the latitude and longitude, respectively, of the point on the sphere.

3. Coherency Matrices of One Polarized Beam of Light

Wiener, in his work on Coherency Matrices and Quantum theory; in making a harmonic analysis of his wave functions, forms a function

$$R_{jk}(v) - R_{jk}(u) = 2\pi \sum_{h=p}^{h=q} A_{jh} \bar{A}_{kh}$$

where

$$f_j(t) = \sum_{1}^{m} A_{jh} e^{i\Lambda_h t}$$

The matrix $R_{jh}(-\Lambda_h + 0) - R_{jh}(-\Lambda_h - 0) = 2\pi A_{jh} \bar{A}_{kh}$

is defined as a coherency matrix.

Therefore, if we take for the $f_j(t)$ functions the parametric wave equations for polarized light the coherency matrix is

$$\begin{vmatrix} A\bar{A} & \bar{A} B e^{i(\psi-\varphi)} \\ A\bar{B} e^{-i(\psi-\varphi)} & B\bar{B} \end{vmatrix}$$

where $A\bar{A} + B\bar{B}$ and Δ are invariant.

$A\bar{A} + B\bar{B}$ as the energy of the two components equals Tuckerman's $2P$

Taking the real part of $e^{i(\psi-\varphi)}$ we then have

$$\begin{vmatrix} A\bar{A} & \bar{A} B \cos(\psi-\varphi) \\ A\bar{B}\cos(\psi-\varphi) & B\bar{B} \end{vmatrix}$$

$\Delta = A\bar{A}\, B\bar{B} \sin^2(\psi-\varphi)$

If $A = \bar{A}$ $B = \bar{B}$

$\Delta = [A B \sin(\psi-\varphi)]^2 = S^2$

that is, Δ equals the mutual energy squared.

V. THEORY OF TWO POLARIZED BEAMS OF LIGHT BY TUCKERMAN'S METHOD

In this part we consider two monochromatic elliptically polarized rays of light falling normally on a series of plane parallel doubly refracting plates.

Let
$$\xi_1 = A_1 e^{i(pt+\varphi_1)} \qquad \xi_2 = A_2 e^{i(pt+\varphi_2)}$$
$$\eta_1 = B_1 e^{i(pt+\psi_1)} \qquad \eta_2 = B_2 e^{i(pt+\psi_2)}$$
$$\xi = \xi_1 + \xi_2 \qquad \eta = \eta_1 + \eta_2$$

Considering the real parts only
$$\xi = A_1 \cos(pt+\varphi_1) + A_2 \cos(pt+\varphi_2)$$
$$\xi = A \cos\alpha \cos pt - A \sin\alpha \sin pt$$
$$\xi = A \cos(pt+\alpha)$$

where
$$A\cos\alpha = A_1\cos\varphi_1 - A_2\cos\varphi_2$$
$$A\sin\alpha = A_1\sin\varphi_1 + A_2\sin\varphi_2$$
$$\therefore\ A^2 = A_1^2 + 2A_1A_2\cos(\varphi_2-\varphi_1) + A_2^2$$
$$= A_{11} + 2A_{12} + A_{22}$$

Similarly
$$\eta = B\cos(pt+\beta)$$

where
$$B\cos\beta = B_1\cos\psi_1 + B_2\cos\psi_2$$
$$B\sin\beta = B_1\sin\psi_1 + B_2\sin\psi_2$$
$$\therefore\ B^2 = B_1^2 + 2B_1B_2\cos(\psi_2-\psi_1) + B_2^2$$
$$= B_{11} + 2B_{12} + B_{22}$$
$$AB\cos(\beta-\alpha) = A_1B_1\cos(\psi_1-\varphi_1) + A_1B_2\cos(\psi_2-\varphi_1) + A_2B_1\cos(\varphi_2-\psi_1) + A_2B_2\cos(\psi_2-\varphi_2)$$
$$= K_{11} + K_{12} + K_{21} + K_{22} = K$$

$$AB \sin(\beta - \alpha) = A_1 B_1 \sin(\psi_1 - \varphi_1) + A_1 B_2 \sin(\psi_2 - \varphi_1) + A_2 B_1 \sin(\varphi_2 - \psi_1) + A_2 B_2 \sin(\psi_2 - \varphi_2)$$

$$K_{11} = A_1 B_1 \cos(\psi_1 - \varphi_1)$$

$$K_{12} = A_1 B_2 \cos(\psi_2 - \varphi_1)$$

$$K_{21} = A_2 B_1 \cos(\varphi_2 - \psi_1)$$

$$K_{22} = A_2 B_2 \cos(\psi_2 - \varphi_2)$$

$$S_{11} = A_1 B_1 \sin(\psi_1 - \varphi_1)$$

$$S_{12} = A_1 B_2 \sin(\psi_2 - \varphi_1)$$

$$S_{21} = A_2 B_1 \sin(\varphi_2 - \psi_1)$$

$$S_{22} = A_2 B_2 \sin(\psi_2 - \varphi_2)$$

$$A_{11} = A_1^2$$

$$A_{12} = A_1 A_2 \cos(\varphi_2 - \varphi_1)$$

$$A_{21} = A_{12}$$

$$A_{22} = A_2^2$$

$$B_{11} = B_1^2$$

$$B_{12} = B_1 B_2 \cos(\psi_2 - \psi_1)$$

$$B_{21} = B_{12}$$

$$B_{22} = B_2^2$$

$$\alpha_{12} = A_1 A_2 \sin(\varphi_2 - \varphi_1)$$

$$\beta_{12} = B_1 B_2 \sin(\psi_2 - \psi_1)$$

Energy density of rays of light

Let $\psi_1 = \xi_1 + i\eta_1$

$\bar{\psi}_1 = \xi_1 - i\eta_1$

$$\psi_1\bar{\psi}_1 = \xi_1\bar{\xi}_1 + \eta_1\bar{\eta}_1 + i(\bar{\xi}_1\eta_1 - \xi_1\bar{\eta}_1)$$

$$= A_1^2 + B_1^2 + 2A_1B_1 \sin(\psi_1 - \varphi_1)$$

$$= 2(P_1 + S_{11})$$

Similarly $\psi_2\bar{\psi}_2 = A_2^2 + B_2^2 + 2A_2B_2 \sin(\psi_2 - \varphi_2) = 2(P_2 + S_{22})$

We therefore find that the energy density of the rays of light is invariant under a rotation. The function $\psi\bar{\psi}$ is defined as light energy function and is similar in this theory to $\psi\bar{\psi}$ function of Schroedinger which is defined in the quantum theory as electric density, and is shown to be an invariant.

In the case of two rays of light, we

Let $\psi = \psi_1 + \psi_2$

$\bar{\psi} = \bar{\psi}_1 + \bar{\psi}_2$

$$\psi\bar{\psi} = \psi_1\bar{\psi}_1 + \psi_2\bar{\psi}_1 + \psi_1\bar{\psi}_2 + \psi_2\bar{\psi}_2$$

$$\psi_1\bar{\psi}_2 = (\xi_1 + i\eta_1)(\bar{\xi}_2 - i\bar{\eta}_2)$$

$$= \xi_1\bar{\xi}_2 + \eta_1\bar{\eta}_2 + i(\eta\bar{\xi}_2 - \bar{\eta}_2\xi_1)$$

$$\bar{\psi}_1\psi_2 = (\bar{\xi}_1 - i\bar{\eta}_1)(\xi_2 + i\eta_2)$$

$$= \bar{\xi}_1\xi_2 + \bar{\eta}_1\eta_2 + i(\bar{\xi}_1\eta_2 - \xi_2\bar{\eta}_1)$$

$$\psi_1\bar{\psi}_2 + \bar{\psi}_1\psi_2 = 2A_1A_2\cos(\varphi_2 - \varphi_1) + 2B_1B_2\cos(\psi_2 - \psi_1)$$

$$+ 2A_1B_2\sin(\psi_2 - \varphi_1) + 2A_2B_1\sin(\varphi_2 - \psi_1)$$

$$= 2A_{12} + 2B_{12} + 2S_{12} + 2S_{21}$$

$$\psi\bar{\psi} = A_{11} + B_{11} + 2S_{11} + 2A_{12} + 2B_{12} + 2S_{12} + 2S_{21}$$

$$+ A_{22} + B_{22} + 2S_{22}$$

$$= A^2 + B^2 + 2S$$

$$= 2(P + S)$$

At this point it is interesting to note that the optical density is equal to the sum of the energies of the two components increased by twice the mutual energies. This relation is analogous to the relation of inductance in coupled circuits where

$$\text{Inductance} = L_1 + L_2 + 2M$$

When L_1 and L_2 represent the separate inductances and M, the mutual inductance, is a function of L_1 and L_2.

Since $\Psi\bar{\Psi}$ is energy density, its relation is more nearly analogous to the energy relations in a directly coupled electrical circuit.

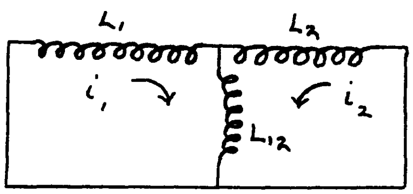


Diagram 1.

$$\text{Energy} = \frac{1}{2} L_{11} i_1^2 + \frac{1}{2} L_{22} i_2^2 + L_{12} i_1 i_2 = W$$

Let $L_{11} = L_1 + L_{12}$

$$L_{22} = L_2 + L_{12}$$

$$W = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_{12}(i_1 + i_2)^2 + \frac{1}{2} L_2 i_2^2$$

$$= \frac{1}{2}(L_1 + L_{12}) i_1^2 + L_{12} i_1 i_2 + \frac{1}{2}(L_2 + L_{12}) i_2^2$$

$$= \frac{1}{2} L_{11} i_1^2 + L_{12} i_1 i_2 + \frac{1}{2} L_{22} i_2^2$$

If the currents are transformed linearly the quadratic form is invariant. Given an infinite system of net works, with same instantaneous current, the total instantaneous energy is the same.

Let $i_1 = a_{11} i_1'$

$$i_2 = a_{21} i_1' + a_{22} i_2'$$

$$\frac{1}{2} L_{11} (a_{11} i_1')^2 + L_{12} (a_{11} i_1')(a_{12} i_1' + a_{22} i_2') + \frac{1}{2} L_{22} (a_{12} i_1' + a_{22} i_2')^2 =$$
$$\frac{1}{2} (i_1')^2 (L_{11} a_{11}^2 + 2 L_{12} a_{11} a_{12} + L_{12} a_{12}^2) +$$
$$i_1' i_2' a_{22} (L_{12} a_{11} + L_{22} a_{12}) + \frac{1}{2} (i_2')^2 L_{22} a_{22}^2$$

Thus the quadratic form is preserved,

new inductance

$$L_{11} = a_{11}^2 L_{11} + 2 L_{12} a_{11} a_{12} + L_{22} a_{12}^2$$
$$L_{12} = a_{22} (L_{12} a_{11} + L_{22} a_{12})$$
$$L_{22} = a_{22}^2 L_{22}$$

We find in the case of two rays of light that instead of four invariants we have six invariants.

$A_{11} + B_{11}$	$A_{12} + B_{12}$	S_{11}
$A_{22} + B_{22}$	$S_{12} + S_{21}$	S_{22}

for $A^2 + B^2 = 2 P_1 + 2 P_2 + 2(A_{12} + B_{12})$

which is invariant, and

$$S = S_{11} + S_{12} + S_{21} + S_{22}$$

and consequently $(A_{12} + B_{12})$ and $(S_{12} + S_{21})$ are invariant.

In the case of two rays of light we have the energy of each component of each beam and the mutual energies between the different components as invariants. The mutual energy between the components of a single beam, and the sum of the mutual energies between non-corresponding components of the two rays are invariant, i. e.,

$$S_{12} + S_{21} \text{ and } A_{12} + B_{12}$$

VI. INTERFERENCE OF LIGHT RAYS

In the problem of two rays of light a new element is introduced - the conditions for interference.

If $\varphi_2 = \varphi_1$ i. e., the rays are in phase, then $A^2 = (A_1 + A_2)^2$ In this case the rays are in resonance.

Suppose that $\varphi_2 = \varphi_1 \pm \pi$ then the rays are in interference

$$A^2 = (A_1 - A_2)^2$$

If $A_1 = A_2$, $A^2 = 0$ and complete interference results.

If $\varphi_2 = \varphi_1 \pm \frac{\pi}{2}$

$$A^2 = A_1^2 + A_2^2$$

In this case these rays are in quadrature.

If the two rays have the same ellipticity

$$\frac{A_1}{B_1} = \frac{A_2}{B_2}$$

Interference may occur in the case of two simple harmonic motions that are in the same straight line. Therefore it is the phase relation between φ_1 and φ_2 and ψ_1 and ψ_2 that must be investigated. For interference

$$\varphi_2 = \varphi_1 \pm \pi \qquad \psi_2 = \psi_1 \pm \pi$$

$$A^2 = (A_1 - A_2)^2 \qquad B^2 = (B_1 - B_2)^2$$

VII. COHERENCY MATRICES FOR TWO SIMULTANEOUS POLARIZED RAYS OF LIGHT

1. Coherent Light

In this case

$$\xi_1 = f_1(t) = A_1 e^{i(pt+\varphi_1)} = A_1 e^{i\varphi_1} e^{ipt}$$

$$\eta_1 = f_2(t) = B_1 e^{i(pt+\psi_1)} = B_1 e^{i\psi_1} e^{ipt}$$

$$\xi_2 = f_3(t) = A_2 e^{i(pt+\varphi_2)} = A_2 e^{i\varphi_2} e^{ipt}$$

$$\eta_2 = f_4(t) = B_2 e^{i(pt+\psi_2)} = B_2 e^{i\psi_2} e^{ipt}$$

$$f_j(t) = \sum_1^4 A_{jh} e^{i\Lambda_h t} \qquad \text{where } \Lambda_h = \Lambda_l = pt$$

$$\varphi_{jk}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{h,l=1}^{m} A_{jh} \bar{A}_{kl} e^{i\Lambda_h \tau} e^{i(\Lambda_h - \Lambda_l)t}\, dt$$

$$= \sum_{h=1}^{m} A_{jh} \bar{A}_{kh} e^{i\Lambda_h \tau} \qquad h = l$$

The general coherency matrix for this case is then

$$\left|\begin{array}{llll} A_1\bar{A}_1 & \bar{A}_1 B_1 e^{i(\psi_1-\varphi_1)} & A_1 A_2 e^{i(\varphi_2-\varphi_1)} & \bar{A}_1 A_2 e^{i(\psi_2-\varphi_1)} \\ A_1 \bar{B}_1 e^{-i(\psi_1-\varphi_1)} & B_1 \bar{B}_1 & A_2 B_1 e^{i(\varphi_2-\psi_1)} & \bar{B}_1 B_2 e^{i(\psi_2-\psi_1)} \\ A_1 \bar{A}_2 e^{-i(\varphi_2-\varphi_1)} & \bar{A}_2 B_1 e^{-i(\varphi_2-\psi_1)} & A_2 \bar{A}_2 & \bar{A}_2 B_2 e^{i(\psi_2-\varphi_2)} \\ A_1 \bar{B}_2 e^{-i(\psi_2-\varphi_1)} & B_1 \bar{B}_2 e^{-i(\psi_2-\psi_1)} & A_2 \bar{B}_2 e^{-i(\psi_2-\varphi_2)} & B_2 \bar{B}_2 \end{array}\right|$$

$A_1\bar{A}_1 + B_1\bar{B}_1 + A_2\bar{A}_2 + B_2\bar{B}_2$ is invariant as well as the determinant of the matrix $\Delta = 0$.

Taking the real part of $e^{i(pt+\psi)}$ the matrix then takes the following form

$$\begin{vmatrix} A_{11} & K_{11} & A_{12} & K_{12} \\ K_{11} & B_{11} & K_{21} & B_{12} \\ A_{12} & K_{21} & A_{22} & K_{22} \\ K_{12} & B_{12} & K_{22} & B_{22} \end{vmatrix} = \Delta$$

$$\Delta = -4 S_{11} S_{12} S_{21} S_{22}$$

We have previously shown that

S_{11}, S_{22} and $S_{12} + S_{21}$ are invariant

$\therefore$ S_{12} and S_{21} are invariant

It has been shown that

$A_{11} + B_{11} + A_{22} + B_{22} + 2A_{12} + 2B_{12}$ is invariant.

In this case then the value of the determinant as well as the sum of the terms in the main diagonal of the matrix, and of the two minors that contain no terms of the main diagonal, are invariant.

2. Incoherent Light

In this case, $h \neq l$ in the $\varphi_{\gamma\kappa}(\tau)$. This means that there is no persistent definite frequency relation between the corresponding components of the two polarized rays of light, i. e., this is the case of two incoherent beams of polarized light.

$$\xi_1 = f_1(t)$$

$$\eta_1 = f_2(t)$$

$$\xi_2' = A_2 e^{i(p't+\varphi_2)}$$

$$\eta_2' = B_2 e^{i(p't+\psi_2)}$$

$$\varphi_{jk}(\tau) = \sum_{h=1}^{m} A_{jl} \bar{A}_{kh} e^{i\lambda_h \tau} \qquad h = l$$

$$= 0 \qquad h \neq l$$

The following eight terms become equal to zero:

$$A_{3h}\bar{A}_{1h} = 0 \qquad A_{1h}\bar{A}_{3h} = 0$$

$$A_{4h}\bar{A}_{1h} = 0 \qquad A_{2h}\bar{A}_{3h} = 0$$

$$A_{3h}\bar{A}_{2h} = 0 \qquad A_{1h}\bar{A}_{4h} = 0$$

$$A_{4h}\bar{A}_{2h} = 0 \qquad A_{2h}\bar{A}_{4h} = 0$$

The coherency matrix takes the following form:

$$\begin{vmatrix} A_1\bar{A}_1 & \bar{A}_1 B_1 e^{i(\psi_1-\varphi_1)} & 0 & 0 \\ A_1\bar{B}_1 e^{-i(\psi_1-\varphi_1)} & B_1\bar{B}_1 & 0 & 0 \\ 0 & 0 & A_2\bar{A}_2 & \bar{A}_2 B_2 e^{i(\psi_2-\varphi_2)} \\ 0 & 0 & A_2\bar{B}_2 e^{-i(\psi_2-\varphi_2)} & B_2\bar{B}_2 \end{vmatrix}$$

where $\Delta = 0$.

Taking the real part, we have

$$\begin{vmatrix} A_{11} & K_{11} & 0 & 0 \\ K_{11} & B_{11} & 0 & 0 \\ 0 & 0 & A_{22} & K_{22} \\ 0 & 0 & K_{22} & B_{22} \end{vmatrix}$$

where $\Delta = S_{11}^2 \; S_{22}^2$, which is an invariant since S_{11} and S_{22} have previously been shown invariant.

Thus it is seen that the value of the determinant of the matrix for two incoherent beams of polarized light equals the product of the value of the determinants of the matrices of each beam of polarized light considered separately. The other invariants for this case are:

$$P_{11} = \frac{1}{2}(A_{11} + B_{11})$$

$$P_{22} = \frac{1}{2}(A_{22} + B_{22})$$

Thus the determinant of the matrix and the sum of its diagonal terms are invariants for this case.

VIII. SIXTEEN FUNDAMENTAL COHERENCY MATRICES

Sixteen matrices representing different types of coherent and incoherent polarized or unpolarized light are made the basis for the following study, the energy of the components taken equal to unity. The matrices I_i represent incoherent light and those called C_i coherent light. At this point it is interesting to note that if one unit is subtracted from each term in the main diagonal, the resulting matrices give the sixteen Dirac matrices.

The rules of combination for these matrices are

$$I_i \; I_i = 2 \; I_i$$

$$C_i \; C_i = 2 \; C_i$$

$$I_i \; I_j = \overline{\widetilde{I_j \; I_i}}$$

$$C_i \; C_j = \overline{\widetilde{C_j \; C_i}}$$

$$I_i \; C_j = \overline{\widetilde{C_j \; I_i}}$$

The resulting matrix represents the combination of four beams of light. It is therefore to be expected physically that $I_i \; I_i = 2 \; I_i$.

Table 1. FUNDAMENTAL MATRICES

$$I_1 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

Two rays plane polarized at 45°

$$I_5 = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

Two rays unpolarized

$$I_2 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

One ray plane polarized at 45°, the other at 135°

$$I_6 = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Two rays polarized at 0

$$I_3 = \begin{vmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \end{vmatrix}$$

Two rays circularly polarized with same sense of rotation

$$I_7 = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

One ray unpolarized

$$I_4 = \begin{vmatrix} 1 & -i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \end{vmatrix}$$

Two rays circularly polarized with opposite sense of rotation

$$I_8 = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

Two rays plane polarized in opposite directions

(Table 1. Cont.)

$$C_1 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}$$

Two rays unpolarized in resonance

$$C_5 \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

Two rays unpolarized plane coupling at 45°

$$C_2 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

Two rays unpolarized one pair of components in resonance, others interfering

$$C_6 \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}$$

Two rays unpolarized plane coupling at 135°

$$C_3 \begin{vmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{vmatrix}$$

Two rays unpolarized both pairs of components in same quadrature

$$C_7 \begin{vmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{vmatrix}$$

Two rays unpolarized circularly coupled

$$C_4 \begin{vmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{vmatrix}$$

Two rays unpolarized both pairs of components in quadrature, the quadrature differing by π

$$C_8 \begin{vmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{vmatrix}$$

Two rays unpolarized circularly coupled with opposite senses

Table 2.

PHASE RELATIONS FOR THE SIXTEEN FUNDAMENTAL MATRICES

	K_{11}	A_{12}	K_{12}	K_{21}	B_{12}	K_{22}
	$\psi_1 - \varphi_1$	$\varphi_2 - \varphi_1$	$\psi_2 - \varphi_1$	$\varphi_2 - \psi_1$	$\psi_2 - \psi_1$	$\psi_2 - \varphi_2$
I_1	$\psi_1 = \varphi_1$					$\psi_2 = \varphi_2$
I_2	$\psi_1 = \varphi_1$					$\psi_2 = \varphi_2 + \pi$
I_3	$\psi_1 = \varphi_1 + \frac{3\pi}{2}$					$\psi_2 = \varphi_2 + \frac{3\pi}{2}$
I_4	$\psi_1 = \varphi_1 + \frac{3\pi}{2}$					$\psi_2 = \varphi_2 + \frac{\pi}{2}$
I_5						
I_6						
I_7						
I_8						
C_1		$\varphi_2 = \varphi_1$			$\psi_2 = \psi_1$	
C_2		$\varphi_2 = \varphi_1$			$\psi_2 = \psi_1 + \pi$	
C_3		$\varphi_2 = \varphi_1 + \frac{3\pi}{2}$			$\psi_2 = \psi_1 + \frac{3\pi}{2}$	
C_4		$\varphi_2 = \varphi_1 + \frac{3\pi}{2}$			$\psi_2 = \psi_1 + \frac{\pi}{2}$	
C_5			$\psi_2 = \varphi_1$	$\varphi_2 = \psi_1$		
C_6			$\psi_2 = \varphi_1 + \pi$	$\varphi_2 = \psi_1$		
C_7			$\psi_2 = \varphi_1 + \frac{3\pi}{2}$	$\varphi_2 = \psi_1 + \frac{3\pi}{2}$		
C_8			$\psi_2 = \varphi_1 + \frac{3\pi}{2}$	$\varphi_2 = \psi_1 + \frac{\pi}{2}$		

IX. THE GROUPS OF TRANSFORMATION MATRICES

The next step is to determine the transformations which will transform the matrix representing one kind of light into a different kind, that is, to determine M in the following relations

$$I_i = M_a I_j M_a^{-1}$$
$$I_i = M_b C_j M_b^{-1}$$
$$C_i = M_c C_j M_c^{-1}$$

These transformation matrices divided into two main types which will be called the β type and the φ type. To the β type belong the β, δ and γ matrices. To the φ type belong the φ, ρ and μ. The η matrices are more nearly like the β matrices but do not belong to the group.

1. The β Class

In determining the matrices of this group it was found that a choice from twenty-four possible types of matrices must be made, that is, in satisfying the conditions imposed by the transformations, twenty-four possible types were involved. These twenty-four formed

Table 3

TRANSFORMATION MATRICES

	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
I_1	β_1	β_2	β_{29}^{22}	β_{31}^{21}	${}^{rv}_{a}\varphi_1$	${}^{rv}_{b}\varphi_2$	${}^{rv}_{c}\varphi_3$	${}^{rv}_{d}\varphi_4$	γ_1	γ_2	γ_{21}^{30}	γ_{22}^{29}	δ_1	δ_5	δ_{22}^{29}	δ_{21}^{30}
I_2	β_2	β_1	β_{31}^{21}	β_{29}^{22}	${}^{rv}_{c}\varphi_3$	${}^{rv}_{d}\varphi_4$	${}^{rv}_{a}\varphi_1$	${}^{rv}_{b}\varphi_2$	γ_2	γ_1	γ_{22}^{29}	γ_{21}^{30}	δ_2	δ_6	δ_{21}^{30}	δ_{22}^{29}
I_3	β_{27}^{17}	β_{25}^{18}	β_1	β_2	${}^{iv}_{a}\varphi_1$	${}^{iv}_{b}\varphi_2$	${}^{iv}_{c}\varphi_3$	${}^{iv}_{d}\varphi_4$	γ_{17}^{26}	γ_{18}^{25}	γ_2	γ_1	δ_{17}^{26}	δ_{21}^{30}	δ_1	δ_2
I_4	β_{25}^{18}	β_{27}^{17}	β_2	β_1	${}^{iv}_{c}\varphi_3$	${}^{iv}_{d}\varphi_4$	${}^{iv}_{a}\varphi_1$	${}^{iv}_{b}\varphi_2$	γ_{18}^{25}	γ_{17}^{26}	γ_1	γ_2	δ_{18}^{25}	δ_{22}^{29}	δ_2	δ_1
I_5	${}^{rh}_{a}\varphi_1$	${}^{rh}_{c}\varphi_3$	${}^{ih}_{a}\varphi_1$	${}^{ih}_{b}\varphi_2$	β_1	η_1	η_2	η_3	${}^{rh}_{a}\rho_1$	${}^{rh}_{b}\rho_2$	${}^{ih}_{a}\rho_1$	${}^{ih}_{b}\rho_2$	${}^{rh}_{a}\mu_1$	${}^{rh}_{b}\mu_2$	${}^{ih}_{a}\mu_1$	${}^{ih}_{b}\mu_2$
I_6	${}^{rh}_{b}\varphi_2$	${}^{rh}_{d}\varphi_4$	${}^{ih}_{c}\varphi_3$	${}^{ih}_{d}\varphi_4$	η_1	β_1	η_3	η_2	${}^{rh}_{b}\rho_2$	${}^{rh}_{a}\rho_1$	${}^{ih}_{b}\rho_2$	${}^{ih}_{a}\rho_1$	${}^{rh}_{c}\mu_3$	${}^{rh}_{d}\mu_4$	${}^{ih}_{c}\mu_3$	${}^{ih}_{d}\mu_4$
I_7	${}^{rh}_{c}\varphi_3$	${}^{rh}_{a}\varphi_1$	${}^{ih}_{b}\varphi_2$	${}^{ih}_{a}\varphi_1$	η_2	η_3	β_1	η_1	${}^{rh}_{c}\rho_3$	${}^{rh}_{d}\rho_4$	${}^{ih}_{c}\rho_3$	${}^{ih}_{d}\rho_4$	${}^{rh}_{b}\mu_2$	${}^{rh}_{a}\mu_1$	${}^{ih}_{d}\mu_4$	${}^{ih}_{c}\mu_3$
I_8	${}^{rh}_{d}\varphi_4$	${}^{rh}_{b}\varphi_2$	${}^{ih}_{d}\varphi_4$	${}^{ih}_{c}\varphi_3$	η_3	η_2	η_1	β_1	${}^{rh}_{d}\rho_4$	${}^{rh}_{c}\rho_3$	${}^{ih}_{d}\rho_4$	${}^{ih}_{c}\rho_3$	${}^{rh}_{d}\mu_4$	${}^{rh}_{c}\mu_3$	${}^{ih}_{b}\mu_2$	${}^{ih}_{a}\mu_1$
C_1	δ_1	δ_2	δ_{11}^{28}	δ_{12}^{27}	${}^{rv}_{a}\rho_1$	${}^{rv}_{b}\rho_2$	${}^{rv}_{c}\rho_3$	${}^{rv}_{d}\rho_4$	β_1	β_2	β_{27}^{12}	β_{31}^{11}	γ_1	γ_3	γ_{11}^{28}	γ_{12}^{27}
C_2	δ_2	δ_1	δ_{12}^{27}	δ_{11}^{28}	${}^{rv}_{b}\rho_2$	${}^{rv}_{a}\rho_1$	${}^{rv}_{d}\rho_4$	${}^{rv}_{c}\rho_3$	β_2	β_1	β_{31}^{11}	β_{27}^{12}	γ_2	γ_4	γ_{12}^{27}	γ_{11}^{28}
C_3	δ_9^{26}	δ_{10}^{25}	δ_2	δ_1	${}^{iv}_{a}\rho_1$	${}^{iv}_{b}\rho_2$	${}^{iv}_{c}\rho_3$	${}^{iv}_{d}\rho_4$	β_{29}^{9}	β_{25}^{10}	β_1	β_2	γ_9^{26}	γ_{11}^{28}	γ_2	γ_1
C_4	δ_{10}^{25}	δ_9^{26}	δ_1	δ_2	${}^{iv}_{b}\rho_2$	${}^{iv}_{a}\rho_1$	${}^{iv}_{d}\rho_4$	${}^{iv}_{c}\rho_3$	β_{25}^{10}	β_{29}^{9}	β_2	β_1	γ_{10}^{25}	γ_{12}^{27}	γ_1	γ_2
C_5	γ_1	γ_3	γ_{12}^{18}	γ_{10}^{20}	${}^{rv}_{a}\mu_1$	${}^{rv}_{b}\mu_2$	${}^{rv}_{c}\mu_3$	${}^{rv}_{d}\mu_4$	δ_1	δ_3	δ_{10}^{20}	δ_{12}^{18}	β_1	β_2	β_{18}^{12}	β_{22}^{10}
C_6	γ_2	γ_4	γ_{11}^{17}	γ_9^{19}	${}^{rv}_{c}\mu_3$	${}^{rv}_{d}\mu_4$	${}^{rv}_{a}\mu_1$	${}^{rv}_{b}\mu_2$	δ_2	δ_4	δ_9^{19}	δ_{11}^{17}	β_2	β_1	β_{17}^{11}	β_{21}^{9}
C_7	γ_9^{19}	γ_{11}^{17}	γ_1	γ_3	${}^{iv}_{a}\mu_1$	${}^{iv}_{b}\mu_2$	${}^{iv}_{c}\mu_3$	${}^{iv}_{d}\mu_4$	δ_9^{19}	δ_{11}^{17}	δ_3	δ_1	β_{21}^{9}	β_{22}^{10}	β_1	β_5^{3}
C_8	γ_{11}^{17}	γ_9^{19}	γ_3	γ_1	${}^{iv}_{d}\mu_4$	${}^{iv}_{c}\mu_3$	${}^{iv}_{b}\mu_2$	${}^{iv}_{a}\mu_1$	δ_{11}^{17}	δ_9^{19}	δ_1	δ_3	β_{17}^{11}	β_{18}^{12}	β_5^{3}	β_1

a non Abelian closed group. This group was resolved into a group of four, three and two. The four group is the vierer gruppe. The types in this group are the four forms of the Dirac matrices. The three group is irreducible except by using imaginaries. The types in this group were selected for the transformation matrices and are called β, γ and δ. No other set of these matrices satisfying the imposed conditions formed a closed group. Each type of the twenty-four represents thirty-two elements. The β group is a closed Abelian group. This group can be resolved into three two group,and one four group.

Table 4.
TYPES OF TRANSFORMATION MATRICES

Reducible to Three Groups: (1,2,3,4) (1, 13, 21) (1,5)
Dirac Matrices 1,2,3,4. Transformation Matrices 1,13,21

1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
0	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0
0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0
1				2				3				4			

1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	0
0	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0
5				6				7				8			

1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0
0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0
9				10				11				12			

1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	0	0	1	0	0	0	1	0	1	0	0	0	1	0	0
0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
0	0	1	0	0	0	1	0	1	0	0	0	1	0	0	0
13				14				15				16			

1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	0	0	1	0	0	0	1	1	0	0	0	1	0	0	0
0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0
0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
17				18				19				20			

1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1
0	0	1	0	0	0	1	0	0	1	0	0	0	1	0	0
0	0	0	1	0	0	0	1	1	0	0	0	1	0	0	0
0	1	0	0	1	0	0	0	0	0	0	1	0	0	1	0
21				22				23				24			

Table 5

GROUP TABLE OF TYPES OF TRANSFORMATION MATRICES

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	2	1	4	3	6	5	8	7	11	15	9	13	12	16	10	14	20	24	21	17	19	23	22	18
3	3	4	1	2	8	7	6	5	14	13	16	15	10	9	12	11	23	19	18	22	24	20	17	21
4	4	3	2	1	7	8	5	6	16	12	14	10	15	11	13	9	22	21	24	23	18	17	20	19
5	5	6	7	8	1	2	3	4	21	22	19	20	17	18	23	24	13	14	11	12	9	10	15	16
6	6	5	8	7	2	1	4	3	19	23	21	17	20	24	22	18	12	16	9	13	11	15	10	14
7	7	8	5	6	4	3	2	1	18	17	24	23	22	21	20	19	15	11	14	10	16	12	13	9
8	8	7	6	5	3	4	1	2	24	20	18	22	23	19	17	21	10	9	16	15	14	13	12	11
9	9	14	11	16	13	10	15	12	1	6	3	8	5	2	7	4	21	22	23	24	17	18	19	20
10	10	13	12	15	14	9	16	11	23	19	17	21	24	20	18	22	8	4	1	5	3	7	6	2
11	11	16	9	14	12	15	10	13	2	5	4	7	6	1	8	3	19	23	22	18	20	24	21	17
12	12	15	10	13	11	16	9	14	20	24	22	18	19	23	21	17	6	1	4	7	2	5	8	3
13	13	10	15	12	9	14	11	16	17	18	23	24	21	22	19	20	5	2	3	8	1	6	7	4
14	14	9	16	11	10	13	12	15	3	7	1	5	8	4	6	2	24	20	17	21	23	19	18	22
15	15	12	13	10	16	11	14	9	22	21	20	19	18	17	24	23	7	3	2	6	4	8	5	1
16	16	11	14	9	15	12	13	10	4	8	2	6	7	3	5	1	18	17	20	19	22	21	24	23
17	17	22	23	20	21	18	19	24	13	14	15	16	9	10	11	12	1	6	7	4	5	2	3	8
18	18	21	24	19	22	17	20	23	7	3	5	1	4	8	2	6	16	12	13	9	15	11	14	10
19	19	24	21	18	20	23	22	17	6	1	8	3	2	5	4	7	11	15	10	14	12	16	9	13
20	20	23	22	17	19	24	21	18	12	16	10	14	11	15	9	13	2	5	8	3	6	1	4	7
21	21	18	19	24	17	22	23	20	5	2	7	4	1	6	3	8	9	10	15	16	13	14	11	12
22	22	17	20	23	18	21	24	19	15	11	13	9	16	12	14	10	4	8	5	1	7	3	2	6
23	23	20	17	22	24	19	18	21	10	9	12	11	14	13	16	15	3	7	6	2	8	4	1	5
24	24	19	18	21	23	20	17	22	8	4	6	2	3	7	1	5	14	13	12	11	10	9	16	15

Table 6. THE β MATRICES
Number 1 of Group of Type Matrices

$$\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{vmatrix}$$

1 2 3 4

$$\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{vmatrix}$$

5 6 7 8

$$\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&i&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&i&0\\0&0&0&-i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&-i&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&-i&0\\0&0&0&-i\end{vmatrix}$$

9 10 11 12

$$\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&i&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&i&0\\0&0&0&-i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-i&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-i&0\\0&0&0&-i\end{vmatrix}$$

13 14 15 16

$$\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&1&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&1&0\\0&0&0&-i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&-1&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&-1&0\\0&0&0&-i\end{vmatrix}$$

17 18 19 20

$$\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&1&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&1&0\\0&0&0&-i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&-1&0\\0&0&0&i\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&-1&0\\0&0&0&-i\end{vmatrix}$$

21 22 23 24

$$\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&-1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&-1\end{vmatrix}$$

25 26 27 28

$$\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&-1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&-i&0\\0&0&0&1\end{vmatrix}\quad\begin{vmatrix}1&0&0&0\\0&-i&0&0\\0&0&-i&0\\0&0&0&-1\end{vmatrix}$$

29 30 31 32

Table 7. THE δ MATRICES
Number 13 of the Group of Type of Matrices

$$\underset{1}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{array}}\quad\underset{2}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&-1&0\end{array}}\quad\underset{3}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&-1&0&0\\0&0&1&0\end{array}}\quad\underset{4}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&-1&0&0\\0&0&-1&0\end{array}}$$

$$\underset{5}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&1&0&0\\0&0&1&0\end{array}}\quad\underset{6}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&1&0&0\\0&0&-1&0\end{array}}\quad\underset{7}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&-1&0&0\\0&0&1&0\end{array}}\quad\underset{8}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&-1&0&0\\0&0&-1&0\end{array}}$$

$$\underset{9}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&i&0&0\\0&0&i&0\end{array}}\quad\underset{10}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&i&0&0\\0&0&-i&0\end{array}}\quad\underset{11}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&-i&0&0\\0&0&i&0\end{array}}\quad\underset{12}{\begin{array}{cccc}1&0&0&0\\0&0&0&1\\0&-i&0&0\\0&0&-i&0\end{array}}$$

$$\underset{13}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&i&0&0\\0&0&i&0\end{array}}\quad\underset{14}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&i&0&0\\0&0&-i&0\end{array}}\quad\underset{15}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&-i&0&0\\0&0&i&0\end{array}}\quad\underset{16}{\begin{array}{cccc}1&0&0&0\\0&0&0&-1\\0&-i&0&0\\0&0&-i&0\end{array}}$$

$$\underset{17}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&1&0&0\\0&0&i&0\end{array}}\quad\underset{18}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&1&0&0\\0&0&-i&0\end{array}}\quad\underset{19}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&-1&0&0\\0&0&i&0\end{array}}\quad\underset{20}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&-1&0&0\\0&0&-i&0\end{array}}$$

$$\underset{21}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&1&0&0\\0&0&i&0\end{array}}\quad\underset{22}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&1&0&0\\0&0&-i&0\end{array}}\quad\underset{23}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&-1&0&0\\0&0&i&0\end{array}}\quad\underset{24}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&-1&0&0\\0&0&-i&0\end{array}}$$

$$\underset{25}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&i&0&0\\0&0&1&0\end{array}}\quad\underset{26}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&i&0&0\\0&0&-1&0\end{array}}\quad\underset{27}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&-i&0&0\\0&0&1&0\end{array}}\quad\underset{28}{\begin{array}{cccc}1&0&0&0\\0&0&0&i\\0&-i&0&0\\0&0&-1&0\end{array}}$$

$$\underset{29}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&i&0&0\\0&0&1&0\end{array}}\quad\underset{30}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&i&0&0\\0&0&-1&0\end{array}}\quad\underset{31}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&-i&0&0\\0&0&1&0\end{array}}\quad\underset{32}{\begin{array}{cccc}1&0&0&0\\0&0&0&-i\\0&-i&0&0\\0&0&-1&0\end{array}}$$

Table 8. THE γ MATRICES

Number 21 of the Type of Matrices

$$\underset{1}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{vmatrix}}\quad\underset{2}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&-1&0&0\end{vmatrix}}\quad\underset{3}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&-1\\0&1&0&0\end{vmatrix}}\quad\underset{4}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&-1\\0&-1&0&0\end{vmatrix}}$$

$$\underset{5}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&1\\0&1&0&0\end{vmatrix}}\quad\underset{6}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&1\\0&-1&0&0\end{vmatrix}}\quad\underset{7}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&-1\\0&1&0&0\end{vmatrix}}\quad\underset{8}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&-1\\0&-1&0&0\end{vmatrix}}$$

$$\underset{9}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&i\\0&i&0&0\end{vmatrix}}\quad\underset{10}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&i\\0&-i&0&0\end{vmatrix}}\quad\underset{11}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&-i\\0&i&0&0\end{vmatrix}}\quad\underset{12}{\begin{vmatrix}1&0&0&0\\0&0&1&0\\0&0&0&-i\\0&-i&0&0\end{vmatrix}}$$

$$\underset{13}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&i\\0&i&0&0\end{vmatrix}}\quad\underset{14}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&i\\0&-i&0&0\end{vmatrix}}\quad\underset{15}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&-i\\0&i&0&0\end{vmatrix}}\quad\underset{16}{\begin{vmatrix}1&0&0&0\\0&0&-1&0\\0&0&0&-i\\0&-i&0&0\end{vmatrix}}$$

$$\underset{17}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&1\\0&i&0&0\end{vmatrix}}\quad\underset{18}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&1\\0&-i&0&0\end{vmatrix}}\quad\underset{19}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&-1\\0&i&0&0\end{vmatrix}}\quad\underset{20}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&-1\\0&-i&0&0\end{vmatrix}}$$

$$\underset{21}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&1\\0&i&0&0\end{vmatrix}}\quad\underset{22}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&1\\0&-i&0&0\end{vmatrix}}\quad\underset{23}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&-1\\0&i&0&0\end{vmatrix}}\quad\underset{24}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&-1\\0&-i&0&0\end{vmatrix}}$$

$$\underset{25}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&i\\0&1&0&0\end{vmatrix}}\quad\underset{26}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&i\\0&-1&0&0\end{vmatrix}}\quad\underset{27}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&-i\\0&1&0&0\end{vmatrix}}\quad\underset{28}{\begin{vmatrix}1&0&0&0\\0&0&i&0\\0&0&0&-i\\0&-1&0&0\end{vmatrix}}$$

$$\underset{29}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&i\\0&1&0&0\end{vmatrix}}\quad\underset{30}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&i\\0&-1&0&0\end{vmatrix}}\quad\underset{31}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&-i\\0&1&0&0\end{vmatrix}}\quad\underset{32}{\begin{vmatrix}1&0&0&0\\0&0&-i&0\\0&0&0&-i\\0&-1&0&0\end{vmatrix}}$$

2. The φ Class

The determination of the matrices of transformation in this group differed from that of the β groups. The φ, ρ and μ types do not form a closed group. The matrices of this class are divided into three main types, each type having two different forms. A matrix of the following type

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \quad \text{is called } {}^{r}\varphi^{v} \text{ where}$$

r stands for real, v for vertical and a matrix of the type

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \text{is called } {}^{i}\varphi^{h} \text{ where}$$

i stands for imaginary and h for horizontal. This terminology is used for the φ, ρ and μ types.

Each one of the φ, ρ and μ in table 3 is an element of a vierergruppe, each vierergruppe being an independent group. A number of types for the φ, ρ and μ matrices satisfied the transformation conditions. All simple types satisfied the following conditions;

$$\varphi^{h}\varphi^{v} = \begin{matrix} a & a & 0 & 0 \\ b & b & 0 & 0 \\ 0 & 0 & c & c \\ 0 & 0 & d & d \end{matrix} \qquad \rho^{h}\rho^{v} = \begin{vmatrix} a & 0 & a & 0 \\ 0 & b & 0 & b \\ c & 0 & c & 0 \\ 0 & d & 0 & d \end{vmatrix} \qquad \mu^{h}\mu^{v} = \begin{vmatrix} a & 0 & 0 & a \\ 0 & b & b & 0 \\ 0 & c & c & 0 \\ d & 0 & 0 & d \end{vmatrix}$$

Table 10 ${}^{r}\varphi^{v}$ GROUPS

(a, b, c, d, form separate groups)

$$\underset{a_1}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{a_2}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{a_3}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}} \quad \underset{a_4}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}}$$

$$\underset{b_1}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}} \quad \underset{b_2}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{b_3}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{b_4}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}}$$

$$\underset{c_1}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{c_2}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{c_3}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}} \quad \underset{c_4}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}}$$

$$\underset{d_1}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}} \quad \underset{d_2}{\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}} \quad \underset{d_3}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix}} \quad \underset{d_4}{\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}}$$

Table 9. THE η MATRICES

$$
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{1}
\quad
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{2}
\quad
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}_{3}
\quad
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{4}
$$

Table 11. THE ${}^{i}\varphi^{v}$ MATRICES

$$
\begin{vmatrix} 0 & -i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \end{vmatrix}_{a_1}
\quad
\begin{vmatrix} 0 & -i & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \end{vmatrix}_{b_2}
\quad
\begin{vmatrix} 0 & -i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \end{vmatrix}_{c_3}
\quad
\begin{vmatrix} 0 & -i & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \end{vmatrix}_{d_4}
$$

Table 12. φ^{h} MATRICES

$$
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{\sim}_{a}\varphi^{h}_{1}}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{\sim}_{b}\varphi^{h}_{2}}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{\sim}_{c}\varphi^{h}_{3}}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{\sim}_{d}\varphi^{h}_{4}}
$$

$$
\begin{vmatrix} 0 & 0 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{i}_{a}\varphi^{h}_{1}}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{i}_{b}\varphi^{h}_{2}}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ i & -1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{i}_{c}\varphi^{h}_{3}}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ i & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{vmatrix}_{{}^{i}_{d}\varphi^{h}_{4}}
$$

Table 13. ρ^{v} MATRICES

$$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

$$^{r}_{a}\rho^{v}_{1} \qquad ^{r}_{b}\rho^{v}_{2} \qquad ^{r}_{c}\rho^{v}_{3} \qquad ^{r}_{d}\rho^{v}_{4}$$

$$\begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -i & 0 & 0 \end{vmatrix}$$

$$^{i}_{a}\rho^{v}_{1} \qquad ^{i}_{b}\rho^{v}_{2} \qquad ^{i}_{c}\rho^{v}_{3} \qquad ^{i}_{d}\rho^{v}_{4}$$

Table 14. ρ^{h} MATRICES

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$^{r}_{a}\rho^{h}_{1} \qquad ^{r}_{b}\rho^{h}_{2} \qquad ^{r}_{c}\rho^{h}_{3} \qquad ^{r}_{d}\rho^{h}_{4}$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$^{i}_{a}\rho^{h}_{1} \qquad ^{i}_{b}\rho^{h}_{2} \qquad ^{i}_{c}\rho^{h}_{3} \qquad ^{i}_{d}\rho^{h}_{4}$$

Table 15. μ^{v} MATRICES

$$
\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
$$

$${}^{r}_{a}\mu^{v}_{1} \qquad {}^{r}_{b}\mu^{v}_{2} \qquad {}^{r}_{c}\mu^{v}_{3} \qquad {}^{r}_{d}\mu^{v}_{4}$$

$$
\begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
$$

$${}^{i}_{a}\mu^{v}_{1} \qquad {}^{i}_{b}\mu^{v}_{2} \qquad {}^{i}_{c}\mu^{v}_{3} \qquad {}^{i}_{d}\mu^{v}_{4}$$

Table 16. μ^{h} MATRICES

$$
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}
$$

$${}^{r}_{a}\mu^{h}_{1} \qquad {}^{r}_{b}\mu^{h}_{2} \qquad {}^{r}_{c}\mu^{h}_{3} \qquad {}^{r}_{d}\mu^{h}_{4}$$

$$
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -i \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -i \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & i \end{vmatrix}
\quad
\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & i \end{vmatrix}
$$

$${}^{i}_{a}\mu^{h}_{1} \qquad {}^{i}_{b}\mu^{h}_{2} \qquad {}^{i}_{c}\mu^{h}_{3} \qquad {}^{i}_{d}\mu^{h}_{4}$$

X. CHARACTERIZATION OF OPTICAL INSTRUMENTS BY MATRICES

It has been shown that any one of the sixteen fundamental coherency matrices can be transformed into any other, by the transformation matrices. Since each of the coherency matrices represents a particular type of light, then the transformation matrices characterize the optical instruments that change one type of light into another. The instruments represented by the β transformation matrices are conservative optical instruments, in that the power of the input is identical with the power of the output. These optical instruments form a closed group. This method thus affords a means of studying the behaviour of the action of different types of light under different optical instruments. From Table 3 it is readily seen that the same instrument can be used to transform several different types of light. These instruments are in general not reversible.

Michelson's Interferometer utilizes two beams of light.

The ray R is split up into two rays of light by G_1 which is thinly silvered. The reflected ray R_1 traverses a distance D_1 passing through a glass G_2 and is reflected by mirror M_1. The transmitted

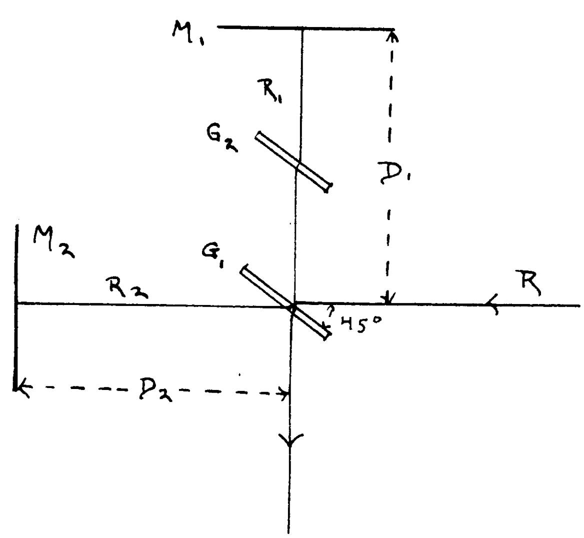


Diagram 3.

ray R_2 is reflected by mirror M_2 which is reflected by glass G_1 and unites with reflected R_1 which is transmitted by G_1. If the paths of these two rays differ by even multiples of π, reinforcement takes place; if by odd multiples of π, interference. The coherency matrix for these two types of emitted rays are

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

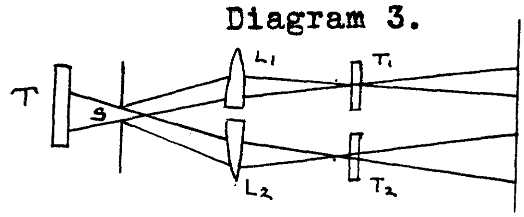
Reinforcement Interference

The matrix $\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ of the form ρ^4

characterizes the action of the interferometer for reinforcement, the matrix $\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$ for interference.

Interference of two beams of light may be caused by the divided lens method.

Diagram 3.



The theory of this instrument is similar to Michelson's Interferometer and is characterized by the same transformation matrices. This instrument can be used in the study of interference of polarized light by placing a tourmaline crystal at T, or one each at T_1 and T_2. Tourmaline transmits light in one direction only, absorbing the light in the other. Therefore, tourmaline is not a conservative instrument. When the tourmaline is placed at T interference takes place as before; if placed at T_1 and T_2 interference does or does not take place according to the position of the axes of the two tourmaline crystals. If these axes are parallel, interference occurs; if at right angles no interference fringes appear. The matrices of the emitted ray of light are

Interference $$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Reinforcement $$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Axes of Tourmaline Crystals at Right Angles $$\begin{vmatrix} 1 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{\gamma} & 0 & 0 & 1 \end{vmatrix}$$ where $\gamma = \pm i$ or ± 1

CONCLUSION

Polarized light can be analyzed by several methods, but the method of coherency matrices is more comprehensive and can be more readily used, in the case of two or more beams of light, Given a coherency matrix representing a ray of light, the type of light and the relation of its components can be immediately stated. The transformations changing one type of light into another are easily determined. The coherency method gives a very simple way of analyzing the many possible combinations of rays of light and optical instruments.

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BIOGRAPHY

Dorothy W. Weeks received the B. A. degree from Wellesley College, June 1916, S. M. from the Massachusetts Institute of Technology, June 1923, and S. M. from Simmons College, June 1925. In addition to this she has studied at Cornell, Harvard and George Washington Universities.

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