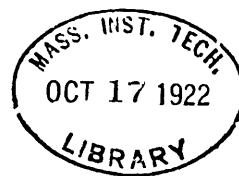


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Thesis case*



H A R M O N I C A N A L Y S I S,

A CRITICAL COMPENDIUM OF METHODS AND DEVICES FOR THE
ANALYSIS OF COMPLEX ALTERNATING CURRENT WAVES WITH
SUGGESTIONS FOR IMPROVEMENTS AND DISCUSSION OF THE
REQUIREMENTS OF SUCH DEVICES.

A THESIS SUBMITTED TO THE FACULTY OF THE MASSACHUSETTS
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Respectfully submitted : -

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DEFINITION OF HARMONIC ANALYSIS.

Fourier in his "Analytic Theory of Heat" (1822) showed for the first time that any given function of a variable x might be expressed as a Trigonometric Series. Series involving only sines and cosines of whole multiples of x , that is series of the form:-

$$y = b_0 + b_1 \cos.x + b_2 \cos.2x + b_3 \cos.3x + \dots$$

$$+ a_1 \sin.x + a_2 \sin.2x + a_3 \sin.3x + \dots$$

are generally known as "Fourier's Series".

As Electrical Engineering deals with electric waves more or less closely approaching a simple sine or cosine function of time, the application of Fourier's Series to Electrical Engineering problems is self evident.

Harmonic Analysis consists of breaking up into its component parts, or the separate determination of each of the component parts, of a complex wave or curve representing the variations of some function with respect to another.

The lowest order term of the series is called the fundamental and the higher order terms are called harmonics, whence the appellation for the procedure of "Harmonic Analysis". While the process is chiefly applied to cyclic functions, it may also be applied to non-cyclic ones as well, the number of terms required for a satisfactory approximation usually being much greater in the latter case. It has been found that in application this conception holds perfectly, and that all phenomena arising from complex waves may be adequately interpreted by considering that the wave is actually several different waves, each a pure sine or cosine function and each having a period or frequency bearing an integral relation to the others.

Usually in Electrical Engineering Harmonic Analysis is applied to Alternating Current Waves which represent the variations in current or potential with respect to time. These waves have different degrees of symmetry from which certain characteristics of the series may be determined. If the waves are symmetrical with respect to the origin, that is the first half cycle is equal to the reversed second half cycle, and the positive and negative halves of the wave are reversed duplicates of each other, no even harmonics will be present, and thus no even terms in the series. This is usually the case with alternating current. The first constant term of the series will also be absent in this case.

If the waves are symmetrical with respect to the mid point of each half cycle, then all the harmonics will be in phase with the fundamental, and no cosine components will be present in the series. If, however, the waves are not symmetrical with respect to the origin, then both even and odd harmonics and corresponding terms may be present. The first constant term is fixed by the areas on each side of the X-axis. If the positive and negative areas are equal the constant term is zero. If, however, the positive and negative areas are not equal, then the value of the constant term is equal to the value of the ordinate of the displacement of the X-axis necessary to make these areas equal. Extreme cases of waves containing all of the symmetrical conditions mentioned above, but still requiring an infinite number of terms in the series to rigorously represent the function are Triangular, Rectangular, Trapezoidal, and other geometrically angular waves, while usual cases are waves with peaked or flattened tops etc.

For certain cyclic functions the number of terms will be infinite, although the function may be approximated by a finite number of terms, usually with sufficient accuracy for practical purposes, and often with greater accuracy than it is possible to obtain in the results of the analysis. The number of terms required to give a sufficient approximation will depend upon the convergency of the series. Thus a rectangular wave requires about 30 terms for approximation, while a triangular wave will be approximated to the same degree by about 15 terms. In most cases of alternating current analysis a smaller number of terms is sufficient. Generally the even harmonics are absent, and in the usual cases of alternating machinery, harmonics above the eleventh are so small that they may be entirely neglected. In fact, since they are usually less than the error of measurement, it is impossible to definitely state whether they exist or not. Including the eleventh harmonic, six terms would be necessary for odd harmonics only. In the case of alternators tooth harmonics will be present in most cases, to a greater or less degree, but they will seldom be higher than the twenty-seventh harmonic, requiring fourteen terms if all intermediate orders are included, or more approximately, six terms up to the eleventh harmonic and two or three additional terms adjacent to and including the tooth harmonic.

Harmonic Analysis is not by any means limited to electrical applications, and is useful in the study of any varying function, some common applications of which are the study of heat flow, sound waves, astronomical observations, tides and other work of similar nature. With functions containing discontinuities the application of the Fourier Series is more difficult but can be used in many cases, the trigonometric relations expressing a continuous function, a portion of which is the function under investigation. By using a certain amount of discrimination in dealing with such cases, the use of the series can be extended to cover a great many different subjects.

PREFACE AND APPRECIATION.

The author does not wish to claim great originality in the preparation of this thesis. It is primarily a compilation of various means and methods and other interesting data in connection with the problems of harmonic analysis. He feels that the subject is one of increasing importance, and that a more detailed and consecutive study of the problems involved will lead to a far simpler and more practical method of performing the analysis than those available at the present time. Therefore the compilation includes nearly all of the various types of analysers proposed from time to time, given with greater or less detail as their importance seemed to warrant.

An attempt has been made to classify the different methods in such a way that they are divided up by the main basic features of construction, and thus group themselves into similar classes for study. The electric analyser described in some detail was developed by the author as an experimental model with the expectation of building a more complete one if its success seemed to warrant it. In the mean time Mr. Woodbury worked in the Research Division under the supervision of Prof. Bush and the author, and succeeded in developing a machine which filled the present needs of the laboratory so satisfactorily, that no further work was done upon the electric machine for the time being.

In order to fix the goal towards which the analysis should aim, the Ideal Analyser has been defined early in the thesis. In the discussion and conclusions regarding the various types the desirable points of the ideal type which have been missed are brought out, as well as such advantages and disadvantages as are most striking.

The author wishes to express his sincere appreciation of the assistance in the preparation of this summary to Dr. Kennelly and Dr. Bush for assistance in obtaining data, to Mr. Woodbury for also sending photographs taken of the development stages of his analyser, and to Miss Sanders for staggering under armfuls of heavy reference books from the library.

INTRODUCTION.

It is very seldom in engineering pursuits that a problem is encountered which is capable of so many methods of solution as Harmonic Analysis, and yet practically all of which are unsatisfactory. Methods giving great precision are extremely tedious, while methods of greater speed are liable to large errors easily or unwittingly made. In some methods errors are unavoidable, in spite of meticulous care in procedure, and may be of serious magnitude.

The question arises as to why harmonic analysis of better standards than those now available should be necessary, since it is true that it is possible to obtain a complete and reasonably accurate solution of any complex wave, by any one of a great variety of methods. It is also true that practically harmonic analysis is not used to any great extent and is looked upon as rather a fanciful trick performed by research men or those who delight to delve in hair splitting minutiae of design, rather than a real robust and practical method of attack upon important problems of development.

The answer to this lies in the fact that as the fundamentals of any problem or science become better familiarized, improvements in the development of the art must, of necessity, lie in improvement of detail. The majority of Alternating Current Theory is based upon the assumptions of sinusoidal wave forms. The methods and procedure with most classes of machinery and applications under this assumption are quite well understood and developed with moderate completion. The practical fact remains, however, that the wave forms obtained from actual machinery are almost never purely sinusoidal and in some cases depart very far from sinusoidal. The phenomena resulting from these deviations, distortions and dissymmetries, give rise to many results which cannot be investigated without an intimate knowledge of the wave analyses. In some cases the direct necessity for analysis is even more pronounced. Thus in the study of sound the quality depends entirely upon the harmonics present; in the study of interference between power transmission lines and telephone lines, the trouble is caused almost entirely by harmonics in the transmission line currents or potentials, and in studying the magnetization data of transformers, harmonics must immediately be considered to account for the very distorted wave shapes of the current required by the cyclic magnetization of steel. Thus there are two general classes of problems which will benefit by better methods of harmonic analysis. First, those problems where present methods are approximate and gives results sufficiently accurate as to make the further expenditure of time and energy upon laborious and lengthy harmonic analysis unwarranted. Second, those problems where harmonics are of necessity basic in the nature of the problem, and where no progress whatever can be made without

harmonic analysis. In the first of these classes most engineers will now admit that they would like the additional information if any reasonable method for obtaining it were available, but the present methods of analysis represent so much time, expense or error, that they are not favorably considered.

It is not the purpose of this thesis to develop a method satisfying all of the desirable qualities which a method of harmonic analysis should have, but rather to tabulate all present methods according to their logical arrangement, in order that they may be better studied and compared; to gather together the many devices and methods described in a great variety of publications, and finally to point the way towards methods that have not been developed which would appear to give promise of good results, and to show some of the results upon one or two such steps that have already been accomplished as a part of this thesis. Naturally there are many duplications, and where such exist an attempt has been made to describe most fully the method developed in priority to the others, merely mentioning the other developments with such added descriptive material as may be necessary to understand any improvements or additions which may have arisen.

As with any experimental subject there are two methods of attack in obtaining a harmonic analysis. First the phenomena may be analysed while it is actually occurring. This is similar to taking meter readings of current or potential, and requires that the source of energy producing the wave under investigation be sufficiently large to supply the necessary power to operate the analysing device without distorting the function analysed, and that the duration of the phenomena be long enough to allow the analysing device to act and to read the results. In general this method is only successful where alternating current waves of similar recurrent cycles are produced. The second method is to first make a record of the phenomena and then to work at leisure from the record. This is the more usual arrangement, and an oscillograph or some allied device as a rule is made use of to trace the original curve. The limitations of this method must then of necessity be at least those of the recording device, but fortunately recording devices are developed to such a high state of perfection that the limitations thus introduced are practically negligible compared with those met in the later methods of analysis.

Since the last mentioned method is used in by far the greater majority of systems for harmonic analysis, the structure of this thesis is built up around such devices, and the first mentioned types are introduced in their appropriate places as analogous or other means of accomplishing the same purpose.

Several possible methods of classifying the different systems or devices present themselves. The mathematical basis is in every case Fourier's Series. This may, however, be solved in several ways, but there are very few variations in the fundamental theory,

the deviations occurring in its method of application. This is a fact which is worthy of special note since it leads to the very pertinent comment that there may be possible improvements in the basic mathematical theory which will allow of great improvements in its application to practice. Classification by mathematical theory, nevertheless, does not appear to be the simplest from the standpoint of comparison, for while it may be the most logical, it is open to the objection that totally different methods, such as graphical and mechanical, may be based upon identical mathematical equations. Also the limitations in variety of the basic mathematical expressions very seriously limits the flexibility of classification by this method. Therefore it appears better to classify by the way in which the result is obtained, and this is followed out in this thesis by using as the three main classification heads:-

- I. Mathematical Methods.
- II. Graphical Methods.
- III. Instrumental and Mechanical Methods.

The third main heading covers the greatest variety of apparatus, and is provided with many sub-headings to re-classify the many ramifications into their similar groupings. In this way the physical conception of the different schemes and devices is emphasized, and the minor objection of repetition of similar mathematical fundamentals sacrificed.

THE IDEAL HARMONIC ANALYSER.

There appear to be two possible ideal analysers. The first a direct reading instrument operated by the function under investigation, and apparently applicable only to electrical engineering. The second a device operating upon a chart, graph or oscillograph record of the function to be analysed, by some form of manipulation, and which would be applicable to any function whatever.

These ideal analysers should have the following characteristics:-

1. A device which can be simply connected to the source of energy by means of binding posts, and which, by means of a multiplicity of dials, or a single dial with a switching device by which readings can be made in rapid succession without readjustment, will indicate at once the values of the various fundamental and harmonic components of the connected wave.

2. A device provided with a tracer arm, which is caused to follow the curve undergoing analysis, by manipulation of the operator, and which, after one trace of the curve will indicate upon some form of scale the various fundamental and harmonic components of the curve. Several arms adjusted to definite ordinates may be substituted for the single tracer arm. The device should include some means of adjusting or compensating for different scales of curves analysed.

In both these devices the values of the components should be in terms of their percentage of the equivalent sine wave, and the phase relations of the components indicated simultaneously. The device should also not be unduly bulky or heavy or expensive. The accuracy should be about 1% of the component unless the latter were very small, when 1/4% of the fundamental would be allowable. The principle should be simple and not require delicate or complicated parts liable to become poorly adjusted or bent. Some simple calibration method must be provided by which the accuracy may be rapidly checked. The size and weight need not necessarily be so small that it is easily portable, but it should be capable of being moved from place to place without requiring complete readjustment. There appears to exist great difficulty in making the components read as percent of equivalent sine wave value, and if the other conditions are fulfilled it would be satisfactory if they read in percent of fundamental or upon an arbitrary scale, since it is generally the ratio of the harmonic and not its absolute value that is of interest.

However, the labor of converting separate sine and cosine components to equivalent sine wave values with the proper phase relations deduced is quite a considerable part of the analysing process, and should thus be either incorporated in the analyser itself, or a separate machine provided for mechanically performing the transposition. The cost is difficult to define and depends in no small measure upon the degree to which the above conditions are fulfilled. If all the conditions are strictly met the desirability of such a device would warrant a large cost, but it is doubtful if its installation would be considered by various concerns if its cost were more than \$500, and it should preferably be nearer \$100. The machines which are now commercially available cost from \$400 to \$1000 approximately, but are subject to criticism upon many of the desirable points covered and so are not installed commercially or otherwise to a very great extent.

CLASSIFICATION OF METHODS OF ANALYSIS.

1. MATHEMATICAL.

- A. General Theory.
- B. Direct Application to Analysis.
- C. Schedule Methods for Saving Time.
- D. Selected Ordinate Methods.

2. GRAPHICAL.

- A. The construction of Derived Curves, the areas of which give the coefficients.
- B. Vectorial Combinations of Selected Ordinates.
- C. Combinations of Selected Areas from Curve.
- D. Transformation of abscissae from Linear to Angular Function by projection upon Cylinder.
- E. Special Scales or Templates for reading values of ordinates, leading to a simple solution by combinations of scalar values.

3. INSTRUMENTAL.

- A. Direct Reading Methods.
 - a. By Resonance.
 - b. Differential Dynamometer.
 - c. Special Circuit Combinations.
- B. Machines operating from Direct Trace of Curve, giving results directly.
 - a. Devices employing a planimeter whose tracer point is caused to follow a curve whose coordinates are:

$$x'y' = f(y, n \cdot \cos \phi).$$
 - b. Devices employing a planimeter whose tracer point follows the curve being investigated, and whose fixed point is given an independent motion involving the position angle of the ordinate subtended by the tracer, or visa-versa.

c.Devices employing planimeter wheels actuated by a special mechanism imparting to the wheels a deflection proportional to the coefficient sought.

C.Machines operating from Direct Trace of Curve, but giving results indirectly.

a.Mechanical Drawing of Secondary Curve, from which the Coefficients may be determined by further measurements.

D.Machines performing calculations based upon Selected Ordinates from the Curve.

a.An Adjustable Framework obtaining solution by Vector Addition.

b.Devices which draw a secondary curve from which the coefficients may be determined by further measurements.

c.Combinations of Electric Circuits, which give the desired coefficients by various adjustments.

d.Systems of cords, pulleys, levers and weights.

e.Hydrostatic Methods.

4.MISCELLANEOUS AND UNTRIED BUT POSSIBLE METHODS.

A.Alignment Charts.

B.Slide Rules.

C.Calculating Machines of Various Types.

D.Other planimeter combinations.

a.Wheel Type.

b.Hatchet or Frytz type.

E.Miscellaneous.

HISTORICAL NOTE.

As already stated Fourier first showed that any given function of a variable x might be expressed as a trigonometric series in his work on the theory of heat in 1822. There does not seem to have been any immediate application of the series to other purposes, and it is probable that Lord Kelvin made the first practical use of the theory in his investigation of tides for the British Admiralty in 1875-1876. It is interesting that his brother, J.J. Thomson, proposed the sphere, cylinder and plate arrangement which Lord Kelvin applied to harmonic analysis, his brother apparently only intending it as a special form of integrator. An eleven element machine of this type was built and used by the Admiralty, and thus the first application of Fourier's Series and the first mechanical analyser were practically coincident. It is also interesting that this first analyser had all the properties of the latest types and obtained several coefficients with one trace of the curve, although it was very heavy and bulky, and doubtless difficult to operate.

Slightly before Kelvin's use of the Fourier's Series Prof. Clifford had developed a graphical method for its solution which was published in 1873, and consisted of projecting the curve upon a cylinder and in this way introducing the trigonometric functions in the abscissae, so that the areas of the reconstructed curves gave the coefficients.

With these two investigations the study of the series at that time seems to have closed, and it was not until the rapid rise to prominence of Alternating Current Power, beginning in about 1890, that more broadspread interest was shown. From that time on the electrical literature is filled with many proposed or actually used schemes, of various types, and each one suited to the particular needs or ideas of its originator, but none of them apparently improving the Graphical method of Clifford or the machine of Kelvin, with the exception of the analytical and mathematical developments of Runge and S.P. Thompson. The developments all appear to improve some one detail of the process at the expense of some other or all other parts. The result is a very large number of possible methods from which it is difficult to pick out any one as being more satisfactory than any other, except for some one special limited application. Mention should also be made of the Houston-Kennelly method and its development more or less independantly by Fischer-Hinnen, which is a convenient development of the mathematical analysis equation (5) already given above.

At the present time there are two mathematical methods, three machines, and one graphical method which appear to stand out as the most common. The mathematical methods are the Schedules of Runge, developed for electric work by S.P. Thompson, and the method of Selected Ordinates attributed to Fischer-Hinnen. The machines are the Henrici-Coradi, a development of the original Kelvin machine, the Chubb-Westinghouse polar analyser, a quite recent development, and the Michelson and Stratton Analyser and Synthetizer. The Graphical method is that of Clifford, developed and improved upon in its technique by Ferry and Slichter. While some of the others appear to be of good design and compare favorably with those in use, they are not commercially available, and do not appear to have been put to use by any except their originators.

The outstanding contributions to the development of the various methods can easily be tabulated as follows:-

MATHEMATICAL METHODS.

Fourier	1822.
Carl Runge	1903-1905.
S.P. Thompson	1905.
F.W. Grover	1913.
P. Kemp	1920

Preliminary work of interesting nature, but not in the right direction was done by S.M. Kintner in 1904, who should have foreseen the Runge Schedule, and a very precise but extremely laborious method was developed by C.P. Steinmetz, published in 1911 and probably of much earlier origin.

Very good examples of these methods are given in Lipka's "Graphical and Mechanical Computation", and the report of the Railroad Commission of the State of California for 1919. The latter includes a schedule for determining 34 coefficients which appears to be the highest order published, and which will determine the harmonics up to the 35th, the curve being divided in 36 parts.

GRAPHICAL METHODS.

Clifford	1873.
Ferry	1892.
Slichter	1909.
These are all projection on cylinder methods.	
Wedmore	1895.
Harrison(Sum of projected Vectors)	1906.
Ashworth(Polygon of Forces)	1911.
Beattie(Special Scales for measuring ordinates)	1911.
Rottenburgh(Derived curves from template chart)	1913.

INSTRUMENTAL METHODS.

Kelvin	1875-76
Sommerfeld and Wiechert	1892
Bashforth	1892
F.A.Laws	1893
Henrici-Coradi	1894
Pupin	1894
Sharp	1895
Yule	1895
Michelson and Stratton.	1898
Mader	1909
Agnew	1909
Beattie	1912
Boucherot	1913
Chubb-Westinghouse	1914
Bush	1920
Dellenbaugh	1921
Woodbury	1921

SELECTED ORDINATE METHODS.

Houston-Kennelly	1898
Fischer-Hinnen	1901
Lincoln	1908

Thus 1892 to 1906 covers the majority of the development, which is natural as this covers the period also of the increase in use of alternating current, and the development of the different methods coincided with the desire to know more about the distorted wave shapes. However, the complexity, time or cost has limited the use of harmonic analysis to a few isolated cases, D.C. Miller in his study of sound being one of the few cases where it has been applied extensively. Since the underlying principles of alternating current are now well understood, and since the details of harmonics, as already stated, are beginning to be of interest and the direction in which research can profitably be undertaken, it appears desirable to have developed a method of analysis which can be applied with the simplicity and directness of other electrical measurements.

MATHEMATICAL THEORY.

(Fourier's Series and Spherical Harmonics. W.E.Byerly. 1893.)

Suppose that we wish to form the series known as Fourier's from a given curve:- $y = f(x)$.

It is clear that the equation:-

$$y = a_1 \sin.x$$

may have a_1 determined so that the curve represented shall pass through any given point. For if we substitute in this equation the coordinates of the point in question we shall have an equation of the first degree in which a_1 is the only unknown and which will therefore give us one and only one value for a_1 .

In like manner the curve:-

$$y = a_1 \sin.x + a_2 \sin.2x$$

may be made to pass through any two arbitrary points whose abscissae lie between 0 and π provided that the abscissae are not equal, &:-

$$y = a_1 \sin.x + a_2 \sin.2x + a_3 \sin.3x + \dots + a_n \sin.nx$$

may be made to pass through any n arbitrarily chosen points whose abscissae lie between 0 and π provided as before that their abscissae are different.

If, then, the given function $f(x)$ is of such a character that for each value of x between $x = 0$ and $x = \pi$ it has one and only one value, and if between $x = 0$ and $x = \pi$ it is finite and continuous or if discontinuous has only finite discontinuities, the coefficients in:-

$$y = a_1 \sin.x + a_2 \sin.2x + a_3 \sin.3x + \dots + a_n \sin.nx$$

can be determined so that the curve represented will pass through any n arbitrarily chosen points of the curve $y = f(x)$ whose abscissae lie between 0 and π and are all different, and these coefficients will have but one set of values.

The coincidence of the series and $f(x)$ will become closer the greater the number of terms taken in the series, provided the series is convergent. Mathematical investigation shows that this is the case, and also that the value of the series at a point of finite discontinuity is equal to one half the sum of the two values which the function approaches as we approach the point in question from opposite sides.

In general the derivative of a Fourier's Series cannot be obtained

by differentiating the series term by term, but its integral can be obtained by integrating the series term by term. The effect of differentiation is to make the series less convergent or even divergent, while the effect of integration is to make the series more convergent. The term by term derivative of a Fourier's Series is itself a Fourier's Series, but the term by term integral of a Fourier's Series is not in general a Fourier's Series. This latter fact is owing to the introduction of a term $b_0 x$ from the integration of the constant term of the original series.

However, further investigation of the differentiation of a Fourier's Series indicates that under certain conditions the term by term derivative will represent the derivative of the series. Let the function $f(x)$ be represented by a Fourier's Series, called S . Let the derivative of $f(x)$ be $f'(x)$ and also satisfy the conditions necessary for a Fourier's Series. The integral of this latter series will be equal to the integral of $f'(x)$, that is, to $f(x)$ plus a constant and one integral will be equal to $f(x)$.

If this integral is a Fourier's Series it must be identical with S . It will be a Fourier's Series only in case the series for $f'(x)$ lacks the constant term b_0 .

But:-

$$b_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) dx$$

Therefore:-

$$b_0 = \frac{1}{\pi} [f(\pi) - f(-\pi)]$$

and it will be zero if:-

$$f(\pi) = f(-\pi)$$

In order that $f'(x)$ shall satisfy the conditions for a Fourier's Series, $f(x)$ while satisfying the same conditions must in addition be finite and continuous between $x = -\pi$ and $x = \pi$

If, then, $f(x)$ is single valued, finite, and continuous, and has only a finite number of maxima and minima between $x = -\pi$ and $x = \pi$, and if $f(\pi) = f(-\pi)$, $f(x)$ can be developed into a Fourier's Series whose term by term derivative will be equal to the derivative of the function. In this case the periodic curve $y = S$ is continuous throughout its whole extent.

It is important to note that this covers the usual cases of alternating current waves.

The evaluation of the Fourier Coefficients may be most rigorously accomplished by a method due to Lagrange and fully developed by Byerly. However, a simpler and more direct, though not so rigorous a method, gives the desired results and is the one usually used. Therefore this latter method will be herein developed, and may be found more fully discussed in Lipka's "Graphical and Mechanical Computation" and other similar treatises.

The Fourier Series is often written in the form:-

$$y = b_0 + c_1 \sin(x + \varphi_1) + c_2 \sin(x + \varphi_2) + c_3 \sin(x + \varphi_3) + \dots + c_n \sin(nx + \varphi_n)$$

which reduces to the form having both sine and cosine terms by substitution of the relations:-

$$c_n = \sqrt{a_n^2 + b_n^2} \quad \varphi_n = \tan^{-1}(a_n/b_n) \quad \cos \varphi_n = b_n / \sqrt{a_n^2 + b_n^2}$$

This is the form in which the constants are most useful for considering after complete analysis, but the form involving separate sine and cosine terms is more convenient for the theoretical determination of the constants.

If in the series:-

$$y = f(x) = b_0 + b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \dots + a_1 \sin x + a_2 \sin 2x + \dots$$

both sides are multiplied by dx and integrated between the limits 0 and 2π the expression given below is obtained.

$$\begin{aligned} \int_0^{2\pi} y dx &= b_0 \int_0^{2\pi} dx + b_1 \int_0^{2\pi} \cos x dx + \dots + b_n \int_0^{2\pi} \cos nx dx + \dots \\ &\quad + a_1 \int_0^{2\pi} \sin x dx + \dots + a_n \int_0^{2\pi} \sin nx dx + \dots \\ &= b_0 \left[x \right]_0^{2\pi} + b_1 \left[\sin x \right]_0^{2\pi} + \dots + b_n \left[\sin nx \right]_0^{2\pi} + \dots \\ &\quad - a_1 \left[\cos x \right]_0^{2\pi} - \dots - a_n \left[\cos nx \right]_0^{2\pi} - \dots \\ &= 2\pi b_0 \quad \text{Since all the other terms vanish} \end{aligned}$$

If both sides are multiplied by $\cos kx dx$ and integrated as before:-

$$\begin{aligned} \int_0^{2\pi} y \cos kx dx &= b_0 \int_0^{2\pi} \cos kx dx + \dots + b_k \int_0^{2\pi} \cos^2 kx dx + \dots \\ \text{Part } &\left(+ b_n \int_0^{2\pi} \cos nx \cos kx dx + \dots + a_n \int_0^{2\pi} \sin nx \cos kx dx + \dots \right. \\ &= b_0/k \left[\sin kx \right]_0^{2\pi} + \dots + b_k/2 \left[x + (\sin 2kx/2k) \right]_0^{2\pi} + \dots \\ &\quad + b_n/2 \left[\frac{\sin(n-k)x}{(n-k)} + \frac{\sin(n+k)x}{(n+k)} \right]_0^{2\pi} - \dots \\ &\quad - a_n/2 \left[\frac{\cos(n-k)x}{(n-k)} + \frac{\cos(n+k)x}{(n+k)} \right]_0^{2\pi} - \dots \\ &= \pi b_k, \quad \text{Since all other terms vanish.} \end{aligned}$$

Similarly if both sides are multiplied by $\sin nx \cdot dx$ and integrated between the limits 0 and π there is obtained:-

$$\begin{aligned} \int_0^{2\pi} y \sin nx \cdot dx &= B_0 \int_0^{2\pi} \sin nx \cdot dx + \dots + b_n \int_0^{2\pi} \cos nx \cdot \sin nx \cdot dx + \dots \\ &+ a_k \int_0^{2\pi} \sin^2 kx \cdot dx + \dots + a_n \int_0^{2\pi} \sin nx \cdot \sin nx \cdot dx + \dots \\ &= -b_0/k \left[\cos nx \right]_0^{2\pi} - \dots + a_k/2 \left[x - (\sin 2kx/2k) \right]_0^{2\pi} + \dots \\ &- b_n/2 \left[[\cos(k-n)x/(k-n)] + [\cos(k+n)x/(k+n)] \right]_0^{2\pi} + \dots \\ &+ a_n/2 \left[[\sin(n-k)x/(n-k) - \sin(n+k)x/(n+k)] \right]_0^{2\pi} + \dots \\ &= \pi a_k \quad \text{since all other terms vanish.} \end{aligned}$$

Collecting the results there is obtained as the value of the different coefficients:-

$$b_0 = 1/2\pi \int_0^{2\pi} y \cdot dx, \quad b_k = 1/\pi \int_0^{2\pi} y \cdot \cos kx \cdot dx, \quad a_k = 1/\pi \int_0^{2\pi} y \cdot \sin kx \cdot dx.$$

Where $k = 1, 2, 3, \dots$ etc. Each coefficient may thus be independently determined and thus each individual harmonic can be calculated without calculating the preceding harmonics. For this method of treatment, however, the function must be known in order to obtain the integral. Therefore where the function is unknown some other method must be used. This indicates the basis for a large number of mechanical analysers, which perform the integration from the graphically constructed curve and thus solve the equations for the coefficients.

Mathematically another method may be used which is developed as follows. This method is applicable when the series has a finite number of terms, or when the function is sufficiently well approximated by a finite number of terms.

Divide the interval from $x=0$ to $x=\pi$ into n equal intervals and measure the first n ordinates; these are represented by the table:-

x	0	$2\pi/n$	$4\pi/n$	$6\pi/n$. . .	$r(2\pi/n)$. . .	$(n-1)2\pi/n$
	x_0	x_1	x_2	x_3	x_r	x_{n-1}
y	y_0	y_1	y_2	y_3	y_r	y_{n-1}

It is required to determine the constants in the equation:-

$$y = b_0 + b_1 \cos x + \dots + b_k \cos kx + \dots + a_1 \sin x + \dots + a_k \sin kx + \dots$$

where the number of terms is n so that the corresponding curve will pass through the n points given in the table. Substituting the n sets of values of x and y in this equation, n linear equations are obtained of the form:-

$$y_r = b_0 + b_1 \cos x_r + \dots + b_k \cos kx_r + \dots + a_1 \sin x_r + \dots + a_k \sin kx_r + \dots$$

where r takes in succession the values $0, 1, 2, \dots, n-1$. These n equations may now be solved for the coefficients.

To determine b_0 it is only necessary to add the n equations, which gives:-

$$\begin{aligned} \sum y_r &= nb_0 + \dots + b_k \sum \cos kx_r + \dots + a_k \sum \sin kx_r + \dots \\ &= nb_0 \quad \text{since all other terms vanish.} \end{aligned}$$

To determine b_k each of the equations is multiplied by the coefficient of b_k in that equation, i.e., by $\cos kx_r$, and the n resulting equations are added with the result:-

$$\begin{aligned} \sum y_r \cos kx_r &= b_0 \sum \cos kx_r + \dots + b_k \sum \cos^2 kx_r + \dots + b_p \sum \cos px_r \cos kx_r + \dots \\ &\quad + a_p \sum \sin px_r \cos kx_r + \dots \end{aligned}$$

Now:-

$$\sum \cos kx_r = 0$$

$$\sum \cos px_r \cos kx_r = \frac{1}{2} \sum \cos (p+k)x_r + \frac{1}{2} \sum \cos (p-k)x_r = 0$$

$$\sum \sin px_r \cos kx_r = \frac{1}{2} \sum \sin (p+k)x_r + \frac{1}{2} \sum \sin (p-k)x_r = 0$$

$$\begin{aligned} \sum \cos^2 kx_r &= \sum \frac{1}{2} (1 + \cos 2kx_r) = \frac{n}{2} + \frac{1}{2} \sum \cos 2kx_r = n/2, \text{ if } k \neq n/2; \\ &= n, \text{ if } k = n/2. \end{aligned}$$

Hence:-

$$\sum y_r \cos kx_r = (n/2) b_k, \text{ except when } k = n/2.$$

$$= nb_k \quad \text{when } k = n/2.$$

To determine a_k each of the n equations is multiplied by the coefficient of a_k in that equation, i.e., by $\sin.kx_r$, and the n resulting equations are added, obtaining:-

$$\begin{aligned} \Sigma y_r \sin.kx_r = & b_0 \Sigma \sin.kx_r + \dots + b_p \Sigma \cos.px_r \sin.kx_r + \dots \\ & + a_k \Sigma \sin^2.kx_r + \dots + a_p \Sigma \sin.px_r \sin.kx_r + \dots \end{aligned}$$

Now:-

$$\Sigma \sin.kx_r = 0;$$

$$\Sigma \cos.px_r \sin.kx_r = \frac{1}{2} \Sigma \sin(k+p)x_r + \frac{1}{2} \Sigma \sin(k-p)x_r = 0;$$

$$\Sigma \sin.px_r \sin.kx_r = \frac{1}{2} \Sigma \cos(p-k)x_r - \frac{1}{2} \Sigma \cos(p+k)x_r = 0;$$

$$\begin{aligned} \Sigma \sin^2.kx_r = \Sigma \frac{1}{2}(1 - \cos 2kx_r) &= \frac{n}{2} - \frac{1}{2} \Sigma \cos 2kx_r = n/2, \text{ if } k \neq n/2 \\ \text{Hence:-} & \\ &= 0 \text{ if } k = n/2 \end{aligned}$$

$$\Sigma y_r \sin.kx_r = (n/2)a_k$$

Collecting the results, the coefficients are determined by:-

$$b_0 = 1/n \Sigma y_r = 1/n (y_0 + y_1 + y_2 + y_3 + \dots + y_{n-1})$$

$$b_{n/2} = 1/n \Sigma y_r \cos(n/2)x_r = 1/n \Sigma y_r \cos.r\pi = 1/n (y_0 - y_1 + y_2 - y_3 + \dots - y_{n-1})$$

$$b_k = 2/n \Sigma y_r \cos.kx_r = 2/n (y_0 \cos.kx_0 + y_1 \cos.kx_1 + \dots + y_{n-1} \cos.kx_{n-1})$$

$$a_k = 2/n \Sigma y_r \sin.kx_r = 2/n (y_0 \sin.kx_0 + y_1 \sin.kx_1 + \dots + y_{n-1} \sin.kx_{n-1})$$

If n is an even integer, the periodic curve is represented by the equation:-

$$\begin{aligned} y = & b_0 + b_1 \cos.x + \dots + b_k \cos.kx + \dots + b_{n/2} \cos(n/2)x + \dots \\ & + a_1 \sin.x + \dots + a_k \sin.kx + \dots + a_{n/2-1} (\sin(\frac{n}{2}-1)x \end{aligned}$$

The n coefficients are determined as above. Thus:-

b_0 is the average of the n ordinates.

$b_{n/2}$ is the average value of the n ordinates taken alternately plus and minus.

a_k or b_k is twice the average value of the products formed by multiplying each ordinate by the cosine or sine of k times the corresponding value of x .

If it was desired to represent the periodic curve by a Fourier's Series containing n terms, but m ordinates were measured, where $m > n$, it would be necessary to determine the coefficients by the method of least squares. It should also be noted that this method assumes a finite number of terms in the series. If only a finite number are taken and the series actually consists of an infinite number, or of a finite number greater than that assumed, then errors are introduced in the resulting coefficients, the magnitude of the errors depending upon the form of the periodic curve and the magnitude and number of the neglected terms of the series.

The labor involved in obtaining the coefficients from the measured ordinates by this method is very great if it is necessary to determine more than one or two values. However, owing to a large number of the mathematical processes involved being duplications of each other it is possible to arrange the work systematically in the form of a schedule, and save a large portion of the time and labor otherwise required. This simplification has been worked out by Carl Runge and S.P. Thompson, and will be dealt with in detail in its proper place in the classification of methods of analysis.

The proof of the absence of certain terms due to symmetry of the periodic curve, may be shown in several ways, and is given below as displayed in Bragstad and La Cour.

The general procedure is as before. The integral forms used are:-

$$\int_0^{2\pi} \sin mx \cdot \sin nx \, dx = \begin{cases} 0 & \text{when } m \neq n \\ 0 & \text{when } m = n = 0 \\ \pi & \text{when } m = n > 0 \end{cases}$$

$$\int_0^{2\pi} \cos mx \cdot \sin nx \, dx = 0$$

$$\int_0^{2\pi} \cos mx \cdot \cos nx \, dx = \begin{cases} 2\pi & \text{when } m = n = 0 \\ \pi & \text{when } m = n > 0 \\ 0 & \text{when } m \neq n \end{cases}$$

where m and n are any positive integers.

The series is multiplied through by one coefficient as before but integrated between $-\pi$ and 0 and then between 0 and $+\pi$:-

Then:-

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \left[\int_{-\pi}^0 f(x) \cos(nx) dx + \int_0^{+\pi} f(x) \cos(nx) dx \right].$$

Substituting $x = -y$ in the first integral we obtain:-

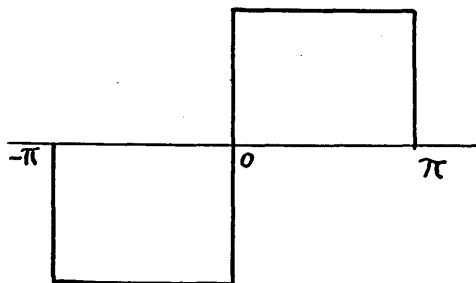
$$\begin{aligned} \int_{-\pi}^0 f(x) \cos(nx) dx &= \int_{\pi}^0 f(-y) \cos(-ny) d(-y) \\ &= \int_0^{\pi} f(-y) \cos(ny) dy. \end{aligned}$$

$$\text{Or:- } \int_{-\pi}^0 f(x) \cos(nx) dx = \int_0^{\pi} f(-x) \cos(nx) dx$$

$$\text{And:- } b_n = \frac{1}{\pi} \int_0^{\pi} [f(x) + f(-x)] \cos(nx) dx$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} [f(x) - f(-x)] \sin(nx) dx$$

Where n is any positive integer. This form is useful with some types of functions. For instance, a rectangular wave as indicated in the sketch.



From $\omega t = 0$ to $\omega t = \pi$, $i = 1$
 From $\omega t = 0$ to $\omega t = -\pi$, $i = -1$

$$\begin{aligned} \text{Then: } b_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} i \cos.n\omega t . d\omega t \\ &= \frac{1}{\pi} \int_0^{\pi} [1 + (-1)] \cos.n\omega t . d\omega t \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{and, } a_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} i \sin.n\omega t . d\omega t = \frac{1}{\pi} \int_0^{\pi} [1 - (-1)] \sin.n\omega t . d\omega t \\ &= 0 \text{ when } n \text{ is even.} \\ &= \frac{4I}{n\pi} \text{ when } n \text{ is odd.} \end{aligned}$$

Thus the series representing the rectangular wave will not contain any cosine terms or any terms of even order.

$$\text{Hence:- } i = \frac{4I}{\pi} \left[\sin\omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \dots \right]$$

With symmetrical curves such that the positive and negative halves are reversed duplicates of each other, we have:-

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \int_0^{+\pi} [f(x) + f(x-\pi) \cos n\pi] \cos(nx) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_0^{+\pi} [f(x) + f(x-\pi) \cos n\pi] \sin(nx) dx$$

For all even values of n , $b_n = 0$ and $a_n = 0$

Since $\cos n\pi = +1$. Thus only odd harmonics are present.

$$\begin{aligned} \text{Considering again, } b_n &= \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx) dx \\ &= \frac{1}{\pi} \int_0^{+\pi} [f(x) + f(-x) \cos(nx)] dx, \end{aligned}$$

b_n is always zero when $f(x) = -f(-x)$.

That is, b_n vanishes and thus the cosine terms are zero when the periodic curve is symmetrical about the origin.

SUMMARY OF BASIC THEORY AS APPLIED TO MACHINES.

The following expressions are developed by the mathematical analysis for the solution of Fourier's Series:

$$b_0 = 1/2\pi \int_0^{2\pi} y \cdot dx \quad (1)$$

$$b_k = 1/\pi \int_0^{2\pi} y \cdot \cos(kx) dx \quad (2)$$

$$a_k = 1/\pi \int_0^{2\pi} y \cdot \sin(kx) dx \quad (3)$$

$$b_0 = 1/n \sum y_r = 1/n (y_0 + y_1 + y_2 + y_3 + \dots + y_{n-1}) \quad (4)$$

$$b_{n/2} = 1/n \sum y_r \cos(n/2)x_r = 1/n \sum y_r \cos.r\pi = 1/n (y_0 - y_1 + y_2 - y_3 + \dots - y_{n-1}) \quad (5)$$

$$b_k = 2/n \sum y_r \cos.kx_r = 2/n (y_0 \cos.kx_0 + y_1 \cos.kx_1 + \dots + y_{n-1} \cos.kx_{n-1}) \quad (6)$$

$$a_k = 2/n \sum y_r \sin.kx_r = 2/n (y_0 \sin.kx_0 + y_1 \sin.kx_1 + \dots + y_{n-1} \sin.kx_{n-1}) \quad (7)$$

$$b_k = 1/\pi \int_0^\pi [f(x) + f(-x)] \cos(kx) dx \quad (8)$$

$$a_k = 1/\pi \int_0^\pi [f(x) - f(-x)] \sin(kx) dx \quad (9)$$

b_0 is thus seen to be the average of the n ordinates and will be zero if the areas of the curve on each side of the time axis are equal.

$b_{n/2}$ is the average value of the n ordinates taken alternately plus and minus.

a_k or b_k is 2x the average value of the products of the ordinates multiplied by their individual position angles.

These various formulae will be frequently referred to by number in describing the various machines.

SCHEDULE METHODS OF ANALYSIS.

SCHEDULE METHODS OF ANALYSIS.

(Byerly, Fourier Series and Sph. Harmonics, 1893.)

(C. Runge, Zeit. für Math. und Phys., Vol. 48; 52, p. 443; 117, 1903-05.)

(S. P. Thompson, Proc. Phys. Soc. Vol. 19, 33; p. 443, 334, 1905-11).

Schedule methods of analysis are the only practical methods of mathematically determining the coefficients in a Fourier Series. Any form arranged systematically may be called a schedule, but the type usually referred to under this name is the form developed by Carl Runge in 1903 and modified by various others to fill particular requirements. The theory is based upon equations (4), (5), (6) and (7), viz:

$$b_0 = 1/n(y_0 + y_1 + y_2 + y_3 + y_4 + \dots + y_{n-1})$$

$$b_{n/2} = 1/n(y_0 - y_1 + y_2 - y_3 + \dots - y_{n-1})$$

$$b_k = 2/n(y_0 \cos.kx_0 + y_1 \cos.kx_1 + y_2 \cos.kx_2 + \dots + y_{n-1} \cos.kx_{n-1})$$

$$a_k = 2/n(y_0 \sin.kx_0 + y_1 \sin.kx_1 + y_2 \sin.kx_2 + \dots + y_{n-1} \sin.kx_{n-1})$$

Thus for the constant term it is only necessary to alternately add and subtract the successive ordinates, the curve being divided up in n even sections and ordinates erected as shown in Fig. 18. The result divided by the number of ordinates will give the average ordinate, or constant term, which is similar to well known methods of making approximate integrations. This part of the analysis is not troublesome; however, and is generally accomplished by ordinary planimeter measurements with great accuracy.

The special case for $b_{n/2}$ is the basis for the Fischer-Hinnen method, separately discussed, and the two following equations will degenerate into this form when $k = n/2$. It is evident that only one coefficient can be determined from this special form for one set of curve ordinates; and the usual

schedule method therefore makes use of the last two of the above equations, and requires the curve to be divided up and ordinates measured just once. It is evident that it is not necessary to have a schedule form, since the same result may be obtained by multiplying the different ordinates by the sine or cosine of the position angle multiplied by k , which is the order of the harmonic coefficient being determined. However, if harmonics of even a moderately high order are thus determined the labor involved becomes very great. It is also evident that the sines and cosines of the various angles multiplied by the various values of k must go through repeated duplicate values, sometimes with the sign changed.

Several attempts have been made to simplify the method by merely tabulating the values of the different sines and cosines for a definite number of ordinates. It is interesting that some very extensive tabulations of this sort were made by S. M. Kintner, and published in the *Electrical World*, Vol. 43, p. 1023, 1904. It is difficult to see how Kintner could have prepared such tables and noted the trigonometric duplications without still further combination into some such form as that developed by Runge at almost the same time.

The best way of indicating the method of producing a schedule is to work it through for a simple case, since there is no additional theory required, but merely the proper arrangement of the various parts of the calculation. Since *Electrical Engineering* deals chiefly with odd harmonics only, this slightly limited case will be taken, as it avoids considerable complication and illustrates the principle adequately. An excellent discussion of the construction of schedules for even as well as odd harmonics is given by Hawley O. Taylor in the *Physical Review*, Vol. 6, No. 4, p. 303, October, 1915.

The condition assumed will be an analysis for odd harmonics up to and including the fifth. This will require the determination of six unknowns, the sine and cosine coefficients of the first, third, and fifth harmonics.

The analysis may be considered as the simultaneous solution of equations, and therefore six equations are necessary. Also the equations above show that ordinates are required of $n-1$ in number. The greatest value of k in this case is 5. Since $k=(n-1)$, $n=6$. Thus the curve must be divided up into six even parts along the X-axis, and the six ordinates read. The ordinates at the beginning and end of the curve will be zero, and so we have read from the curve the values:

$$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6$$

It should be noted that since the even harmonics are assumed absent, the two halves of the wave will be similar and it is therefore only necessary to consider the ordinates for one half of the complete period. If even harmonics are present then it is necessary to construct ordinates for the complete period in a similar manner.

As one half the period corresponds to 180° , the ordinates will occur at angles successively increasing by increments of $180/6=30^\circ$. Thus the values of x are:

$$x_0=0 \quad x_1=30^\circ \quad x_2=60^\circ \quad x_3=90^\circ \quad x_4=120^\circ \quad x_5=150^\circ \quad x_6=180^\circ$$

It is convenient, but not necessary, to choose x_0 or y_0 at a point where the curve crosses the X-axis; and thus $y_0=0$. Then: y_6 also = 0 (For no even harmonics present).

The values thus obtained are now substituted in the equations for the evaluation of the coefficients; and we obtain:

$$3b_1 = y_1 \cos 30^\circ + y_2 \cos 60^\circ + y_3 \cos 90^\circ + y_4 \cos 120^\circ + y_5 \cos 150^\circ$$

$$3b_3 = y_1 \cos 90^\circ + y_2 \cos 180^\circ + y_3 \cos 270^\circ + y_4 \cos 360^\circ + y_5 \cos 450^\circ$$

$$3b_5 = y_1 \cos 150^\circ + y_2 \cos 300^\circ + y_3 \cos 450^\circ + y_4 \cos 600^\circ + y_5 \cos 750^\circ$$

$$3a_1 = y_1 \sin 30^\circ + y_2 \sin 60^\circ + y_3 \sin 90^\circ + y_4 \sin 120^\circ + y_5 \sin 150^\circ$$

$$3a_3 = y_1 \sin 90^\circ + y_2 \sin 180^\circ + y_3 \sin 270^\circ + y_4 \sin 360^\circ + y_5 \sin 450^\circ$$

$$3a_5 = y_1 \sin 150^\circ + y_2 \sin 300^\circ + y_3 \sin 450^\circ + y_4 \sin 600^\circ + y_5 \sin 750^\circ$$

All of these sines and cosines may be expressed as functions of 30°, 60° or 90°. Several of them become zero. Substituting equivalent values and gathering terms the following equations result:

$$3b_1 = (y_2 - y_4) \sin 30^\circ + (y_1 - y_5) \sin 60^\circ$$

$$3b_3 = -(y_2 - y_4) \sin 90^\circ$$

$$3b_5 = (y_2 - y_4) \sin 30^\circ - (y_1 - y_5) \sin 60^\circ$$

$$3a_1 = (y_1 + y_5) \sin 30^\circ + (y_2 + y_4) \sin 60^\circ + y_3 \sin 90^\circ$$

$$3a_3 = (y_1 - y_3 + y_5) \sin 90^\circ$$

$$3a_5 = (y_1 + y_5) \sin 30^\circ - (y_2 + y_4) \sin 60^\circ + y_3 \sin 90^\circ$$

By inspection of the symmetry of the coefficients it appears that all of the ordinates appear in combination with one or more other ordinates except y_3 , that the cosine coefficients depend upon differences of ordinates, and that the sine coefficients depend upon sums of coefficients. Therefore the ordinates are written out as below and added and subtracted:

	y_1	y_2	y_3
	y_5	y_4	
Sum	s_1	s_2	s_3
Diff.	d_1	d_2	

The processes may then be arranged in a table as follows:

Multipliers.	Cosine terms.		Sine Terms.	
Sin 30° = 0.5	d_2		s_1	
Sin 60° = 0.866		d_1		s_2
Sin 90° = 1.0			$-d_2$	s_3
Sum 1st Col.				s_1
Sum 2nd Col.				s_3
SUM		$3b_1$	$3b_3$	$3a_1$
DIFF.		$3b_5$		$3a_5$
				$3a_3$

Checks: $b_1 + b_3 + b_5 = 0$
 $a_1 - a_3 + a_5 = y_3$

The same general arrangement will be found to hold for any number of harmonics. To operate the ordinates are read, arranged as indicated and the sums and differences taken. For higher orders of harmonics further combinations are found convenient. The values of s_k and d_k are then multiplied by the multiplier found in the left hand column of the schedule and inserted in the indicated place in the tabular form. The two columns are then added up, and the sums and differences of the totals of the columns give the coefficients sought times $n/2$. As pointed out by P. Kemp, Journal Inst. Elec. Engrs. Vol. 57, p. 85, 1920, the schedule may still further be slightly simplified by introducing the term $n/2$ into the multipliers, when the final division to obtain the value is unnecessary.

Figs. 1 to 16 give schedules for the following harmonics:

ODD HARMONICS ONLY:	Six Points. Harmonics to the 5th.
	Twelve Points. Harmonics to the 11th.
	Eighteen Pts. Harmonics to the 17th.
	Thirty Six Pt. Harmonics to the 35th.
EVEN AND ODD HARMONICS:	Six Points. Harmonics to the 3rd.
	Twelve Points. Harmonics to the 6th.
	Eighteen Pts. Harmonics to the 9th.
	Twenty Four Pt. Harmonics to the 11th.

Each Schedule is followed by an example worked through, for the odd harmonic types only, and not including the 36 point schedule, which would require too much space. The examples given, however, will undoubtedly make the method clear.

Sometimes it is desirable to determine ordinates between those chosen for the analysis, particularly for checking a curve with sudden peaks or peculiarities. Fig. 12 gives an inverted

schedule for determining the intermediate ordinates from the previously determined coefficients. This is also useful for constructing a curve from assumed coefficients.

The subscripts of the intermediate ordinates determined by this schedule indicate their position with respect to the original y_0 in degrees. Thus y_{40} is 40° away from y_0 and its complement y_{140} is 140° distant from y_0 .

Figs. 1 to 6, 12 to 15 and 17 are taken from Frederick W. Grover, Analysis of Alternating Current Waves by the Method of Fourier, Bulletin of the Bureau of Standards, Vol. 9, p. 567, 1913.

This article goes into the question of limitations and accuracy of the schedule methods very carefully. There are two main sources of error in the schedule method.

1> All harmonics above the number determined are neglected, and must be really negligible or will otherwise introduce large errors in the results.

2> The cosine terms are determined by the differences of ordinates, and thus the magnitude of error in the result may be many times the error of reading the ordinates themselves. This is also true of all the coefficients determined by the differences of the two columns; since the totals may be nearly the same, and the difference due entirely to errors in the determinations.

In order to show about the expected magnitude of the errors of the first class Grover computes the table given in Fig. 17. The curves referred to are taken from an alternator with a distorted wave shape. The schedules with the greatest number of points are of course the most accurate and show the errors obtained by too few points. For errors of the second class Grover made successive analyses with the same schedule for the the same curve, taking y_0 at different points. These results show about 3° maximum error in phase and a few percent error in magnitude, the errors being greatest for the smallest

harmonics and the schedules using the smallest number of points. If the apparent magnitude of the harmonic is only two or three percent of the fundamental, then the errors may be very large; 50 percent or more, in the magnitude of the harmonic, but only a small percentage of the fundamental. A fair accuracy to expect is probably about two percent of the fundamental as the maximum error for any term. This means that harmonics determined which give values less than two percent of the fundamental may be neglected since they may be only the result of errors in the method and not actually exist at all.

C. P. Steinmetz in his books on Engineering Mathematics and Elements of Electrical Engineering gives a sort of schedule method for determining the coefficients with far greater accuracy than by the more customary schedules. His method also requires far greater labor. The basic theory is the same. The difference lies in subtracting the coefficients as obtained from the function, and then continuing the analysis with the remainder. By this method the instantaneous values of the function are always of the same magnitude as the coefficients being determined. That is first b_0 is calculated, and then subtracted from y leaving $y' = y - b_0$. The a_1 and b_1 are determined and subtracted giving:

$$y'' = y_1 - (a_1 \sin \theta + b_1 \cos \theta) = y - (b_0 + a_1 \sin \theta + b_1 \cos \theta)$$

As an illustration he gives the determination of the first three harmonics of a pulsating current curve. The results give the value of the 7 terms $b_0, b_1, a_1, b_2, a_2, b_3, a_3$. In order to determine these coefficients he requires 36 entries in each of 12 columns, making 432 entries, most of which require some additional work such as multiplication or division etc.. When this is compared with the Runge Schedule for obtaining the same results it is safe to assume that the added accuracy is not worth the added labor required.

TABLE 1 Six Point Schedule

Measured ordinates	Sums	Diffs.		Sine terms		Cosine terms	
				1st and 5th	3d	1st and 5th	3d
y_0		d_0					
$y_1 \ y_5$	s_1	d_1	$\sin 30^\circ$	s_1		d_1	
$y_2 \ y_4$	s_2	d_2	" 60"	$s_2 \ s_2$		d_1	
y_3	s_3		" 90"	s_3	$s_1 - s_2$	d_0	$d_2 - d_1$
			sums	$S_0 \ S_6$	S	$D_0 \ D_6$	D
				$A_1 = \frac{S_0 + S_6}{3}$	$A_2 = \frac{S}{3}$	$B_1 = \frac{D_0 + D_6}{3}$	$B_2 = \frac{D}{3}$
				$A_3 = \frac{S_0 - S_6}{3}$		$B_3 = \frac{D_0 - D_6}{3}$	

TABLE 2 Twelve Point Schedule

Measured ordinates	Sums	Diffs.		Sine terms		
				1st and 11th	3d and 9th	5th and 7th
y_0		d_0				
$y_1 \ y_{11}$	s_1	d_1	$\sin 15^\circ$	s_1		s_1
$y_2 \ y_{10}$	s_2	d_2	" 30"	$s_2 \ s_2$		s_2
$y_3 \ y_9$	s_3	d_3	" 45"	s_3	d_1	$-s_2$
$y_4 \ y_8$	s_4	d_4	" 60"	s_4		$-s_1$
$y_5 \ y_7$	s_5	d_5	" 75"	s_5	d_2	s_2
y_6	s_6		" 90"	s_6	d_3	s_3
				$S_0^I \ S_6^I$	$S_0 \ S_6$	$S_0^{II} \ S_6^{II}$
				$A_1 = \frac{S_0^I + S_6^I}{6}$	$A_2 = \frac{S_0 + S_6}{6}$	$A_3 = \frac{S_0^{II} + S_6^{II}}{6}$
				$A_{11} = \frac{S_0^I - S_6^I}{6}$	$A_4 = \frac{S_0 - S_6}{6}$	$A_7 = \frac{S_0^{II} - S_6^{II}}{6}$
				Cosine terms		
				1st and 11th	3d and 9th	5th and 7th
			$\sin 15^\circ$	d_1		d_1
			" 30"	d_1		d_1
			" 45"	d_2	d_1	$-d_1$
			" 60"	d_2		$-d_2$
			" 75"	d_3	d_2	d_2
			" 90"	d_0	d_3	d_3
				$D_0^I \ D_6^I$	$D_0 \ D_6$	$D_0^{II} \ D_6^{II}$
				$B_1 = \frac{D_0^I - D_6^I}{6}$	$B_2 = \frac{d_2 + d_1}{6}$	$B_3 = \frac{D_0^{II} + D_6^{II}}{6}$
				$B_{11} = \frac{D_0^I - D_6^I}{6}$	$B_4 = \frac{d_2 - d_1}{6}$	$B_7 = \frac{D_0^{II} - D_6^{II}}{6}$

FIGURES No. 1 and 2.

TABLE 5
Analysis of a Curve from Six Ordinates

Measured ordinates	s.	d.	Sine terms					Cosine terms						
			1st and 5th				3d	1st and 5th				3d		
			+	-	+	-		+	-	+	-			
0	-1.3	-1.3												
1	15.1 17.3	32.4	-2.2						0.425					
2	28.0 28.85	56.85	-0.85			49.233		32.400			1.905		-1.300	
3	32.35	32.35				32.350		32.350		1.300			0.850	
Sums			48.550		49.233		0.050		1.725		1.905		-0.450	
Sc			49.233						1.905					
A ₁ =			97.783		-0.683				-3.630		0.180			
			32.594		A ₂ =-0.228		A ₃ =0.017		B ₁ =-1.210		B ₂ =0.060		B ₃ =-0.150	

CHECK

B ₁ +B ₂	-1.150	A ₁ +A ₂	32.366	(A ₁ -A ₂)	32.822	(B ₁ -B ₂)	-1.270
B ₂	-0.150	-A ₂	-0.017	× z	65.644	× z	-2.540
Sum	-1.300	Sum	32.349		56.291		-2.165
y ₀	-1.3	y ₁	32.35		.554		.035
					.003		
A ₁ +A ₂ =	32.366			z(A ₁ -A ₂)	56.848	z(B ₁ -B ₂)	-2.200
A ₁ -A ₂ =	32.822			× sin 60°		× sin 60°	
B ₁ +B ₂ =	-1.150			z	56.85	d ₁	-2.2
B ₁ -B ₂ =	-1.270						

CALCULATION OF THE AMPLITUDES AND PHASES

$C_1 = \sqrt{32.594^2 + 1.210^2}$	$C_2 = \sqrt{0.017^2 + 0.150^2}$	$C_3 = \sqrt{0.228^2 + 0.060^2}$
= 32.616	= 0.151	= 0.236
$\tan \theta_1 = \frac{1.210}{32.594}$	$\tan \theta_2 = \frac{0.150}{0.017}$	$\tan \theta_3 = \frac{-0.060}{-0.228}$
= 0.0371	= 8.8	= 0.263
$\theta_1 = 2^\circ 8'$	$\theta_2 = 83^\circ 5'$	$\theta_3 = 194^\circ 7'$
	$\psi_2 = 27^\circ 8'$	$\psi_3 = 38^\circ 9'$

FIGURE No. 3.

TABLE 6
Example of Analysis of Curve from Twelve Measured Ordinates

		Sine terms						Cosine terms					
		1st and 7th		3d and 9th		5th and 11th		1st and 7th		3d and 9th		5th and 11th	
0	0.3	+	+	+	+	+	+	+	+	+	+	+	+
1	8.7	4.432	16.350	0.778	0.450	16.228	32.951	16.614	32.842	32.951	48.600	49.233	49.233
2	14.3	32.951	49.233	0.778	0.450	-0.109	60.564	16.614	-0.109	32.951	-0.653	-0.653	-0.653
3	20.6	60.564	32.250	0.450	1.228	0.328	97.967	32.250	0.450	0.328	-0.742	-0.742	0.524
4	26.15	97.967	97.833	0.184	0.022	0.328	97.967	97.833	0.184	0.022	0.328	0.124	0.524
5	29.8	195.800	195.800	-0.111	-0.022	0.328	97.833	32.250	-0.111	-0.022	0.328	0.124	0.524
6	32.25	32.250	32.250	0.184	0.022	0.328	97.833	32.250	0.184	0.022	0.328	0.124	0.524
Sums		97.967	97.833	0.184	0.022	0.328	97.833	32.250	0.184	0.022	0.328	0.124	0.524
S_0		97.967	97.833	0.184	0.022	0.328	97.833	32.250	0.184	0.022	0.328	0.124	0.524
S_1		97.833	195.800	-0.111	-0.022	0.328	97.833	32.250	-0.111	-0.022	0.328	0.124	0.524
Amplitudes		$A_1=32.633$	$A_1=32.633$	$A_1=0.184$	$A_1=0.022$	$A_1=0.328$	$A_1=0.184$	$A_1=0.022$	$A_1=0.184$	$A_1=0.022$	$A_1=0.328$	$A_1=0.124$	$A_1=0.087$
0	0.3	+	+	+	+	+	+	+	+	+	+	+	+
1	17.2	0.300	0.300	0.802	0.802	0.802	0.300	0.300	0.802	0.802	0.802	0.300	0.300
2	46.6	0.300	0.300	2.275	2.275	2.275	0.300	0.300	2.275	2.275	2.275	0.300	0.300
3	63.8	0.300	0.300	3.551	3.551	3.551	0.300	0.300	3.551	3.551	3.551	0.300	0.300
4	62.7	0.300	0.300	0.193	0.193	0.193	0.300	0.300	0.193	0.193	0.193	0.300	0.300
5	1.1	0.300	0.300	4.813	4.813	4.813	0.300	0.300	4.813	4.813	4.813	0.300	0.300
6	0.300	0.300	0.300	4.813	4.813	4.813	0.300	0.300	4.813	4.813	4.813	0.300	0.300
Sums		0.300	0.300	4.813	4.813	4.813	0.300	0.300	4.813	4.813	4.813	0.300	0.300
D_0		0.300	0.300	4.813	4.813	4.813	0.300	0.300	4.813	4.813	4.813	0.300	0.300
D_1		-5.576	-5.576	4.813	4.813	4.813	-5.576	-5.576	4.813	4.813	4.813	0.300	0.300
D_2		-4.813	-4.813	10.719	10.719	10.719	-4.813	-4.813	10.719	10.719	10.719	0.300	0.300
D_3		-10.339	-10.339	-0.713	-0.713	-0.713	-10.339	-10.339	-0.713	-0.713	-0.713	0.300	0.300
Amplitudes		$B_1=1.786$	$B_1=1.786$	$B_1=-0.119$	$B_1=-0.119$	$B_1=-0.119$	$B_1=1.786$	$B_1=1.786$	$B_1=-0.119$	$B_1=-0.119$	$B_1=-0.119$	$B_1=0.391$	$B_1=0.134$

CHECK

		A_1+A_{11}		A_3+A_9		A_5+A_{11}		B_1-B_{11}		B_3-B_{11}		B_5-B_{11}	
+	1.842	32.611	0.361	32.611	0.361	32.611	0.361	32.611	0.361	32.611	0.361	32.611	0.361
-	1.616	-0.150	0.211	-0.150	0.211	-0.150	0.211	-0.150	0.211	-0.150	0.211	-0.150	0.211
B_1+B_{11}	0.525	$-(A_3+A_9)$		$-(A_5+A_{11})$		$-(A_3+A_9)$		$-(A_5+A_{11})$		$-(A_3+A_9)$		$-(A_5+A_{11})$	
B_3+B_{11}	2.141	0.361	0.361	0.361	0.361	0.361	0.361	0.361	0.361	0.361	0.361	0.361	0.361
B_5+B_{11}	0.299	32.250	32.250	32.250	32.250	32.250	32.250	32.250	32.250	32.250	32.250	32.250	32.250
Actual value	0.3	Actual value	32.25	Actual value	32.25	Actual value	32.25	Actual value	32.25	Actual value	32.25	Actual value	32.25

CALCULATION OF THE AMPLITUDES AND PHASES.

$C_1 = \sqrt{32.633^2 + 1.723^2 - 32.678}$
 $C_2 = \sqrt{0.205^2 + 1.786^2 - 1.758}$
 $C_3 = \sqrt{0.124^2 + 0.391^2 - 0.410}$
 $C_4 = \sqrt{0.087^2 + 0.134^2 - 0.160}$
 $C_5 = \sqrt{0.055^2 + 0.170^2 - 0.179}$
 $C_{11} = \sqrt{0.119^2 + 0.022^2 - 0.121}$

$\tan \theta_1 = \frac{1.723}{32.633}$
 $\tan \theta_2 = \frac{-1.786}{0.205}$
 $\tan \theta_3 = \frac{-0.391}{0.124}$
 $\tan \theta_4 = \frac{-0.134}{0.087}$
 $\tan \theta_5 = \frac{0.170}{0.055}$
 $\tan \theta_{11} = \frac{0.119}{0.022}$

$\theta = 3.2^\circ$
 $\theta_2 = 276.6^\circ$
 $\theta_3 = 252.4^\circ$
 $\theta_4 = 303.0^\circ$
 $\theta_5 = 72.1^\circ$
 $\theta_{11} = 79.55^\circ$

$\sum X_1 \sin 45^\circ$
 $\sum X_2$
 $\sum X_3$
 $\sum X_4$
 $\sum X_5$
 $\sum X_{11}$

$B_1 - B_{11}$
 $-(B_3 - B_{11})$
 $-(B_5 - B_{11})$
 $\sum X_1 \sin 45^\circ$
 $\sum X_2$
 $\sum X_3$
 $\sum X_4$
 $\sum X_5$
 $\sum X_{11}$

FIGURE No. 4.

TABLE 3
Eighteen Point Schedule

Measured ordinates	Sums	Diffs.	Sine terms					
			1st and 17th	3d and 15th	5th and 13th	7th and 11th	9th	
y_0		d_0						
y_1 y_{17}	s_1	d_1	s_1		$-s_7$	$-s_6$		
y_2 y_{16}	s_2	d_2	s_2		$-s_4$	$-s_4$		
y_3 y_{15}	s_3	d_3	s_3	d_1	s_2	$-s_2$	s_2	
y_4 y_{14}	s_4	d_4	s_4		s_4	s_4		
y_5 y_{13}	s_5	d_5	s_5		s_1	s_7	s_2	
y_6 y_{12}	s_6	d_6	s_6		$-s_5$	s_4	s_4	
y_7 y_{11}	s_7	d_7	s_7		$-s_6$	s_1	$-s_1$	
y_8 y_{10}	s_8	d_8	s_8		s_7	$-s_2$	$-s_2$	
y_9	s_9	d_9	s_9	d_2	s_7	$-s_2$		Σ
$d_1 = s_1 + s_9 - s_7$ $d_2 = s_2 + s_8 - s_6$ $d_3 = s_3 - s_5$			$\frac{So^I + Se^I}{9}$ $\frac{So^I - Se^I}{9}$	$\frac{So^{II} + Se^{II}}{9}$ $\frac{So^{II} - Se^{II}}{9}$	$\frac{So^{III} + Se^{III}}{9}$ $\frac{So^{III} - Se^{III}}{9}$	$\frac{So^{IV} + Se^{IV}}{9}$ $\frac{So^{IV} - Se^{IV}}{9}$	$\frac{So^I + Se^I + So^{II} + Se^{II} + So^{III} + Se^{III} + So^{IV} + Se^{IV}}{9}$ $\frac{So^I - Se^I + So^{II} - Se^{II} + So^{III} - Se^{III} + So^{IV} - Se^{IV}}{9}$	$\frac{\Sigma}{9}$
			Cosine terms					
			1st and 17th	3d and 15th	5th and 13th	7th and 11th	9th	
d_0			d_0		$-d_2$	d_1		
d_1			d_1		$-d_5$	d_1		
d_2			d_2	d_2	d_4	d_4		
d_3			d_3		d_1	$-d_7$		
d_4			d_4		$-d_3$	$-d_3$		
d_5			d_5	d_1	$-d_1$	$-d_5$		
d_6			d_6		d_7	d_5		
d_7			d_7	d_6	d_6	d_6	Δ	
$\Delta = d_7 - d_5$ $d_1 = d_1 - d_5 - d_7$ $d_2 = d_2 - d_4 - d_6$ $d_3 = d_3 - d_1$			$\frac{Do^I + De^I}{9}$ $\frac{Do^I - De^I}{9}$	$\frac{Do^{II} + De^{II}}{9}$ $\frac{Do^{II} - De^{II}}{9}$	$\frac{Do^{III} + De^{III}}{9}$ $\frac{Do^{III} - De^{III}}{9}$	$\frac{Do^{IV} + De^{IV}}{9}$ $\frac{Do^{IV} - De^{IV}}{9}$	$\frac{\Delta}{9}$	

FIGURE No. 5.

TABLE 7
 Example of Analysis of a Curve from Eighteen Measured Ordinates

Measured ordinates	Sine terms																	
	<i>s</i>	<i>u</i>	1st and 17th		3d and 15th		5th and 13th		7th and 11th		9th							
0	0.1	0.1	+	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
1	4.6	6.3	10.9	-1.7	1.893	7.319	1.150	10.732	16.075	20.145	16.150	37.859	53.433	47.341	10.243	33.000	57.584	58.388
2	10.2	11.2	21.4	-1.0	16.150	30.211	8.227	8.349	49.992	7.349	13.755	53.433	53.433	16.286	97.100	94.100	67.188	66.431
3	13.9	18.4	32.3	-4.5	40.753	53.433	58.004	33.000	21.075	10.243	33.000	57.584	58.388	16.286	97.100	94.100	67.188	66.431
4	22.25	24.75	47.0	-2.5	58.074	58.004	33.000	21.075	10.243	33.000	57.584	58.388	16.286	97.100	94.100	67.188	66.431	66.431
5	26.8	28.4	53.2	0.4	149.870	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
6	29.35	32.35	61.7	-3.0	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837
7	33.4	38.4	61.8	5.0	149.870	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
8	28.1	30.8	58.9	-2.7	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
9	33.0	33.0	33.0		148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
Sum			148.967		148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
<i>S₀</i>			148.967		148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
<i>S₁</i>			298.837		298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837	298.837
Amplitudes			148.967		148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967	148.967
			<i>A₁</i>		<i>A₂</i>	<i>A₃</i>		<i>A₄</i>		<i>A₅</i>		<i>A₆</i>		<i>A₇</i>		<i>A₈</i>		<i>A₉</i>
			0.903		0.864	0.964		1.533		0.817		0.817		0.817		0.817		0.817

Cosine terms																	
Measured ordinates	1st and 17th		3d and 15th		5th and 13th		7th and 11th		9th								
	0	+	+	+	+	+	+	+	+	+	+						
1	0.459	1.710	2.100	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
2	1.500	0.257	2.100	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
3	1.915	3.897	3.897	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
4	0.940	1.674	3.100	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
5	0.100	1.967	5.371	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
6	4.824	5.200	5.200	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
7	-4.724	5.200	5.200	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
8	-3.604	-6.149	-6.149	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
9	-8.328	-1.120	-0.949	6.149	6.149	6.149	6.149	6.149	6.149	6.149							
Amplitudes				<i>B₁</i>		<i>B₂</i>		<i>B₃</i>		<i>B₄</i>							
				0.124		0.105		0.738		0.948							

CHECK																	
Measured ordinates	1st and 17th		3d and 15th		5th and 13th		7th and 11th		9th								
	0	+	+	+	+	+	+	+	+	+	+						
1	1.049	33.304	33.304	6.801	6.801	6.801	6.801	6.801	6.801	6.801							
2	0.210	0.100	0.100	6.801	6.801	6.801	6.801	6.801	6.801	6.801							
3	0.122	0.179	0.333	6.801	6.801	6.801	6.801	6.801	6.801	6.801							
4	1.381	33.816	33.816	6.801	6.801	6.801	6.801	6.801	6.801	6.801							
5	0.101	33.000	33.000	6.801	6.801	6.801	6.801	6.801	6.801	6.801							
6	0.1	33.0	33.0	6.801	6.801	6.801	6.801	6.801	6.801	6.801							
Amplitudes				<i>C₁</i>		<i>C₂</i>		<i>C₃</i>		<i>C₄</i>							
				0.100		0.100		0.100		0.100							

CALCULATION OF THE AMPLITUDES AND PHASES.

$$C_1 = \sqrt{33.204^2 + 0.925^2} = 33.217 \quad \tan \theta_1 = 0.925 / 33.204 \quad \theta_1 = 1^{\circ}36'$$

$$C_2 = \sqrt{0.964^2 + 0.105^2} = 0.970 \quad \tan \theta_2 = 0.105 / 0.964 \quad \theta_2 = 6^{\circ}3'$$

$$C_3 = \sqrt{1.533^2 + 0.738^2} = 1.702 \quad \tan \theta_3 = -0.738 / 1.533 \quad \theta_3 = 208^{\circ}8'$$

$$C_4 = \sqrt{0.005^2 + 0.218^2} = 0.218 \quad \tan \theta_4 = -0.218 / 0.005 \quad \theta_4 = 268^{\circ}1'$$

$$C_5 = \sqrt{0.333^2 + 0.122^2} = 0.355 \quad \tan \theta_5 = 0.122 / 0.333 \quad \theta_5 = 20^{\circ}8'$$

$$C_6 = \sqrt{0.174^2 + 0.108^2} = 0.205 \quad \tan \theta_6 = -0.108 / 0.174 \quad \theta_6 = 211^{\circ}8'$$

$$C_7 = \sqrt{0.817^2 + 0.948^2} = 1.251 \quad \tan \theta_7 = 0.948 / 0.817 \quad \theta_7 = 49^{\circ}3'$$

$$C_8 = \sqrt{0.864^2 + 1.201^2} = 1.528 \quad \tan \theta_8 = -1.201 / 0.864 \quad \theta_8 = 235^{\circ}6'$$

$$C_9 = \sqrt{0.100^2 + 0.124^2} = 0.159 \quad \tan \theta_9 = 0.124 / 0.100 \quad \theta_9 = 51^{\circ}1'$$

89420-13. (To face p. 594.)

FIGURE No. 6.

CURVE NO. _____

VIBRATOR NO. _____

18. Ordinate Analysis, Sheet No. 1.

ORIGINAL DATA				RESULTS $F = \sqrt{\sum \frac{(A_n^2 + B_n^2)}{0.707 F} = G}$					
n	Y _n	SIN X	S _n	n	d _n	A _n	B _n	C _n	100 C _n /F
0				0					
1				1		1			
2				2		3			
3				3		5			
4				4		7			
5				5		9			
6				6		11			
7				7		13			
8				8		15			
9				9		17			

$s'_1 = s_1 + s_5 - s_7 =$
 $s'_2 = s_2 + s_4 - s_8 =$
 $s'_3 = s_3 - s_9 =$
 $s''_1 = s'_1 - s'_3 =$
 $d'_0 = d_0 - d_6 =$
 $d'_1 = d_1 - d_5 - d_7 =$
 $d'_2 = d_2 - d_4 - d_8 =$
 $d''_0 = d'_0 - d'_2 =$

CHECK ON HARMONIC ANALYSIS

SINE TERMS				COSINE TERMS				CHECK	
n	A _n	n	+ - n	n	B _n	n	+ - n	ORD. COMP.	ORD. DATA
1	1	1		1	1	1		n	0
3	5	8		3	3	3		n	3
5	5	5		5	5	5		n	6
7	5	-7		7	7	7		n	9
9	5	-9		9	9	9		n	12
11	5			11	11	11		n	15
13	5			13	13	13		n	18
15	5			15	15	15		n	15
17	5			17	17	17		n	18

$D = A - 2F =$ $A =$ $F =$
 $Y_0 = H + J + K =$ $B =$ $Y_3 = 0.5A + B + 0.866M =$ $H =$ $Y_6 = 0.866D + 0.5Y_0 - 1.5K =$
 $M = H - J =$ $Y_{15} = -Y_0 =$ $Y_{18} = 0.866D - 0.5Y_0 + 1.5K =$ $Y_0 = A - B =$
 Computed by _____ Date _____

Fig. No 7.

CURVE NO. _____ VIBRATOR NO. _____

18-Ordinate Analysis, Sheet No. 2.

	17	8	15	5	13	7	11	9
SIN 10°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 20°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 30°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 40°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 50°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 60°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 70°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 80°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
" 90°	S ₁	S ₂	S ₃	S ₄	S ₅	S ₆	S ₇	S ₈
I = SUM 1st COLS.	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈
II = " 2d COLS.	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈
I + II	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈
I - II	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈
1/9 (I + II)	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈
1/9 (I - II)	A ₁	A ₂	A ₃	A ₄	A ₅	A ₆	A ₇	A ₈
SIN 10°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 20°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 30°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 40°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 50°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 60°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 70°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 80°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
" 90°	d ₁	d ₂	d ₃	d ₄	d ₅	d ₆	d ₇	d ₈
I = SUM 1st COLS.	B ₁	B ₂	B ₃	B ₄	B ₅	B ₆	B ₇	B ₈
II = " 2d COLS.	B ₁	B ₂	B ₃	B ₄	B ₅	B ₆	B ₇	B ₈
I + II	B ₁	B ₂	B ₃	B ₄	B ₅	B ₆	B ₇	B ₈
I - II	B ₁	B ₂	B ₃	B ₄	B ₅	B ₆	B ₇	B ₈
1/9 (I + II)	B ₁	B ₂	B ₃	B ₄	B ₅	B ₆	B ₇	B ₈
1/9 (I - II)	B ₁	B ₂	B ₃	B ₄	B ₅	B ₆	B ₇	B ₈

Computed by _____
Date _____

Fig. No 8.

PIF 10000

CURVE NO.----- VIBRATOR NO.----- REMARKS -----

ORIGINAL DATA					RESULTS $F = \sqrt{2(A_n^2 + B_n^2)}$ ----- $G = 0.707 F$ -----				
n	Yn	tn	Sn	Dn	n	An	Bn	Cn	100 Cn/F
1		35			1				
2		34			3				
3		33			5				
4		32			7				
5		31			9				
6		30			11				
7		29			13				
8		28			15				
9		27			17				$S_1' = S_1 + S_{11} - S_{13} =$
10		26			19				$S_2' = S_2 + S_{10} - S_{14} =$
11		25			21				$S_3' = S_3 + S_9 - S_{15} =$
12		24			23				$S_4' = S_4 + S_8 - S_{16} =$
13		23			25				$S_5' = S_5 + S_7 - S_{17} =$
14		22			27				$S_6' = S_6 - S_{18} =$
15		21			29				$S_7' = S_7 + S_5 - S_5' =$
16		20			31				$S_8' = S_8 - S_6' =$
17		19			33				$d_1' = d_0 - d_{12} =$
18					35				$d_2' = d_1 - d_{11} - d_{13} =$
19									$d_3' = d_4 - d_{10} - d_{14} =$
20					0				$d_4' = d_5 - d_9 - d_{15} =$
21									$d_5' = d_6 - d_8 - d_{16} =$
22									$d_6' = d_7 - d_7 - d_{17} =$
23									$d_7' = d_1' - d_5' - d_5' =$
24									$d_8' = d_5' - d_5' =$

CURVE NO.----- VIB. NO.-----

COMPUTED BY----- DATE:-----

FIG. No 9.

TECHNICAL REPORT NO. 62

SIN 5 DEG	1	35	3	33	5	31	7	29	9	27	11	25	13	23	15	21	17	19	
5	0.0872				S ₇														
10	0.1736					S ₁₄													
15	0.2598						S ₂₁												
20	0.3420							S ₂₈											
25	0.4226								S ₃₅										
30	0.5000									S ₄₂									
35	0.5736										S ₄₉								
40	0.6428											S ₅₆							
45	0.7071												S ₆₃						
50	0.7660													S ₇₀					
55	0.8192														S ₇₇				
60	0.8660															S ₈₄			
65	0.9063																S ₉₁		
70	0.9414																	S ₉₈	
75	0.9709																		S ₁₀₅
80	0.9949																		S ₁₁₂
85	1.0132																		S ₁₁₉
90	1.0261																		S ₁₂₆
SUM 11° COS																			
I = II																			
I - II																			
A (I - II)	A ₁			A ₅			A ₉			A ₁₃			A ₁₇			A ₂₁			
A (I + II)	A ₁₂₅			A ₁₃₅			A ₁₄₅			A ₁₅₅			A ₁₆₅			A ₁₇₅			

SIN 5 DEG	1	35	3	33	5	31	7	29	9	27	11	25	13	23	15	21	17	19	
5	0.0872					d ₇													
10	0.1736						d ₁₄												
15	0.2598							d ₂₁											
20	0.3420								d ₂₈										
25	0.4226									d ₃₅									
30	0.5000										d ₄₂								
35	0.5736											d ₄₉							
40	0.6428												d ₅₆						
45	0.7071													d ₆₃					
50	0.7660														d ₇₀				
55	0.8192															d ₇₇			
60	0.8660																d ₈₄		
65	0.9063																	d ₉₁	
70	0.9414																		d ₉₈
75	0.9709																		d ₁₀₅
80	0.9949																		d ₁₁₂
85	1.0132																		d ₁₁₉
90	1.0261																		d ₁₂₆
SUM 11° COS																			
I = II																			
I - II																			
B (I - II)	B ₁			B ₅			B ₉			B ₁₃			B ₁₇			B ₂₁			
B (I + II)	B ₁₂₅			B ₁₃₅			B ₁₄₅			B ₁₅₅			B ₁₆₅			B ₁₇₅			

CHECK ON 36 ORDINATE ANALYSIS

SINE TERMS

OSC. NO. _____ VIB. NO. _____

	+	-		+	-		+	-
A ₁			A ₃			A ₅		
-A ₂₃			-A ₂₁			-A ₁₉		
A ₂₅			A ₂₇			A ₂₉		
A			B			C		

	+	-		+	-		+	-
A ₇			A ₉			A ₁₁		
-A ₁₇			-A ₁₅			-A ₁₃		
A ₃₁			A ₃₃			A ₃₅		
D			E			F		

n	ORDINATES	
	COMP	DATA
0		
6		
9		
12		
18		
24		
27		
30		
36		

	+	-		+	-		+	-		
A			B			C			C	
-F			-E			-D			-E	
									-F	
H			J			K			L	

COSINE TERMS

	+	-		+	-		+	-
B ₁			B ₃			B ₅		
B ₂₃			B ₂₁			B ₁₉		
B ₂₅			B ₂₇			B ₂₉		
M			N			O		

	+	-		+	-		+	-
B ₇			B ₉			B ₁₁		
B ₁₇			B ₁₅			B ₁₃		
B ₃₁			B ₃₃			B ₃₅		
P			Q			R		

$\omega = H + K =$
 $\beta = \alpha + J - 2L =$
 $\gamma = \alpha - 2K =$
 $\delta = S - U =$
 $\epsilon = Y_0 - 2V =$
 $Y_0 = S + T + U$
 $Y_6 = 0.5\alpha + J + 0.866\delta$
 $Y_9 = 0.707(\beta + \epsilon)$
 $Y_{12} = 0.866\gamma + 0.5 Y_0 - 1.5T$
 $Y_{18} = \alpha - J$

REVERSE SIGNS OF COSINE TERMS FOR SECOND QUADRANT

	+	-		+	-		+	-		+	-
M			N			O			N		
R			Q			P			O		
									R		
S			T			U			V		

FIG. No 11.

TABLE 4
For the Calculation of the Ordinates of Points Intermediate to Those Used in the Analysis

SIX-POINT SCHEDULE

sin 15°	A ₁ B ₃		A ₃ B ₁
" 45°	A ₂ B ₂	(A ₁ +A ₃ -A ₂) (B ₁ -B ₂ -B ₃)	-A ₂ -B ₂
" 75°	A ₃ B ₁		A ₁ B ₃
Sums	M ₁ N ₁ y ₁₅ = M ₁ +N ₁ y ₁₄₅ = M ₁ -N ₁	M ₂ N ₂ y ₄₅ = M ₂ +N ₂ y ₁₃₅ = M ₂ -N ₂	M ₃ N ₃ y ₇₅ = M ₃ +N ₃ y ₁₀₅ = M ₃ -N ₃

TWELVE-POINT SCHEDULE

sin 10°	A ₁	-B ₅	(B ₇ +B ₁₁)	-(A ₇ +A ₁₁)	-A ₅	B ₁	B ₁
" 20°	(B ₇ -B ₁₁)	A ₁	-A ₅	-B ₅	-A ₂	B ₁	-(A ₇ +A ₁₁)
" 30°	A ₂	B ₂	-B ₂	A ₂	-A ₂	-B ₂	-B ₂
" 40°	B ₂	(A ₇ -A ₁₁)	A ₁	B ₁	B ₁	-(B ₇ -B ₁₁)	A ₅
" 50°	A ₃	-(B ₇ +B ₁₁)	B ₁	A ₁	(A ₇ +A ₁₁)	-B ₂	B ₅
" 60°	B ₂	A ₂	A ₂	-B ₂	-B ₂	-B ₂	-A ₂
" 70°	(A ₇ +A ₁₁)	B ₁	-B ₂	-A ₂	A ₁	B ₅	-(B ₇ +B ₁₁)
" 80°	B ₁	A ₂	-(A ₇ -A ₁₁)	(B ₇ -B ₁₁)	B ₂	B ₅	A ₁
" 90°	A ₂	-B ₂	B ₂	A ₂	-A ₂	-B ₂	B ₂
Sums	M ₁ N ₁	M ₂ N ₂	M ₃ N ₃	M ₄ N ₄	M ₅ N ₅	M ₆ N ₆	M ₇ N ₇
M+N	y ₁₀	y ₃₀	y ₅₀	y ₇₀	y ₉₀	y ₁₁₀	y ₁₃₀
M-N	y ₁₇₀	y ₁₅₀	y ₁₃₀	y ₁₁₀	y ₉₀	y ₇₀	y ₅₀

EIGHTEEN-POINT SCHEDULE

sin 15°	a ₁ b ₂		a ₂ b ₁
" 45°	a ₂ b ₂	α β	-a ₂ -b ₂
" 75°	a ₁ b ₁		a ₁ b ₂
Sums	M ₁ N ₁ y ₁₅ = M ₁ +N ₁ y ₁₆₅ = M ₁ -N ₁	α β y ₄₅ = α+β y ₁₃₅ = α-β	M ₂ N ₂ y ₇₅ = M ₂ +N ₂ y ₁₀₅ = M ₂ -N ₂

where

a ₁ = A ₁ +A ₁₁ -A ₁₃	b ₁ = B ₁ -B ₁₁ -B ₁₇	α = a ₁ +a ₂ -a ₃
a ₂ = A ₂ +A ₉ -A ₁₁	b ₂ = B ₂ -B ₂ -B ₁₁	β = b ₁ -b ₂ -b ₃
a ₃ = A ₃ +A ₇ -A ₁₇	b ₃ = B ₃ -B ₇ -B ₁₇	

FIGURE No. 12.

SCHEDULE 1

Analysis of a Curve, Involving Even Harmonics and a Constant Term, from Six Ordinates

Arrange the measured ordinates according to the following scheme and take the sums and differences indicated:

	y_0	y_1	y_2	y_3		
	y_0	y_1	y_2	y_3		
Sums	s_0	s_1	s_2	s_3		$s_0 + s_2 = I_0$
Diffs	d_0	d_1	d_2	d_3		$s_1 + s_3 = I_1$

(the sum $s_0 = d_0 = y_0$, but this nomenclature is adhered to for uniformity.)
 The coefficients are those given in the schedule below, the arrangement being the same as in the previous schedules—Table 1, 2, and 3:

	sine terms		cosine terms			
	A ₁ and A ₂		B ₁ and B ₂		B ₀ and B ₂	
sin 30°			-B ₂	B ₁		
sin 60°	s ₁	s ₂	s ₀	-B ₂	I ₀	I ₁
sin 90°						
Sums	S _{0'}	S _{2'}	S _{0''}	S _{2''}	S _{0'''}	S _{2'''}
	$A_1 = \frac{S_0' + S_2'}{3}$		$B_1 = \frac{S_0'' + S_2''}{3}$		$B_0 = \frac{S_0''' + S_2'''}{6}$	
	$A_2 = \frac{S_0' - S_2'}{3}$		$B_2 = \frac{S_0'' - S_2''}{3}$		$B_2 = \frac{S_0''' - S_2'''}{6}$	

CHECKS

$$\begin{aligned}
 s_0 &= (B_0 + B_2) + (B_1 + B_2) \\
 s_2 &= 2(B_0 + B_2) - (B_1 + B_2) \\
 s_0 + s_2 &= 3(B_0 + B_2) \quad 2s_0 - s_2 = 3(B_1 + B_2) \\
 s_1 &= 2(B_0 - B_2) + (B_1 - B_2) \\
 s_3 &= (B_0 - B_2) - (B_1 - B_2) \\
 s_1 + s_3 &= 3(B_0 + B_2) \quad s_1 - 2s_3 = 3(B_1 - B_2) \\
 d_1 &= 2(A_1 + A_2) \sin 60^\circ \\
 d_3 &= 2(A_1 - A_2) \sin 60^\circ
 \end{aligned}$$

The first equation checks the sums of the B's. If it is not fulfilled, equations may be used in which these sums appear singly. The same procedure is to be adopted with the differences of the B's. The sum and difference of the A's occur singly. These checks serve as a control on the values of S₀ and S₂.

FIGURE No. 13.

SCHEDULE 2

Analysis of a Curve, Involving Even Harmonics and a Constant Term, from Twelve Ordinates

MEASURED ORDINATES

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}
Sums												Sums Σ_0				Sums Σ_1				Sums Σ_2				Sums Σ_3											
Diffs.												Diffs. Δ_0				Diffs. Δ_1				Diffs. Δ_2				Diffs. Δ_3											
sine terms																																			
												A ₁ and A ₃				A ₂ and A ₄				A ₅															
												σ_1				σ_2				δ_1				δ_2				$(\sigma_1 - \sigma_2)$							
												σ_3																							
Sums												S_0'				S_2''				S_0''				S_2''				S_0'''							
												$A_1 = \frac{S_0' + S_2'}{6}$				$A_3 = \frac{S_0' - S_2'}{6}$				$A_2 = \frac{S_0'' + S_2''}{6}$				$A_4 = \frac{S_0'' - S_2''}{6}$				$A_5 = \frac{S_0''' + S_2'''}{6}$							
												$A_3 = \frac{S_0' - S_2'}{6}$				$A_2 = \frac{S_0'' - S_2''}{6}$																			
cosine terms																																			
												B ₁ and B ₃				B ₂ and B ₄				B ₅				B ₆ and B ₇											
												Δ_3				$-\Sigma_3$				Σ_1								$(\Sigma_0 + \Sigma_2)$				$(\Sigma_1 + \Sigma_3)$			
												Δ_0				Δ_1								$(\Delta_0 - \Delta_2)$											
Sums												D_0'				D_2''				D_0''				D_2''				D_0'''				D_2'''			
												$B_1 = \frac{D_0' + D_2'}{6}$				$B_3 = \frac{D_0' - D_2'}{6}$				$B_2 = \frac{D_0'' + D_2''}{6}$				$B_4 = \frac{D_0'' - D_2''}{6}$				$B_5 = \frac{D_0''' + D_2'''}{6}$				$B_6 = \frac{D_0''' - D_2'''}{6}$			
												$B_3 = \frac{D_0' - D_2'}{6}$				$B_2 = \frac{D_0'' - D_2''}{6}$				$B_4 = \frac{D_0'' - D_2''}{6}$				$B_6 = \frac{D_0''' - D_2'''}{6}$				$B_7 = \frac{D_0''' - D_2'''}{6}$							

CHECKS

$$\begin{aligned} \Sigma_0 &= (B_1 + B_3) + (B_1 + B_3) + (B_1 + B_3) + B_5 \\ \Sigma_2 &= 2[(B_1 + B_3) + (B_1 + B_3)] \\ \Delta_0 &= 2[(B_1 + B_3) + B_5] \\ \Sigma_1 &= 4(B_2 - B_4) + 2(B_2 - B_4) \\ \Delta_1 &= 4(B_1 - B_3) \sin 60^\circ \\ \sigma_1 &= 2(A_1 + A_3) + 4A_5 \\ \delta_1 &= 4(A_2 + A_4) \sin 60^\circ \\ \sigma_2 &= 4(A_1 - A_3) \sin 60^\circ \\ \delta_2 &= 4(A_2 - A_4) \sin 60^\circ \end{aligned}$$

FIGURE No. 14.

SCHEDULE 3

Analysis of a Curve Involving Even Harmonics and a Constant Term from Eighteen Ordinates

MEASURED ORDINATES

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	$d_1 - d_3 + d_5 = \Delta_1$	$\Sigma_0 + \Sigma_2 = \delta_0$
y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}	y_{17}	y_{18}	y_{19}	y_{20}	$d_2 - d_4 + d_6 = \Delta_2$	$\Sigma_1 + \Sigma_3 = \delta_1$
Sums s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9										$s_0 + s_2 = \Sigma_0$	
Diffs d_0 d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 d_9										$s_3 + s_9 = \Sigma_3$	
										$s_1 + s_3 + s_7 = \Sigma_1$	
										$s_2 + s_4 + s_6 = \Sigma_2$	

	Sine terms			
	A_1 and A_3	A_2 and A_7	A_5 and A_6	A_4 and A_8
$\sin 20^\circ$	d_1 d_3	$-d_5$ d_7		$-d_7$ d_9
$\sin 40^\circ$	d_2 d_4	d_1 $-d_6$		d_8 $-d_1$
$\sin 60^\circ$	d_3 d_5	d_2 $-d_8$	Δ_1 Δ_2	$-d_3$ d_6
$\sin 80^\circ$	d_4 d_6	$-d_7$ d_9		d_1 $-d_4$
Sums	D_0 D_2 $A_1 = \frac{D_0 + D_2}{9}$ $A_3 = \frac{D_0 - D_2}{9}$	D_0' D_2' $A_2 = \frac{D_0' + D_2'}{9}$ $A_7 = \frac{D_0' - D_2'}{9}$	D_0'' D_2'' $A_5 = \frac{D_0'' + D_2''}{9}$ $A_6 = \frac{D_0'' - D_2''}{9}$	D_0''' D_2''' $A_4 = \frac{D_0''' + D_2'''}{9}$ $A_8 = \frac{D_0''' - D_2'''}{9}$

	Cosine terms				
	B_1 and B_3	B_2 and B_7	B_5 and B_6	B_4 and B_8	B_0 and B_9
$\sin 10^\circ$	s_4 $-s_5$	s_2 s_7		s_6 s_1	
$\sin 30^\circ$	$-s_6$ s_3	$-s_4$ $-s_2$	$-\Sigma_2$ Σ_1	$-s_4$ $-s_3$	
$\sin 50^\circ$	s_2 $-s_7$	s_6 s_1		s_4 s_3	
$\sin 70^\circ$	$-s_6$ s_3	$-s_4$ $-s_2$		$-s_4$ $-s_3$	
$\sin 90^\circ$	s_0 $-s_9$	s_6 s_1	Σ_0 $-\Sigma_3$	s_6 s_1	δ_0 δ_1
Sums	S_0 S_2 $B_1 = \frac{S_0 + S_2}{9}$ $B_3 = \frac{S_0 - S_2}{9}$	S_0' S_2' $B_2 = \frac{S_0' + S_2'}{9}$ $B_7 = \frac{S_0' - S_2'}{9}$	S_0'' S_2'' $B_5 = \frac{S_0'' + S_2''}{9}$ $B_6 = \frac{S_0'' - S_2''}{9}$	S_0''' S_2''' $B_4 = \frac{S_0''' + S_2'''}{9}$ $B_8 = \frac{S_0''' - S_2'''}{9}$	S_0'''' S_2'''' $B_0 = \frac{S_0'''' + S_2''''}{18}$ $B_9 = \frac{S_0'''' - S_2''''}{18}$

CHECKS

$$s_9 = (B_0 + B_3) + (B_1 + B_4) + (B_2 + B_7) + (B_5 + B_6) + (B_4 + B_8)$$

$$s_7 = (B_0 - B_3) - (B_1 - B_4) + (B_2 - B_7) - (B_5 - B_6) + (B_4 - B_8)$$

$$d_3 = 2[(A_1 + A_3) + (A_2 + A_7) - (A_4 + A_8)] \sin 60^\circ$$

$$d_5 = 2[(A_1 - A_3) - (A_2 - A_7) + (A_4 - A_8)] \sin 60^\circ$$

FIGURE No. 15.

Schedule for Harmonic Analysis: $y = 3_0 \sin \omega t + 3_1 \sin 2\omega t + 3_2 \sin 3\omega t + \dots + 3_{11} \sin 11\omega t$ (13 terms) at 15° intervals. 3_0 is the mean value.

- Divide a full period into 24 equal parts of 15° each, and assign the 24 successive ordinates $y_1, y_2, y_3, \dots, y_{24}$. y_0 should correspond to y_0 , the initial ordinate, which is not scheduled.
- Arrange these 24 ordinates into two rows, as below, write down their sums and differences.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}	
	y_{24}	y_{23}	y_{22}	y_{21}	y_{20}	y_{19}	y_{18}	y_{17}	y_{16}	y_{15}	y_{14}	y_{13}	
Sums	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}
Differences	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}		

- Enter the above sums and differences, with their proper signs, in the following schedule, multiplying each by the sine of the angle set down in the left hand of each row. Find the totals of the first and second columns and take their sums and differences, preserving the signs of the latter. Divide throughout by 12.

Sine Terms. (A)

Angle	Sine	1st or Fundamental	2nd Harmonic	3rd Harmonic	4th Harmonic	5th Harmonic	6th Harmonic
15°	0.2588	$d_1 + d_{11}$					$d_5 + d_7$
30°	0.5000	$d_2 + d_{10}$	$d_1 + d_5$	$d_7 + d_{11}$			$d_4 + d_{10}$
45°	0.7070	$d_3 + d_9$		$d_1 + d_5$ $d_7 + d_{11}$	$d_5 + d_7$		$d_3 + d_9$
60°	0.8660	$d_4 + d_8$	$d_2 + d_4$	$d_6 + d_{10}$		$d_1 + d_5$ $d_7 + d_{11}$	$d_4 + d_8$
75°	0.9659	$d_5 + d_7$			$d_1 + d_5$ $d_7 + d_{11}$		$d_1 + d_{11}$
90°	1.0000	d_6	d_3	d_9	$d_2 + d_{10}$	d_6	$d_1 + d_5$ $+ d_7 + d_{11}$
		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
		$D_1 = \Delta_1 - \Delta'_1$	$D_2 = \Delta_2 - \Delta'_2$	$D_3 = \Delta_3 - \Delta'_3$	$D_4 = \Delta_4 - \Delta'_4$	$D_5 = \Delta_5 - \Delta'_5$	$D_6 = \Delta_6 - \Delta'_6$
		$A_1 = D_1 / 12$	$A_2 = D_2 / 12$	$A_3 = D_3 / 12$	$A_4 = D_4 / 12$	$A_5 = D_5 / 12$	$A_6 = D_6 / 12$

Angle	Sine	7th Harmonic	8th Harmonic	9th Harmonic	10th Harmonic	11th Harmonic
15°	0.2588	$d_5 + d_7$				$d_1 + d_{11}$
30°	0.5000		$d_2 + d_{10}$		$d_1 + d_5$	$d_7 + d_{11}$
45°	0.7070		$d_3 + d_9$	$d_1 + d_5$ $d_7 + d_{11}$	$d_5 + d_7$	$d_3 + d_9$
60°	0.8660	$d_4 + d_8$	$d_1 + d_5$ $d_7 + d_{10}$	$d_2 + d_4$	$d_6 + d_{10}$	$d_4 + d_8$
75°	0.9659	$d_1 + d_{11}$				$d_5 + d_7$
90°	1.0000	d_6	d_3	d_9	$d_2 + d_{10}$	d_6
		Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}
		$D_7 = \Delta_7 - \Delta'_7$	$D_8 = \Delta_8 - \Delta'_8$	$D_9 = \Delta_9 - \Delta'_9$	$D_{10} = \Delta_{10} - \Delta'_{10}$	$D_{11} = \Delta_{11} - \Delta'_{11}$
		$A_7 = D_7 / 12$	$A_8 = D_8 / 12$	$A_9 = D_9 / 12$	$A_{10} = D_{10} / 12$	$A_{11} = D_{11} / 12$

Cosine Terms. (B)

Angle	Sine	1st or Fundamental	2nd Harmonic	3rd Harmonic	4th Harmonic	5th Harmonic	6th Harmonic
15°	0.2588	s_5	s_7				s_1
30°	0.5000	s_4	s_6			s_3	s_9
45°	0.7070	s_3	s_9	$s_1 + s_5$ $s_7 + s_{11}$	$s_5 + s_7$ s_{11}		s_3
60°	0.8660	s_2	s_{10}	$s_1 + s_{11}$	$s_5 + s_7$		s_{10}
75°	0.9659	s_1	s_{11}				s_5
90°	1.0000	s_0	s_{12}	$s_0 + s_{12}$	s_6	$s_4 + s_{12}$	s_0
		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
		$B_1 = \Delta_1 - \Delta'_1$	$B_2 = \Delta_2 - \Delta'_2$	$B_3 = \Delta_3 - \Delta'_3$	$B_4 = \Delta_4 - \Delta'_4$	$B_5 = \Delta_5 - \Delta'_5$	$B_6 = \Delta_6 - \Delta'_6$
		$B_1 = B_1 / 12$	$B_2 = B_2 / 12$	$B_3 = B_3 / 12$	$B_4 = B_4 / 12$	$B_5 = B_5 / 12$	$B_6 = B_6 / 12$

Angle	Sine	7th Harmonic	8th Harmonic	9th Harmonic	10th Harmonic	11th Harmonic
15°	0.2588	s_{11}	s_1			s_7
30°	0.5000	s_4	s_8	$s_1 + s_5 + s_9 + s_{11}$ $s_7 + s_{11}$	$s_2 + s_{10}$	$s_4 + s_8$
45°	0.7070	s_3	s_9	$s_1 + s_5$ $s_7 + s_{11}$	$s_5 + s_7$ s_{11}	s_3
60°	0.8660	s_{10}	s_2		$s_0 + s_{12}$	$s_1 + s_{11}$
75°	0.9659	s_7	s_6			s_{11}
90°	1.0000	s_0	s_{12}	$s_0 + s_{12}$ $s_6 + s_{12}$	$s_0 + s_{12}$	s_0
		Δ_7	Δ_8	Δ_9	Δ_{10}	Δ_{11}
		$B_7 = \Delta_7 - \Delta'_7$	$B_8 = \Delta_8 - \Delta'_8$	$B_9 = \Delta_9 - \Delta'_9$	$B_{10} = \Delta_{10} - \Delta'_{10}$	$B_{11} = \Delta_{11} - \Delta'_{11}$
		$B_7 = B_7 / 12$	$B_8 = B_8 / 12$	$B_9 = B_9 / 12$	$B_{10} = B_{10} / 12$	$B_{11} = B_{11} / 12$

$B_0 = \frac{\Sigma s_i}{24}$; i.e., the sum of all the 13 values of sums $s_0 \dots s_{11}$, divided by 24.

- Checks:
- $y_{24} = s_0 = \Sigma B$ or the sum of all the B coefficients from B_0 to B_{11} inclusive.
 - $y_{12} = s_{12} = \Sigma B' - \Sigma B$ where $\Sigma B'$ represents the sum of the even coefficients from B_0 to B_{10} inclusive, and ΣB represents the sum of the odd coefficients from B_1 to B_{11} inclusive.
 - $y_6 = A_1 - A_3 + A_5 - A_7 + A_9 - A_{11} + B_0 - B_2 + B_4 - B_6 + B_8 - B_{10}$
 - $y_{18} = -A_1 + A_3 - A_5 + A_7 - A_9 + A_{11} + B_0 - B_2 + B_4 - B_6 + B_8 - B_{10}$

Assumption involved in computations: there are no harmonics higher than the 11th. *Rev. Ed. 1916*

TABLE 8

Comparison of the Results of Analyses of the Same Curves Using Different Numbers of Ordinates

Curve of Example 1

	6 points	12 points	18 points		6 points	12 points	18 points
C ₁	32.616	32.641	32.648	ϕ_1	2°8'	1°55'	1°52'
C ₂	0.151	0.103	0.096	ϕ_2	2798	2693	3595
C ₃	0.236	0.291	0.288	ϕ_3	3899	3692	3790
C ₄		0.085	0.082	ϕ_4		3098	2990
C ₅		0.073	0.075	ϕ_5		1298	1574
C ₁₁		0.108	0.124	ϕ_{11}		6198	595
C ₁₂			0.019	ϕ_{12}			675
C ₁₆			0.032	ϕ_{16}			393
C ₁₇			0.011	ϕ_{17}			1192

Curve of Example 2

	6 points	12 points	18 points		6 points	12 points	18 points
C ₁	32.663	32.678	32.773	ϕ_1	3°14'	3°2'	3°22'
C ₂	1.623	1.798	1.775	ϕ_2	9198	9292	9391
C ₃	0.566	0.410	0.507	ϕ_3	4996	5095	5198
C ₄		0.160	0.183	ϕ_4		4393	4493
C ₅		0.179	0.221	ϕ_5		890	698
C ₁₁		0.121	0.180	ϕ_{11}		792	098
C ₁₂			0.145	ϕ_{12}			2494
C ₁₆			0.064	ϕ_{16}			1694
C ₁₇			0.113	ϕ_{17}			590

Curve of Example 3

	6 points	12 points	18 points		6 points	12 points	18 points
C ₁	34.240	33.194	33.217	ϕ_1	2°50'	1°42'	1°36'
C ₂	1.059	1.051	0.970	ϕ_2	8598	-098	291
C ₃	1.653	1.680	1.702	ϕ_3	4293	4092	4197
C ₄		0.268	0.218	ϕ_4		3491	3893
C ₅		1.619	0.355	ϕ_5		-492	292
C ₁₁		1.279	0.205	ϕ_{11}		1390	1993
C ₁₂			1.251	ϕ_{12}			398
C ₁₆			1.528	ϕ_{16}			1597
C ₁₇			0.159	ϕ_{17}			890

FIGURE No. 17.

F I S C H E R - H I N N E N

M E T H O D O F A N A L Y S I S .

FISCHER-HINNEN METHOD OF ANALYSIS.

(Fischer-Hinnen, Elektrotech. Zeit. Vol. 22, p. 396, 1901 >)

(P. M. Lincoln, Electric Journal, Vol. 5, p. 386, 1908)

(S. P. Thompson, Proc. Phys. Soc. of London, Vol. 23, p. 334, 1911 >)

This method depends upon a development of equation (5) and the Houston-Kennelly method. The period of the curve to be analysed is divided into n equal intervals of width $2\pi/n$. The $n-1$ ordinates are measured at the beginning of each interval. Substituting these values of x and y in the Fourier Series, n equations are obtained, which added together give:

$$\sum y_r = n b_0 + b_1 \sum \cos x_r + \dots + b_k \sum \cos kx_r + \dots + a_k \sum \sin kx_r$$

Where the summation is carried from $r=0$ to $r=n-1$.

If the intervals are started at $x_0=0$, then $x_r=r(2\pi/n)$ and the summations above may be expressed:

$$\begin{aligned} \sum \cos kx_r &= \sum \cos(0+kr \cdot 2\pi/n) = 0, \text{ except when } k=n, 2n, 3n, \dots \\ &= n \cdot \cos 0 = n, \text{ when } k=n, 2n, 3n, \dots \end{aligned}$$

$$\sum \sin kx_r = \sum \sin(0+kr \cdot 2\pi/n) = 0, \text{ for all values of } k.$$

$$\text{Thus: } \sum y_r = n(b_0 + b_n + b_{2n} + b_{3n} + b_{4n} + \dots)$$

If, on the other hand the intervals are started at $x_0^1 = \pi/n$,

Then $x_r^1 = \pi/n + r \cdot 2\pi/n$, and:

$$\begin{aligned} \sum \cos kx_r^1 &= \sum \cos[k\pi/n + kr \cdot 2\pi/n] = 0, \text{ except when } k=n, 2n, 3n, \dots \\ &= n \cdot \cos(k\pi/n) = n, \text{ when } k=2n, 4n, 6n, \dots \\ &= -n, \text{ when } k=n, 3n, 5n, \dots \end{aligned}$$

$$\sum \sin kx_r^1 = \sum \sin[k\pi/n + kr \cdot 2\pi/n] = 0, \text{ for all values of } k.$$

$$\text{Thus: } \sum y_r^1 = n(b_0 - b_n + b_{2n} - b_{3n} + b_{4n} - \dots)$$

Subtracting the first set of ordinates from the second set:

$$\sum y_r - \sum y_r^1 = \sum (y_r - y_r^1) = 2n(b_n + b_{3n} + b_{5n} + \dots)$$

Thus: $b_n + b_{3n} + b_{5n} + \dots = 1/2\pi(y_0 - y_0' + y_1 - y_1' + \dots + y_{n-1} - y_{n-1}')$

From this it may be stated that:- If, starting at $x=0$ we measure $2n$ ordinates at intervals of π/n , the average of these ordinates taken alternately plus and minus is equal to the amplitude of the n -th, $3n$ -th, $5n$ -th, ... cosine components.

By similar treatment it may be shown that starting at $x_0'' = \pi/2n$

$$a_n - a_{3n} + a_{5n} - a_{7n} + \dots = 1/2n(y_0'' - y_0'' + y_1'' - y_1'' + \dots + y_{n-1}'' - y_{n-1}'')$$

Whence:- If, starting at $x = \pi/2n$, we measure $2n$ ordinates at intervals of π/n , the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes, taken alternately plus and minus, of the n -th, $3n$ -th, $5n$ -th, ... sine components.

Thus the individual harmonics are not separated, but the sums of multiple harmonics obtained. By determining the higher harmonics first, then it is possible to subtract them from values found for lower values of n and thus determine all the harmonics individually. Fig. 19 shows a curve divided up in the proper way to determine b_n and a_n . The second set of ordinates lies midway between the first set.

For ODD HARMONICS ONLY the case is simplified. Similar reasoning develops the expressions:

$$\begin{aligned} b_n + b_{3n} + \dots &= 1/2n(2y_0 - 2y_0' + \dots + 2y_{n-1} - 2y_{n-1}') \\ &= 1/n(y_0 - y_1' + \dots + y_{n-1} - y_{n-2}') \end{aligned}$$

Thus it may be stated:- If, starting at $x=0$, we measure n ordinates at intervals of π/n , the average of these ordinates taken alternately plus and minus is equal to the sum of the amplitudes of the n -th, $3n$ -th, $5n$ -th, ... cosine components. If, starting at $x = \pi/2n$, exactly the same procedure is followed, the sum of the amplitudes of the n -th, $3n$ -th, $5n$ -th, ... sine components taken alternately plus and minus, is obtained.

In order to determine the value of the fundamental it is only necessary to make use of the equations:

$$b_1 + b_3 + b_5 + \dots = 0$$

$$a_1 - a_3 + a_5 - \dots = y \text{ at } 90^\circ$$

The operation of the Fischer-Hinnen method may be graphically illustrated as in Fig. 20. It is clearly seen how the ordinates chosen pass through the peaks of the 3rd, 9th, 15th etc. sine components, and through the zero points of the cosine components. The sum of the intercepts of the ordinates with all harmonics except the odd multiples of three sine components thus vanishes when the sum of the ordinates is taken alternately plus and minus.

Beattie (Electrician, Vol. 67, p. 847, 1911) proposes an extension of this method which increases the accuracy and reduces the number of higher harmonics included in the solution. This method is graphically shown in Fig. 21, and consists of dividing the base into $4n$ instead of $2n$ equi-distant points. The initial ordinate is then taken $\pi/4n$ from the origin. The ordinates are then taken in pairs, and added alternately plus and minus as before, but the signs changed for each pair instead of each ordinate, the scheme thus being $+-+--$ etc. The sum will then be: $(a_n + a_{3n} - a_{5n} - a_{7n} + a_{9n} + a_{11n} - a_{13n} \text{ etc.})\sqrt{2}$

For the cosine components the same ordinates are used, but added according to the scheme: $+--+-$ etc. When the sum of ordinates gives: $(b_n - b_{3n} - b_{5n} + b_{7n} + b_{9n} - b_{11n} \dots)\sqrt{2}$

This does not offer much advantage except greater accuracy and the same ordinates being used for sine and cosine terms. But comparing the signs of the coefficients obtained with the $2n$ and $4n$ ordinate systems it will be seen that a combination of the two will eliminate many of the harmonics included in both. If the mean of the results from the $2n$ and $4n$ measurements be taken, and the value called S_n'' for sine and C_n'' for cosine coefficients, the expressions become:

$$S_n'' = a_n - a_{7n} + a_{9n} - a_{15n} \text{ etc.}$$

$$C_n'' = b_n + b_{7n} + b_{9n} + b_{15n} \text{ etc.}$$

Which means that the next harmonic included is seven times the one being sought.

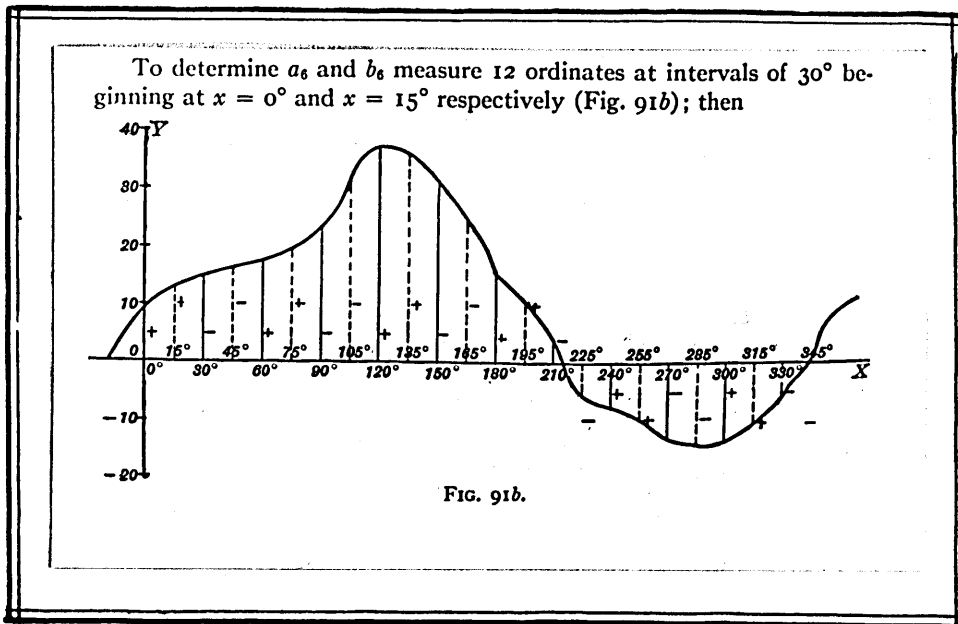
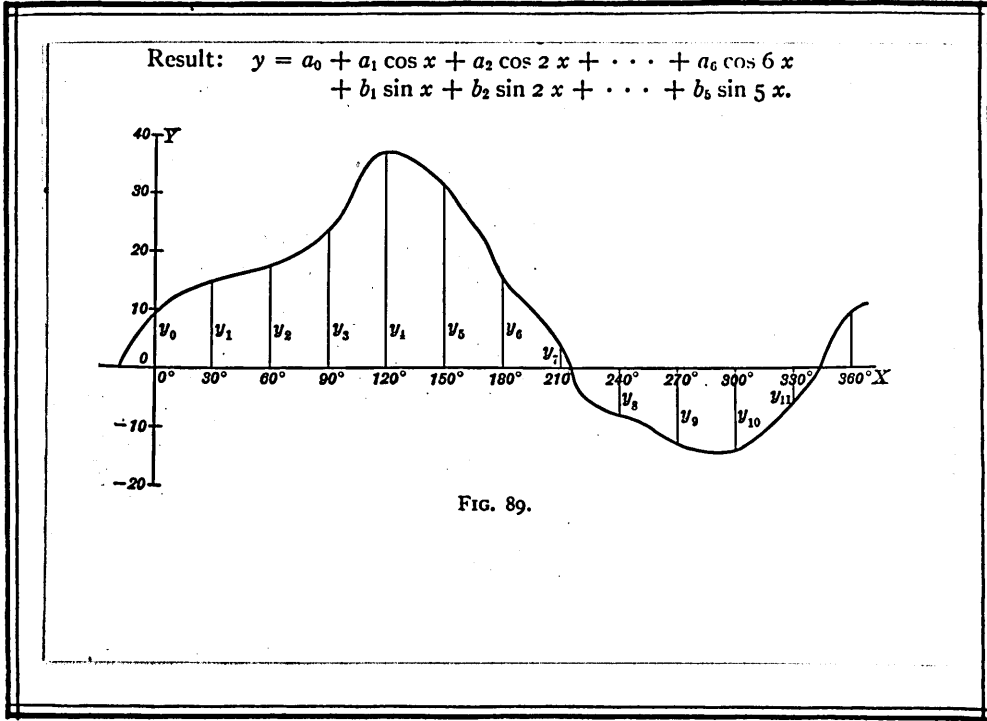


FIGURE No. 18. Division of curve for Schedule.

FIGURE No.19. Division of Curve for Fischer-Hinnen.

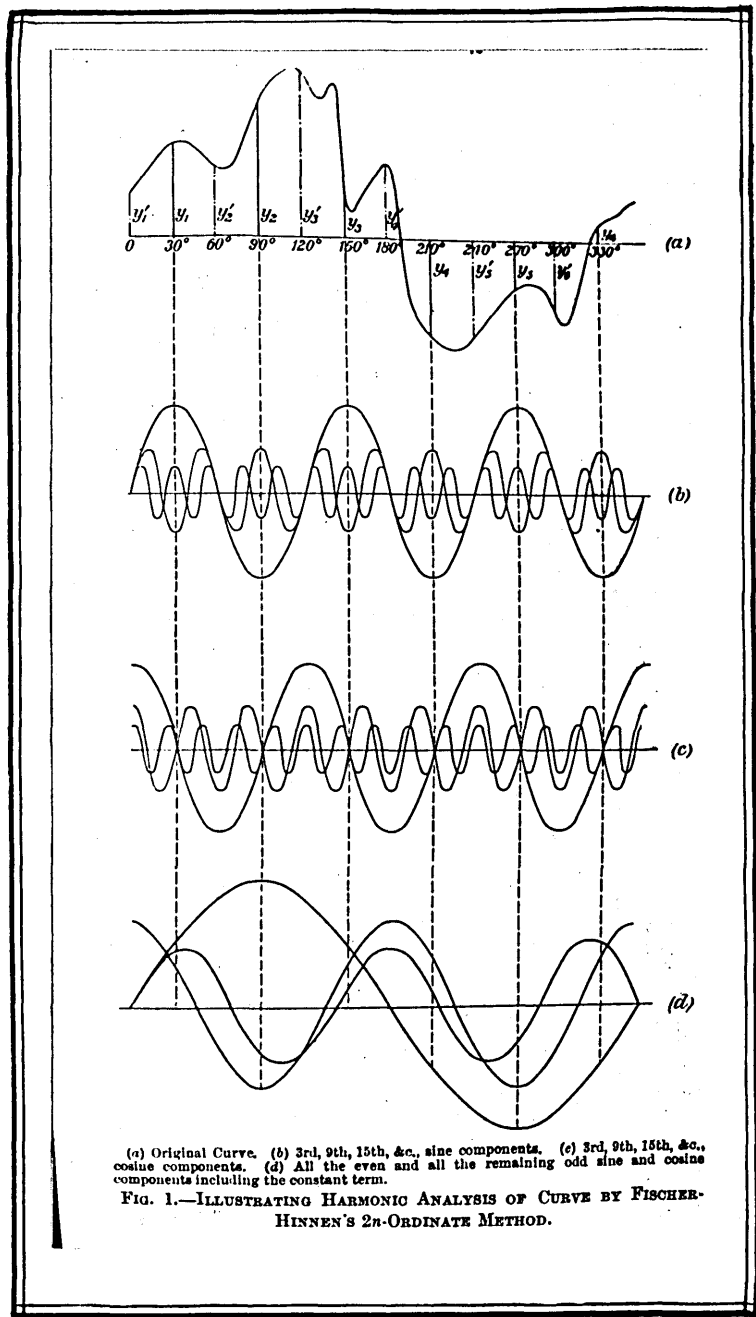


FIGURE No. 20. Harmonic Analysis of Curve by Fischer-Hinnen Method with 2n ordinates.

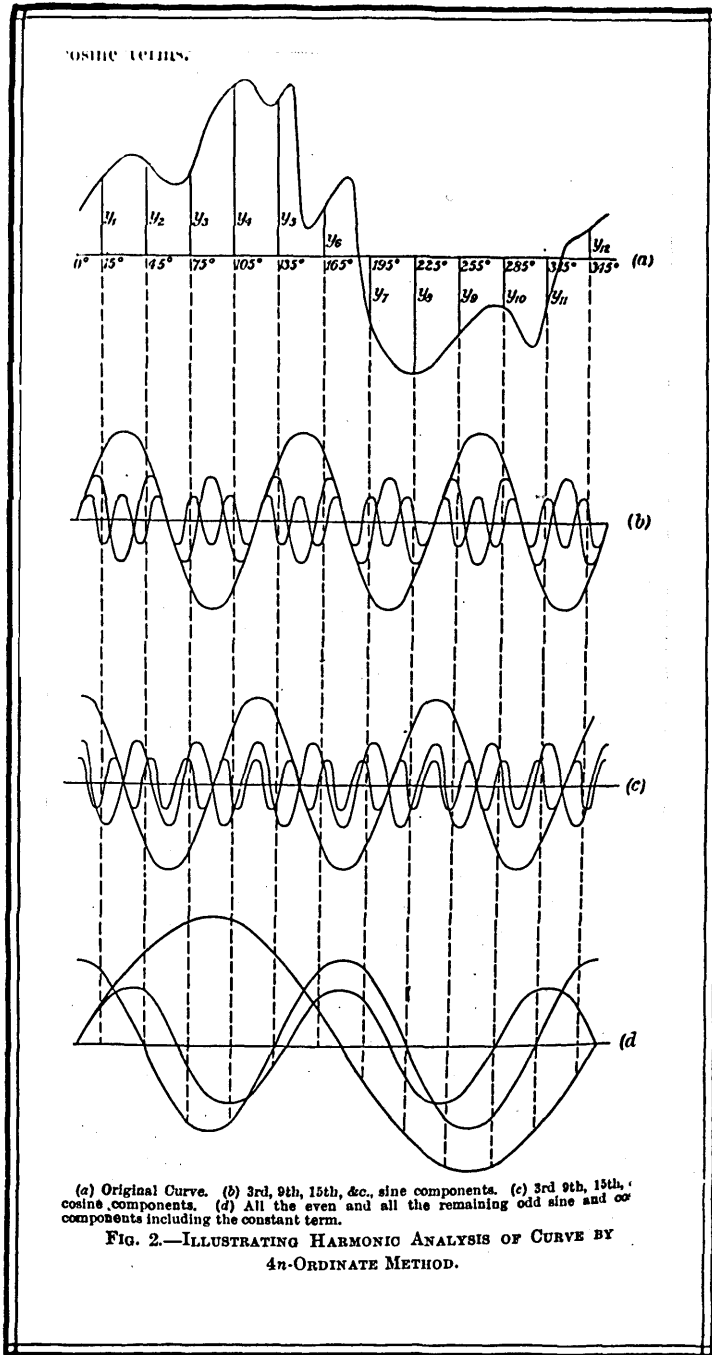


FIGURE No. 21. Analysis of curve by Fischer-Hinnen Method with 4n Ordinates.

GRAPHICAL METHODS OF ANALYSIS.

CLIFFORD-PERRY-SLICHTER METHOD OF GRAPHICAL ANALYSIS.

(Clifford, Proc. London Math. Soc., Vol. 5, p. 11, 1873.)

(Perry, The Electrician, Vols. 28, 35, pp. 362, 285, 1892-95.)

(Slichter, Electrical World, Vol. 54, p. 146, 1909.)

This method was first proposed by Clifford, and used extensively by Prof. Perry. More recently C. S. Slichter prepared special coordinate paper for simplifying its use, the theory remaining the same throughout.

The method consists of replotting the curve to be analysed, which is assumed to be in cartesian form, in the form of the projection of the curve wrapped around a cylinder. Assume that the cylinder is of such diameter that its periphery is just equal to the length of one period of the curve. The curve is then divided up into a convenient number of ordinates, and a circle drawn upon the charting paper, divided up into the same number of segments. The vertical segment is taken as corresponding to the position of y_0 on the curve. Perpendiculars are then erected at the ends of the radii forming the segments of the circle, a horizontal line drawn representing the X-axis, and the curve replotted on these perpendiculars, using the ordinates of the original curve in the same numbered rotation as the radii of the circle. The area of the curve thus obtained is measured with a planimeter, and the result is proportional to the sine term coefficient of the fundamental.

This can easily be seen, since the perpendiculars used to plot the curve measured by the planimeter are spaced amounts proportional to the projection of the radii of the construction circle upon the line perpendicular to the radius corresponding to y_0 , which was used as the X-axis in replotting. In other words the abscissae are now $\sin.x$ instead of x . Thus the coordinates

of the reconstructed curve are $y, \sin x$. The planimeter will measure the integral:

$$\int y \cdot d(\sin x) = -\int y \cdot \cos x \cdot dx$$

The latter form evaluating the fundamental cosine coefficient according to (2).

If the construction were made with one of the horizontal radii taken as proportional to y_0 , then the spacing of the vertical lines would have been proportional the cosine of the angle and by similar reasoning the cosine component of the fundamental would have been evaluated.

The simplification introduced by Slichter consists of using coordinate paper already divided with the vertical lines spaced sinusoidally and the horizontal lines spaced equally. Then it is unnecessary to perform the construction of the coordinates each time, but the curve may immediately be plotted upon the special coordinate paper. For the fundamental the curve is divided up into the same number of spaces along the X-axis as there are in the coordinate paper. The ordinates are plotted vertically, starting at the middle of the left hand line (If middle is taken as zero) for the sine coefficients, and starting with the middle of the vertical line for the cosine coefficients.

For the value of the second harmonic coefficient the same procedure is followed, except that the curve must be plotted to angles having twice the value of the ordinate position angle, and so every other vertical line on the coordinate paper is used, and the curve drawn will cross it twice. For the k-th harmonic every k-th vertical line is used etc.. The coordinate paper is divided up so that 36, 48 or 72 parts may be used as divisions of the complete period of the original wave. Figs. 22 to 27 show waves replotted in this way, and Fig. 28 shows a sample of the coordinate paper for this purpose. Fig. 29 is coordinate paper sinusoidal in both directions included as a matter of interest but not directly applicable to harmonic analysis.

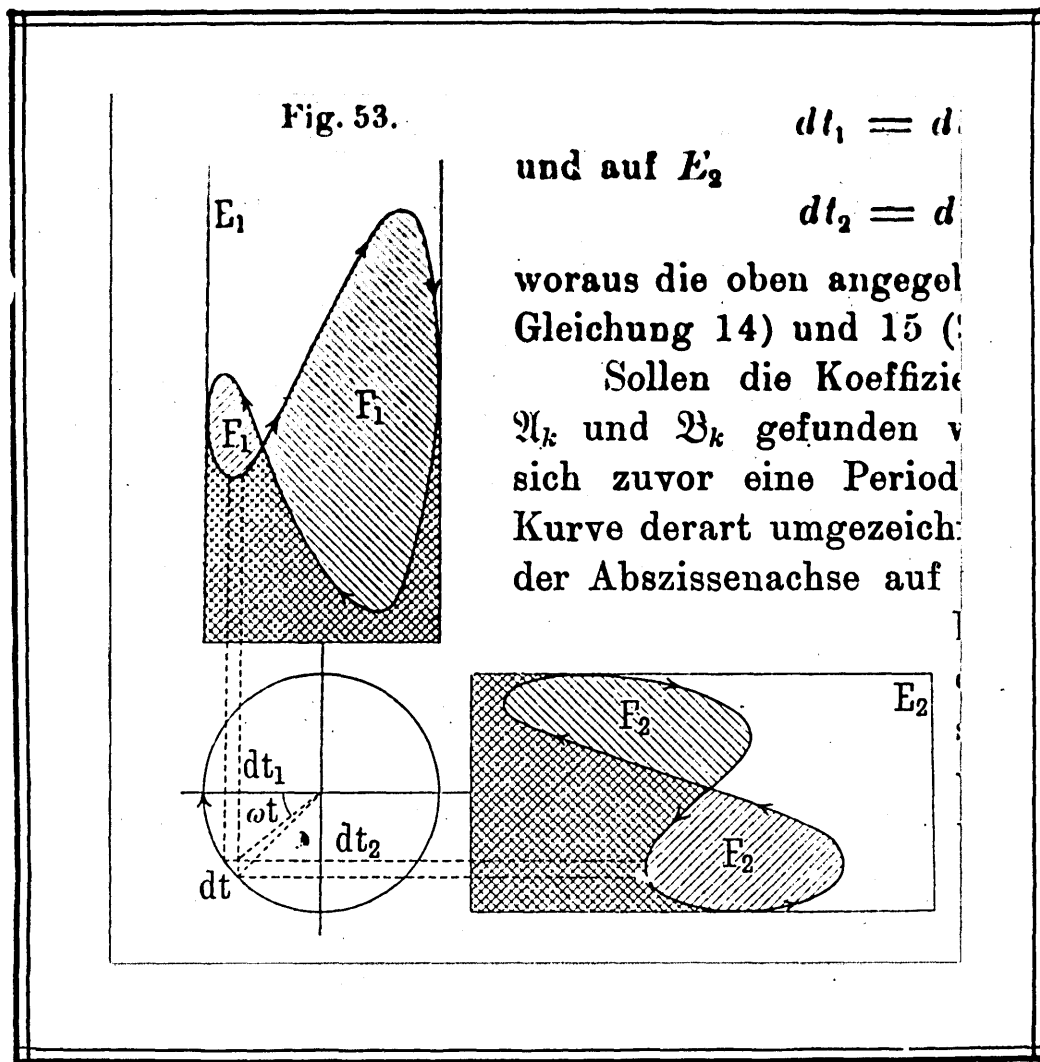
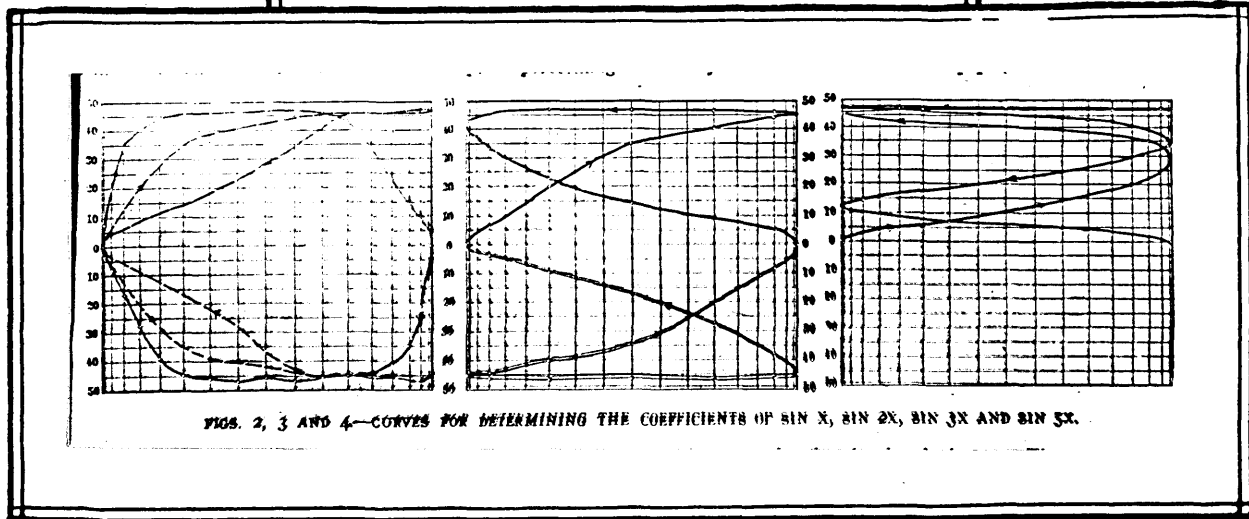
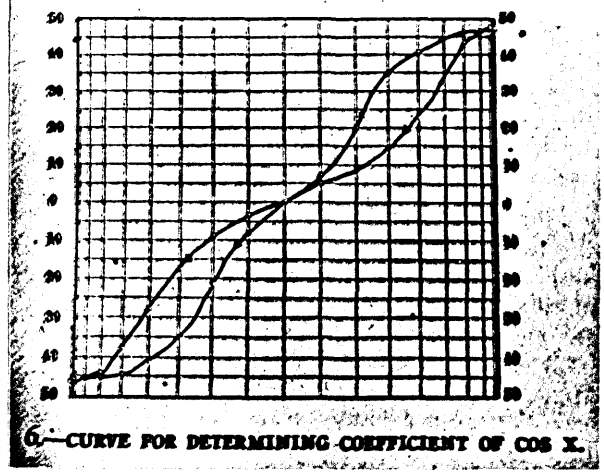
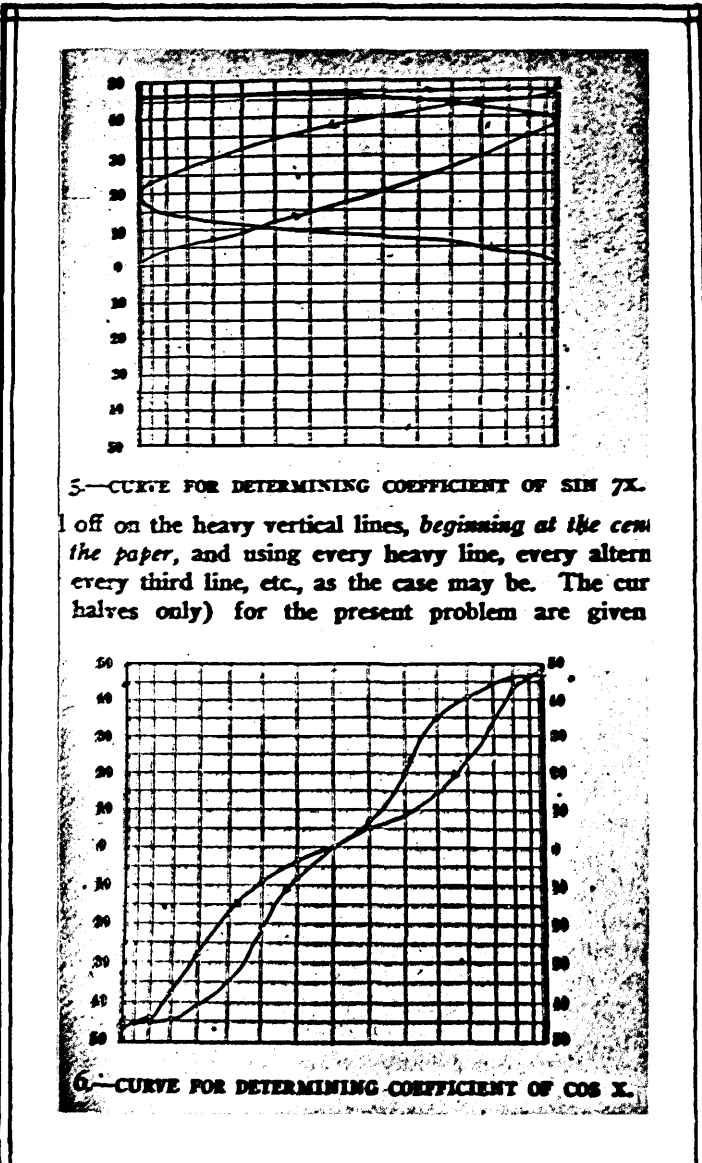
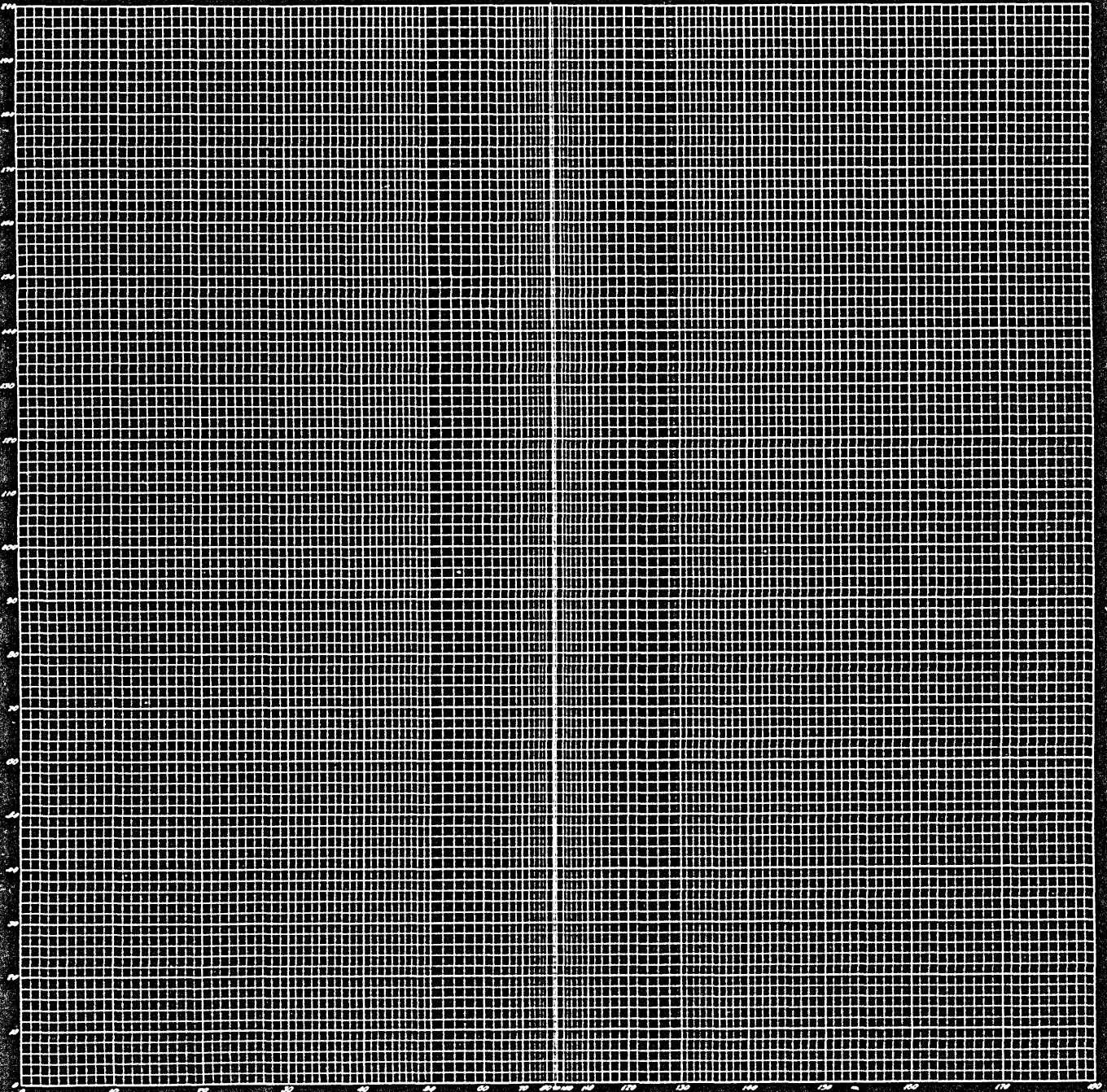


FIGURE No. 22. Construction of curves for determining the Sine and Cosine Components of an Harmonic by the Perry Graphical Method.

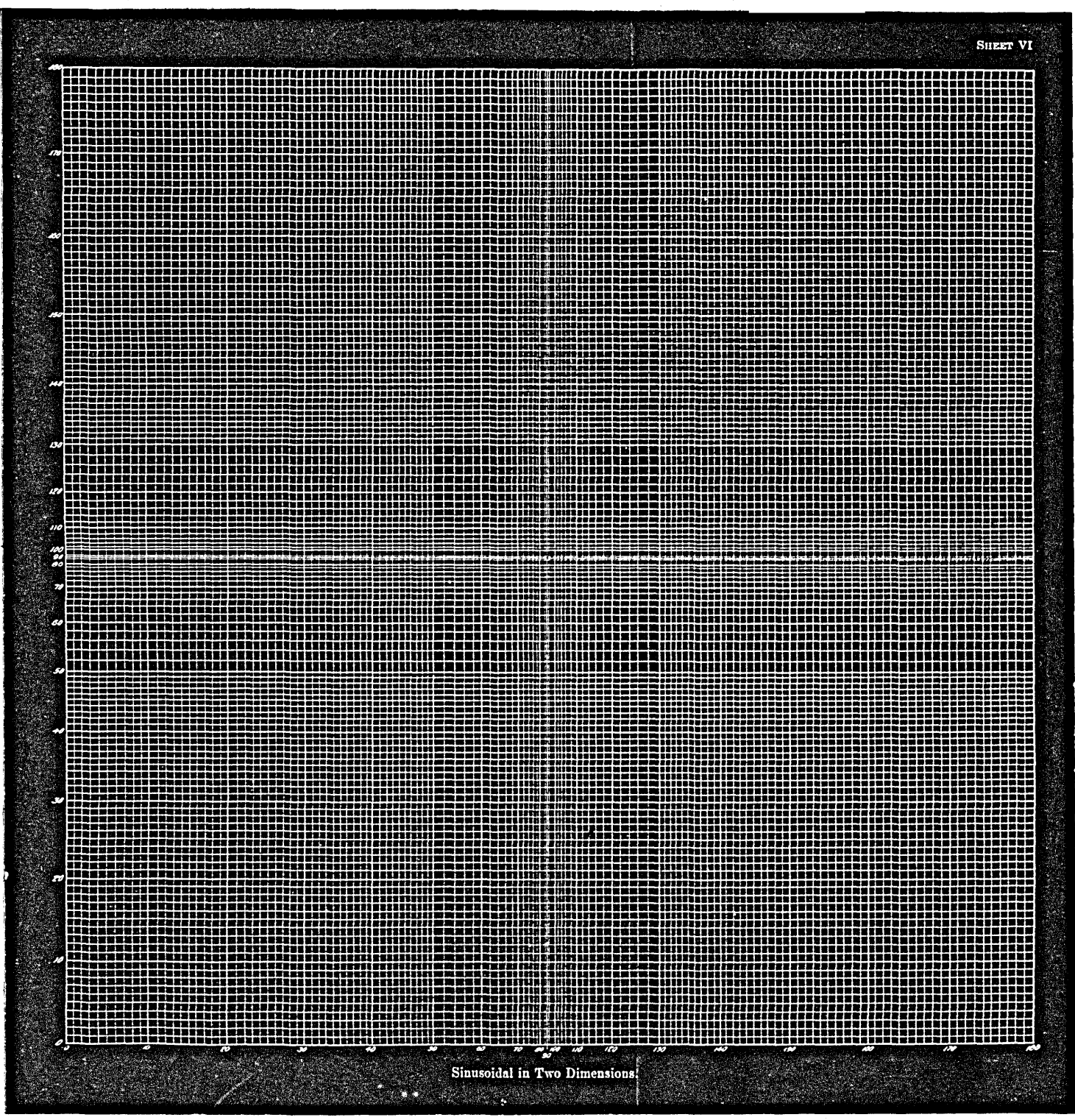


FIGURES No. 23,24,25,26,27. Construction of curves for Harmonic Analysis by the Slichter Method.



Sinusoidal in One Dimension.

FIGURE No. 28. Slighter Graph Paper for Analysis.



Sinusoidal in Two Dimensions

FIGURE No. 29. Slichter Graph Paper.

HOUSTON-KENNELLY METHOD OF GRAPHICAL ANALYSIS.

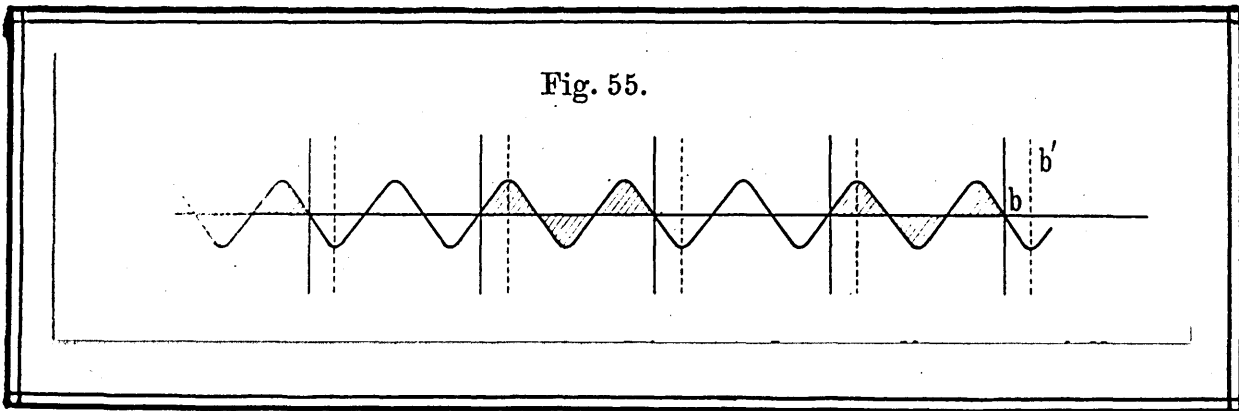
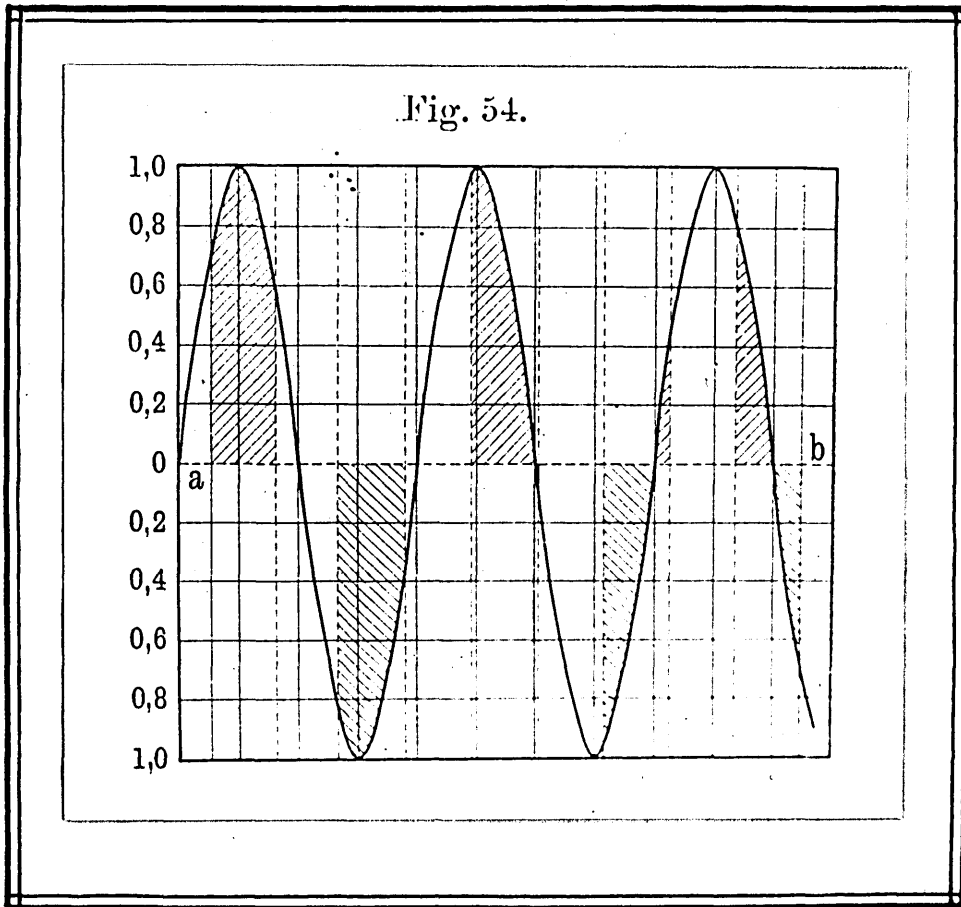
(Houston-Kennelly, Electrical World, Vol. 31, p. 580, 1898)

This method determines the coefficients by dividing the curve up into strips chosen much in the same manner as the method later developed by Fischer-Hinnen. Instead of dealing with the ordinates only, as does Fischer-Hinnen, the Houston-Kennelly method makes use of the summations of areas.

In Fig. 30 let w , an odd number of semi-wave lengths, be divided into p equal strips. If $p > 1$ and prime to w the difference between the sums of the areas in alternate strips is zero, provided the wave is purely sinusoidal. In the Figure 5 semi-wave lengths are divided up into 9 strips. Then the sum of the shaded areas is equal to the sum of the unshaded areas. i.e. The sum of the odd strips minus the sum of the even strips = 0. If however p is some multiple of w , as would be the case for certain harmonics as indicated in Fig. 31, and the strips commence at the zero line, then the odd areas minus the even areas = $p \times$ the area of one semi-wave. Thus for determining the third harmonic the curve would be divided into three strips. If S = the summation difference of areas and L = the length of a complete period of the given wave, Then:

$$a_3 = \pi S / L$$

For the cosine coefficient the divisions are started a triple frequency semi-wave length from zero and the area summations repeated. The determination of any coefficient will also include the value of the coefficients of all higher harmonics which are odd multiples of 3 of the one being sought. Thus the third will really be the 3rd+9th+15th etc. Therefore the higher ones must be determined or assumed negligible and subtracted from the values found.



FIGURES No. 30 and 31. Illustrating Houston-Kennelly Method of Harmonic Analysis.

ASHWORTH-HARRISON METHOD OF GRAPHICAL ANALYSIS.

(Electrician, Vol. 67, p. 888, Engineering, Vol. 81, p. 201, 1911 and 1906)

Harrison, 1906 and Ashworth, 1911 proposed similar schemes known as the Coplanar Force, or Vector method of analysis. It is evident that any method which will multiply a definite set of ordinates by successive sine or cosine values will give the coefficients desired in accordance with equations (6) and (7).

This method accomplishes this by vector addition. The various ordinates are considered as vectors at an angle given by their position, or harmonic multiple. If the ordinates are then added as vectors, the vector sum will be $n/2$ times the desired coefficient.

Fig. 32 shows the vector addition for the fundamental coefficients. The curve is divided in twelve parts for the complete period, each ordinate representing an increment of 30° on the X-axis. The values of the ordinates are laid off to scale, each making an angle of 30° with the preceding one. The line OP represents the vector sum. This is the value of the amplitude of the fundamental term. The angle $POB = \phi_1$ is the phase angle of the fundamental with respect to the zero point of the ordinates chosen, and the projections of OP upon the X and Y axes give the sine and cosine components. Thus one advantage of this method is that the harmonic is completely determined with one graphical construction, and it is not necessary to make separate constructions for the sine and cosine components. It also eliminates the usual necessity of combining the sine and cosine components into the true magnitude of the harmonic amplitude and its phase relation after the components have been determined.

HAZELTINE GRAPHICAL METHOD OF ANALYSIS.

(Hazeltine, Electrical Review, Volume 50, page 235, 1907.)

The Hazeltine Graphical Method, while not differing greatly from others, deserves a note as introducing one or two points of originality.

The curve to be analyzed is plotted in polar form, either directly by means of polar oscillograph, or indirectly by replotting from rectangular components. Another curve is then constructed from the polar curve. The abscissae for this final curve are the angular positions of the radii chosen from the polar curve, while the ordinates are the projections of the corresponding points of the polar curve upon the X or Y axis. Thus two curves are really obtained, and the actual ordinates of these curves are values of the original ordinates times sine or cosine of the position angles. Thus the area of these final curves measured with a planimeter will be the integral of $y \cdot \sin \theta$ or $y \cdot \cos \theta$, which is the value of the fundamental sine and cosine coefficients.

For higher harmonics the polar curve must be replotted giving the angles chosen k times the actual angle of the original curve, the final curves are then re-plotted from this, and the area of these curves with a planimeter will then give the sine and cosine components of the k -th harmonic.

It will be seen that the number of plottings necessary is excessive, but it employs the underlying idea in the Chubb analyser, and is a sort of complementary form to that of Clifford, Slichter etc.

BEATTIE METHOD OF GRAPHICAL ANALYSIS.

(R. Beattie, The Electrician, Vol. 67, p. 326, 370, June 9, 1911.)

This method, in its application, is sort of a cross between the schedule and Fischer-Hinnen types. The curve is divided up into one set of parts, but the readings made by special scales so that only addition and subtraction of the results is necessary to obtain the coefficients.

The theory depends directly upon the Fourier Series solution given in formulas (4), (6), (7). In this expression the successive ordinates must be multiplied by the sine or cosine of their respective position angles. If the value of the ordinate is measured with a special scale for each ordinate, the adjustment of the scale divisions may be made to perform the multiplication by the proper trigonometric function. This is the idea of the method Beattie developed.

The special scales are drawn upon transparent medium such as tracing cloth, or better, celluloid, a special set of scales being required for each coefficient to be determined. Beattie calls these "Reciprocal Sine, or cosine, Scales". Figs. 33 to 38 show such scales for the coefficients b_0 , a_5 , b_5 , and a_1 . It is possible to lessen the required number of charts by putting two sets of scales upon one sheet as in Fig. 33 giving the arrangement for determining b_0 and a_5 . Or it may be that the curve is not available but only certain ordinates, which, if properly chosen, may be read from the uniform part of the scale and the values multiplied by the $\sin 5\theta$ read from the reciprocal scale.

If the number of divisions used per period is n , and the ordinates read from the reciprocal sine scales are y_p' , and from the reciprocal cosine scales are y_p'' , then the values

of the coefficients are given by the expressions:

$$b_k = 2/n(y_1'' + y_2'' + y_3'' + y_4'' + \dots + y_{n-1}'')$$

$$a_k = 2/n(y_1' + y_2' + y_3' + y_4' + \dots + y_{n-1}')$$

Where the proper scale is used for the k-th harmonic.

It is evident, though not mentioned by Beattie, that the scales could be constructed to include the constant multiplier $2/n$ similar to the schedule of Kemp. (Schedule methods).

By means of reciprocal sine curves, Fig. 33, Beattie claims a greater generality but not as much simplicity. The curves are constructed by assuming a large number of reciprocal sine scales to be constructed, and the similarly numbered divisions connected by lines. The curves thus determined correspond to the general equation:

$$y = \pm k / \sin. n\theta$$

The procedure is to first draw the curve so that period is equal to the distance $S_1 - S_3$. The base is then divided up into any number of convenient equal parts, and ordinates erected at the mid-points of the divisions. The curve is then placed over the sheet of reciprocal curves, or visa-versa, and the points where the ordinates intersect the reciprocal curves read. The base must of course coincide with $S_1 - S_3$. The sums of these readings give $n/2$ times the coefficient of the sine component. For the cosine component the curve is shifted 90° and the same process repeated. The particular arrangement shown gives the second harmonic. This system has the advantage of being able to use any number of ordinates with corresponding accuracy. Fig. 38 already mentioned is in reality a Sine Scale chart. The curve is redrawn by means of the sine scales instead of linear scales. The area of the resulting curve gives the coefficient desired. This is a complementary construction to that of Slichter, Clifford, etc.

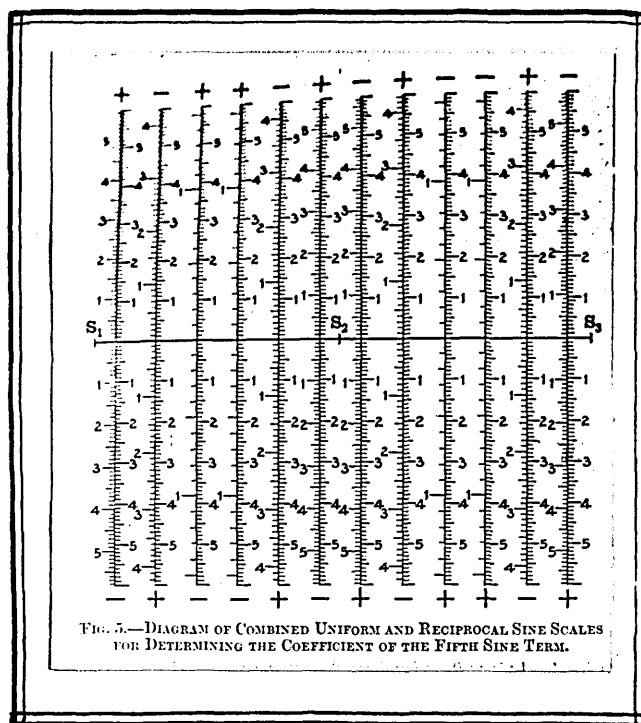
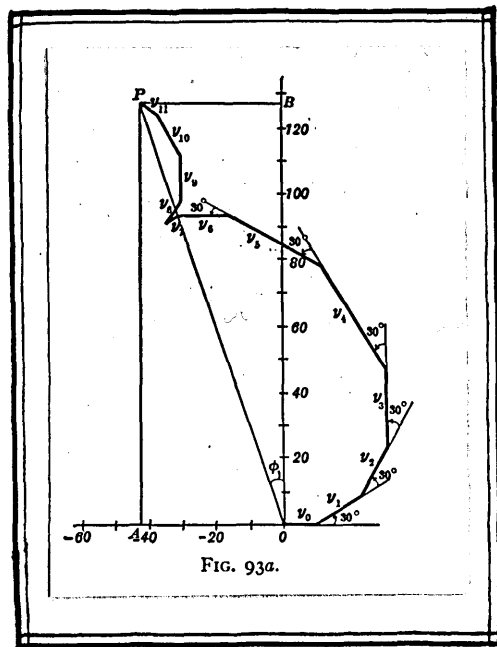


FIGURE No. 32. Analysis by Coplanar Forces.

FIGURE No. 33. Beattie Chart.

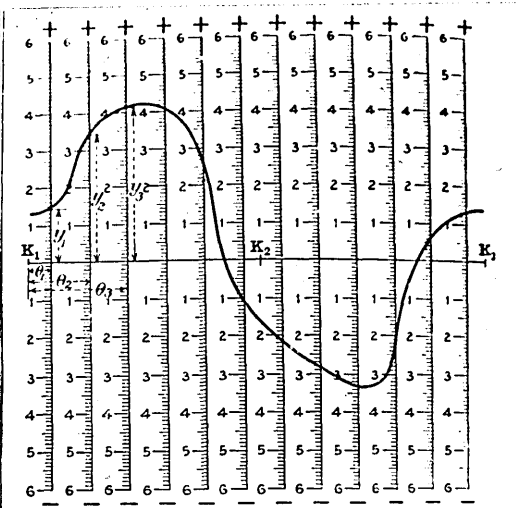


FIG. 1.—DIAGRAM OF UNIFORM SCALES FOR DETERMINING THE CONSTANT TERM IN THE FOURIER SERIES FOR A GIVEN CURVE.

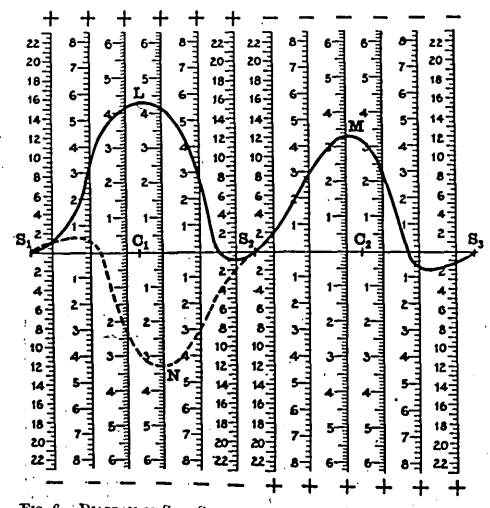


FIG. 6.—DIAGRAM OF SINE SCALES FOR DETERMINING THE COEFFICIENT OF THE FUNDAMENTAL SINE TERM.

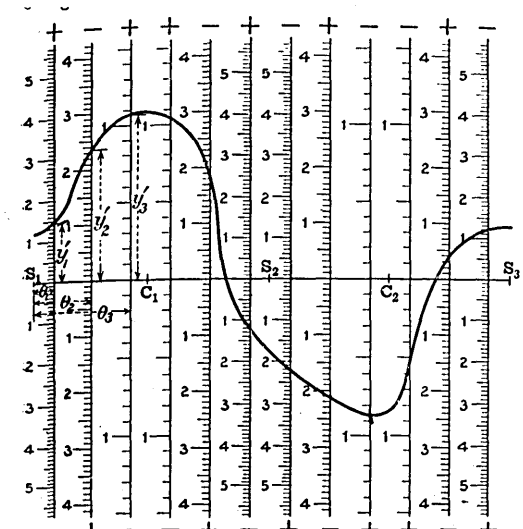


FIG. 2.—DIAGRAM OF RECIPROCAL-SINE SCALES FOR DETERMINING THE COEFFICIENT OF THE FIFTH SINE TERM.

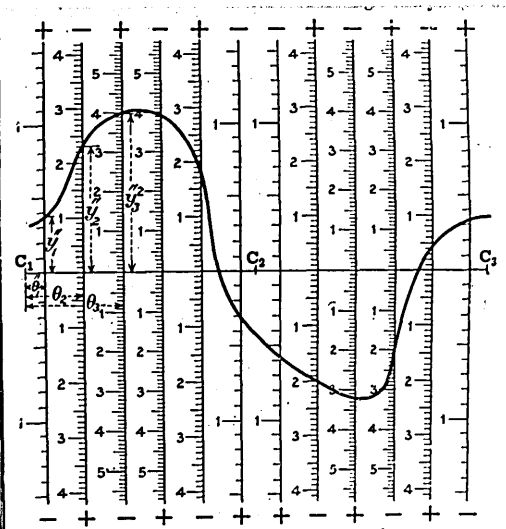


FIG. 3.—DIAGRAM OF RECIPROCAL-COSINE SCALES FOR DETERMINING THE COEFFICIENT OF THE FIFTH COSINE TERM.

FIGURES No. 34,35,36,37. Beattie Charts for Analysis.

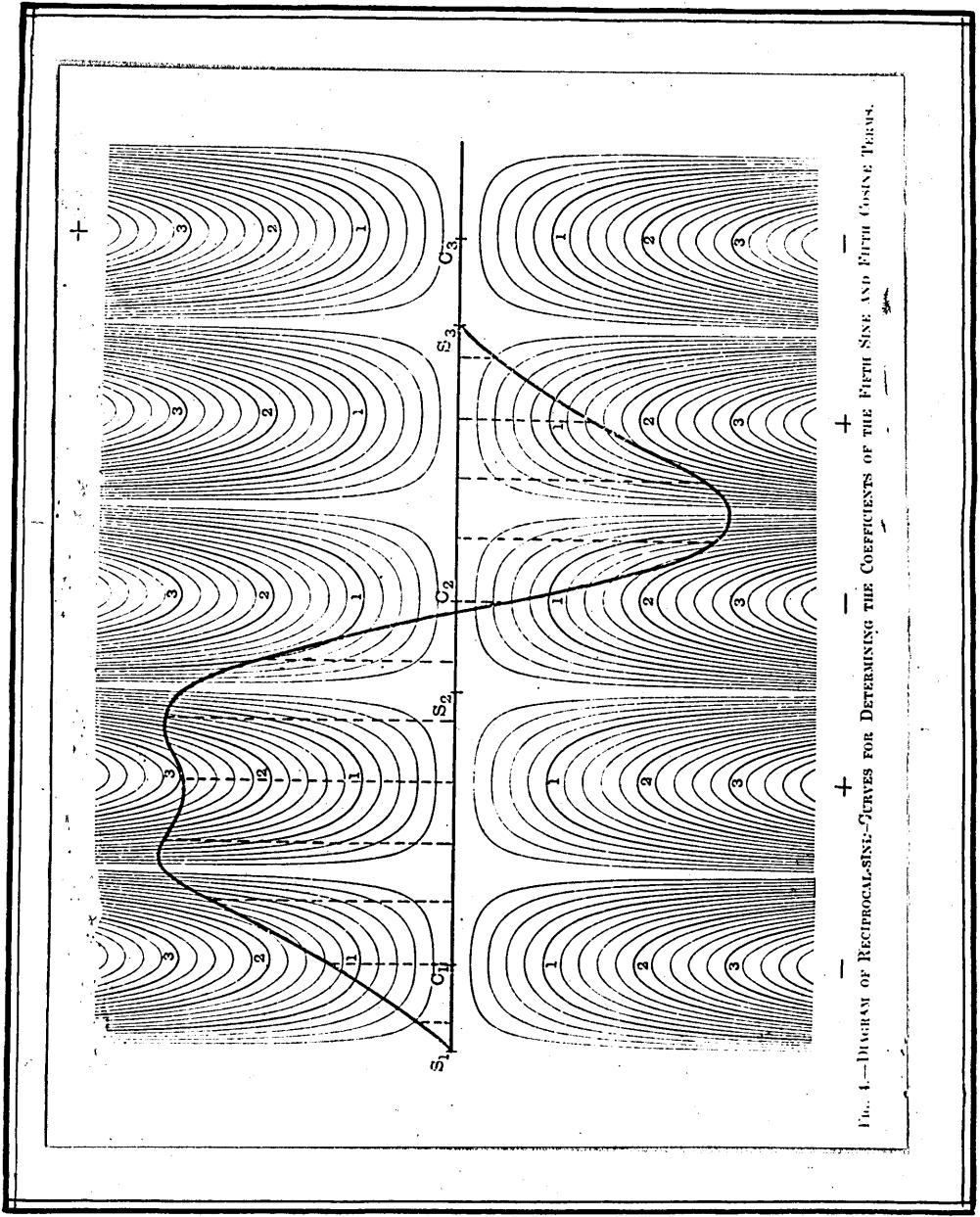


FIGURE No. 38. Beattie Diagram of Reciprocal Sine Curves for Fifth Sine and Cosine Components.

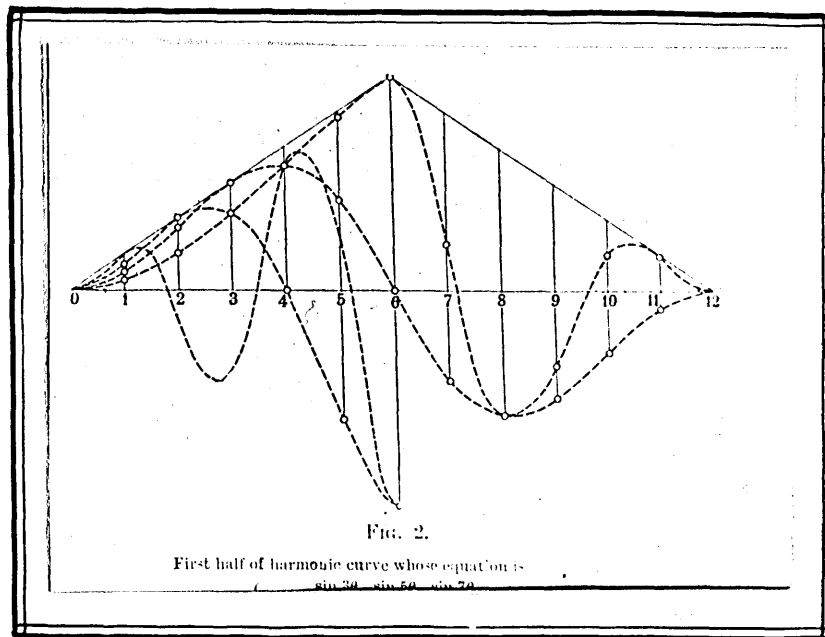
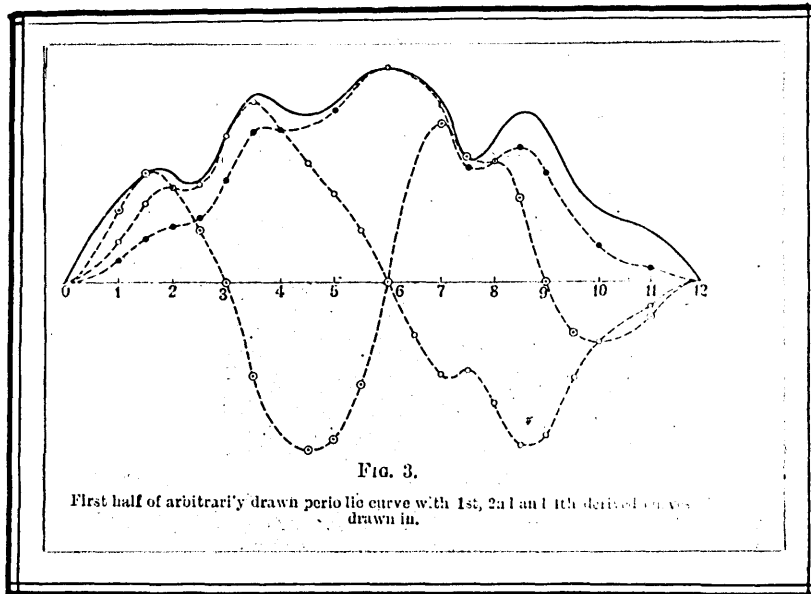
ROTTENBURG METHOD OF GRAPHICAL ANALYSIS.

(H.Rottenburg, A Multiplying Chart for the Graphical Analysis of Curves into their component Harmonics.) (The Electrician; Vol. 70, p. 1141; 1913.)

This method depends directly upon the fact that the coefficients as determined by the integrals of equations (2) and (3) represent the areas of curves whose ordinates are the same as the original curve multiplied by the sine or cosine of their position angle, or harmonic multiple of their position angle, the abscissae remaining the same. Thus the ordinates may be read, multiplied by the value $\sin.kx$ or $\cos.kx$, and plotted to the same base. Then the areas of these derived curves will give the harmonic coefficients, and may be determined with an ordinary planimeter.

Two curves analysed by this method are shown in Figs. 39 and 40. The first is an arbitrarily drawn curve and shows the derived curves for the 1st, 2nd, and 4th harmonics. Unless it is known that no even harmonics exist it is necessary to plot the curves for a complete period in order that the area of the even derived curves should become zero. The second curve is a triangle, and shows the 1st, 2nd, 3rd, 5th and 7th derived curves drawn in for one quarter of the period. As the curve is obviously symmetrical it is not necessary to construct more than this part of the derived curves for this case.

In order to facilitate construction Rottenburg uses a chart consisting of radiating lines on tracing cloth, arranged so that when the bottom is on the X-axis and the top line runs through the curve, then the value of the ordinate multiplied by one of the chosen position angles will be indicated by one of the other lines, which may then be pricked through and the derived curve ultimately drawn through the various prick marks.



FIGURES No. 39 and 40. Rottenburg Construction.

D I R E C T R E A D I N G M E T H O D S O F A N A L Y S I S .

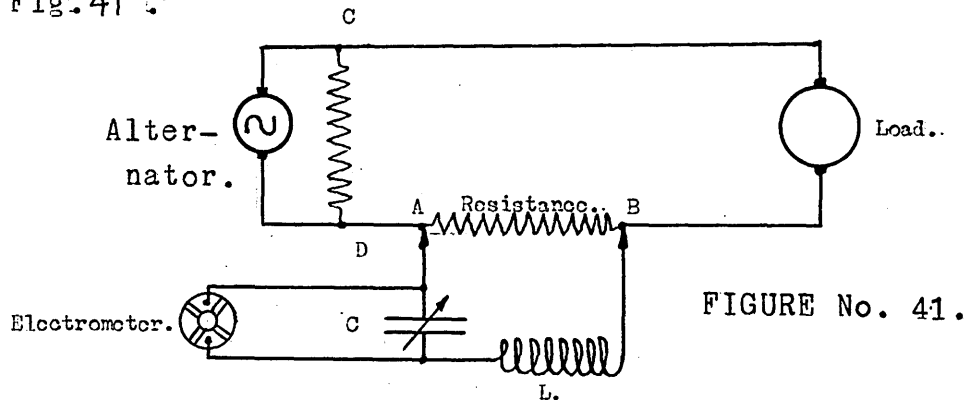
ANALYSIS BY RESONANCE.

(M.I.Pupin,American Jour.of Science.Vol.148,p.379,1894)

(Beattie.Electrician,Vol.69;p.63,1912.)

(Railway Commission Report,State of California,1919.)

As Pupin did much to develop the theory of electrical resonance,he was naturally the first to propose its use for harmonic analysis.It is of course a direct reading method.A resonant circuit is connected as shown in Fig.41.



The circuit consists of a low resistance air core inductance L and a low loss adjustable condenser C . An electrostatic voltmeter is attached to the condenser terminals and indicates when resonance has been obtained. The terminals of the resonant circuit are connected across resistance $A-B$ for measuring harmonics in the current wave, and across a portion of resistance $C-D$ for measuring harmonics in the potential wave. The desired connection is made and the condenser is slowly moved through its range. Every time the proper combination of L and C is obtained to resonate with a frequency in the current going through the shunt being used, it will be indicated by the electrostatic voltmeter. It is a qualitative rather than a quantitative method, but the relative values of the harmonics may be closely approximated. The theory for this as given by Pupin is as follows:

If the current in the main circuit is:

$$y = a_1 \sin \theta + a_3 \sin 3\theta + \dots + a_{2n+1} \sin(2n+1)\theta + \dots$$

Then the drop between the terminals of the resonator is:

$$e = b_1 \sin \theta + \dots + b_{2n+1} \sin(2n+1)\theta + \dots$$

Where: $b_{2n+1} = a_{2n+1} \cdot r$ $r =$ ohmic resistance between resonator terminals.

If: $L =$ Inductance of resonator circuit.

$C =$ Capacity of resonator circuit.

$R =$ Resistance of resonator circuit.

The current in the resonator will be: $\theta = \omega t$

$$y_r = \sum_0^{\infty} (b_{2n+1}) \sin[(2n+1)\theta + \phi_{2n+1}] / \sqrt{(2n+1)^2 \omega^2 \left\{ \frac{1}{(2n+1)^2 \omega^2 C} + L \right\}^2 + R^2}$$

If therefore the capacity C be adjusted so that:

$[1/(2n+1)^2 \omega^2 C] - L = 0$ then the circuit will be in resonance

with the harmonic of frequency: $(2n+1)\omega/2\pi$, and if L is large compared to R , the current y_r will be given by the following expression to within a very small fraction of one percent error:

$$y_r = [b_{2n+1}/R] \sin(2n+1)\theta$$

The voltmeter will read: $P_{2n+1} = [(2n+1)\omega L/R] b_{2n+1}$ (Amplitude)

The fundamental frequency will be given by: $P_1 = \omega L b_1 / R$

The ratio of any harmonic will then be :

$$(P_{2n+1})/P_1 = (2n+1)(b_{2n+1}/b_1)$$

Beattie and several others have proposed circuit modifications to simplify the tests, but the principle is the same in all the methods.

The difficulties lie in the fact that the inductance and capacity must be of large value and reasonably high grade. This makes the apparatus required heavy, bulky and expensive. The electrostatic voltmeter is a rather delicate piece of apparatus, and if it is sensitive enough to determine small values of harmonics, it requires special mounting which cannot be obtained outside of a laboratory. Also the method requires power drawn from the circuit being investigated. The power is not large measured by ordinary power standards, but may be very large compared to that available in many types of investigations, and the amount drawn at resonance may be more than enough to distort the conditions from those that would occur if the resonant shunt were not connected. Further the function being investigated must remain absolutely constant for the complete duration of test, a condition very seldom obtained even in laboratory work.

The work of the California Railroad Commission in investigating Telephone Interference gives the latest data upon the use of this type of instrument. It was modified for use in the field by using a telephone in series with the resonant circuit as indicator, a separate oscillator was connected so that the telephone could be rapidly thrown from resonant circuit to oscillator circuit. The resonant circuit was then adjusted for maximum sound, and the oscillator circuit adjusted until the sound in the telephone when connected to it was identical with that obtained from the resonant shunt. By means of calibration of the oscillator the amplitude and frequency of the different harmonics were thus determined. The accuracy obtained was about 10 percent error in large values of harmonics, although some small ones were discovered which could not be determined with oscillograph curves.

CLASS 3. TYPE A. FORM b.

LAWS' DYNAMOMETRICAL ANALYSER.

(F.A.Laws, Technology Quarterly, Volume VI, p.252, 1893).

This device employs a dynamometer of conventional design, through one coil of which is passed the current to be analysed, and through the other coil is passed a sinusoidal current whose period and phase can be controlled. If the complex current is in phase with the sine current, the deflection of the dynamometer will then be proportional to the sine component of the harmonic in the complex current having the same frequency as that of the sinusoidal current. If the two currents are 90° out of phase then the deflection will be proportional to the cosine component of the same harmonic.

Furthermore, if the phase of the controlled sine current is shifted until the maximum reading is obtained, then the deflection is proportional to the modulus of the harmonic, or its actual amplitude, and the departure of the controlled sine current from coincidence in phase with the complex current is identical with the phase relation of the harmonic to the complex wave. The phase relation of the harmonic may be more accurately determined by adjusting for zero deflection and correcting by 90° . The correctness of this is easily shown as follows:

At any instant the force acting on the movable coil is:

$$KK_1 f(\omega t) \sin. m\omega t$$

Where K is an instrumental constant and K_1 is the maximum value of the sine current. The dynamometer thus performs the integration:

$$D = KK_1 / 2\pi \int_{-\pi}^{+\pi} f(\omega t) \sin. m\omega t =: (KK_1/2) A_m$$

Thus $A_m = 2D/KK_1$ if the two currents are in phase, and similarly for B_m if the two currents are 90° out of phase. In this case m is any odd integer determining the order of the harmonic derived, A and B are respectively the sine and cosine term coefficients, and D is the deflection of the dynamometer. It will at once be seen that the integral is identical with that obtained from the solution of Fourier's Series.

For the second method of operation:

$$D = KK_1/2\pi \int_{-\pi}^{+\pi} f(\omega t) \sin(m\omega t + \phi) d\omega t = KK_1/2 [A_m \cos\phi + B_m \sin\phi]$$

$$= (KK_1/2) \sqrt{A_m^2 + B_m^2} \sin(\phi + \xi)$$

$\tan\xi_m = A_m/B_m$ and D will have its maximum value when $\phi + \xi = 90^\circ$

Furthermore: $\tan\psi_m = \cot\xi_m$ Whence: $\psi_m = \phi_m$ Thus:

ϕ is known from the displacement in phase of the controlled sine current, $|\psi_m|$ being $\tan^{-1} B_m/A_m$.

Prof. Laws suggests the extension of this method to recorded curves by arranging a slide wire with a contact moved by a template cut from the recorded curve and thus giving a current reproducing that which originally was recorded. However, this introduces such complexity that it would appear to limit the method to direct application.

The method should be accurate and the dynamometer is simple of construction, but considerable apparatus would be required to obtain the controlled sine wave, since not only must the frequency and phase be under control, but also the frequency must be maintained a synchronous multiple of the complex wave measured. Fortunately very small power is required, and it might be possible to arrange vacuum tubes excited by the source of the complex wave and filter circuits segregated harmonics, though it is not clear how a simple method of controlling the phase could be developed.

M E C H A N I C A L M E T H O D S O F A N A L Y S I S .

ANALYSER DUE TO LORD KELVIN.

(Kelvin, Proceedings Royal Society, Vols. 24, 27, p. 266, 371, 1876-78).

The machine used by Lord Kelvin as an Analyser was first proposed by his brother, J. J. Thomson, as an ordinary integrator, and Lord Kelvin showed that by certain modifications it could also be used as a harmonic analyser. In both cases the general construction was the same and is shown in Fig. 42. It consists of a disc DD mounted on the axle AA. The disc is inclined at an angle so that a sphere B lies partly on the disc and partly on the cylinder CC, which is mounted by the pivots EE, but left free to rotate. The mounting is adjusted so that as the ball rolls across the face of the disc it will pass through one point where its point of contact with the disc is at the center of the disc. As an integrator linkages are introduced so that the disc DD is revolved an amount proportional to the abscissae of the curve being integrated, while the sphere B is displaced from the central position an amount proportional to the ordinates of the curve. Thus the motion imparted to the recording cylinder CC is proportional to $y \cdot dx$ and the machine performs an integration of the form $\int f(\theta) d\theta$, which gives the area.

However, if the disc DD is given an independent motion, which may be $\phi(\theta)$, then the integration will be of the form $\int f(\theta)\phi(\theta)d\theta$. Suppose a point on the circumference of DD is given a displacement of $\phi(\theta)d\theta$, the radius of the point being R. Then if the distance between the center of the disc and its point of contact with the sphere be $f(\theta)$, the point of contact of the sphere with the disc and therefore the point of contact of the sphere with the cylinder will move through a distance $f(\theta)\phi(\theta)d\theta/R$. Hence if the radius of the integrating cylinder CC be r; the angle $d\omega$ turned through by this cylinder is $f(\theta)\phi(\theta)d\theta/Rr$, and the total angle after complete trace of curve is $1/Rr \int_0^\theta f(\theta)\phi(\theta)d\theta$.

If $\varphi(\theta)$ be made $\sin.k\theta$ or $\cos.k\theta$ the integration then evaluates the required integral for the determination of the harmonic coefficient according to equations (2), (3).

A sketch of the analyser designed for the British Admiralty by Lord Kelvin is shown in Fig. 43. There are eleven discs, the spheres being guided by yokes from the common member across the top, to which is attached a stylus with which the curve to be analysed is followed. The curve itself is wrapped around a drum in the center of the apparatus. Thus the radii at which all the spheres roll are proportional to the ordinates of the curve. The mechanism imparting the proper motion to the discs is not clearly shown. One way of doing this was to have a separate cylinder upon which was wrapped a sheet carrying a true sine curve. This independent cylinder was geared to the cylinder carrying the curve to be analysed so that it traced one period of the sine wave for one period of the unknown wave. The different discs were then geared to a rack with the proper harmonic ratios, and the rack operated by a stylus caused to follow the sine curve. In order to avoid the difficulties of guiding two stylus points, a later model substituted a crank pin and guide motion for the sine curve, the crank pin being driven by the cylinder carrying the unknown curve. The integrating cylinders across the front then gave the coefficients upon a series of graduated discs which can be seen along the front.

This analyser involved one principle of mechanism which Lord Kelvin considered very important for work of this sort, namely, there is no sliding between the various parts of the integrating mechanism, the sphere having a free rolling motion, and thus eliminating errors which are present to a greater or less degree in planimeter types of analysers. In the form described above, however, it was a very large, heavy and bulky apparatus, being practically a fixture in any place where it was installed.

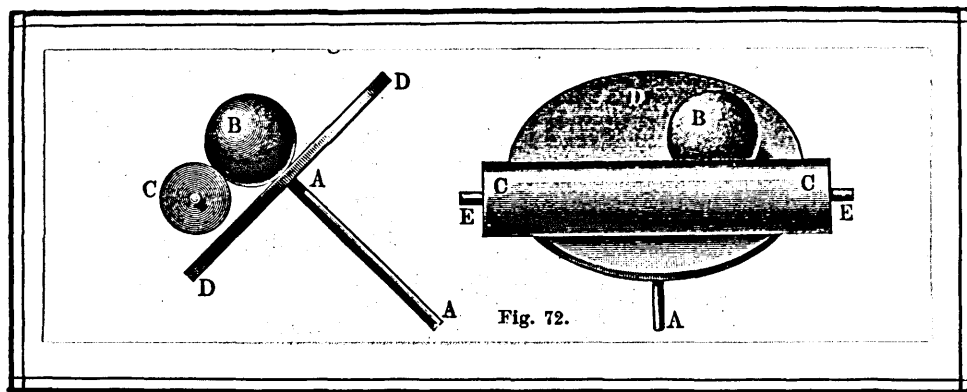


FIGURE No. 42.

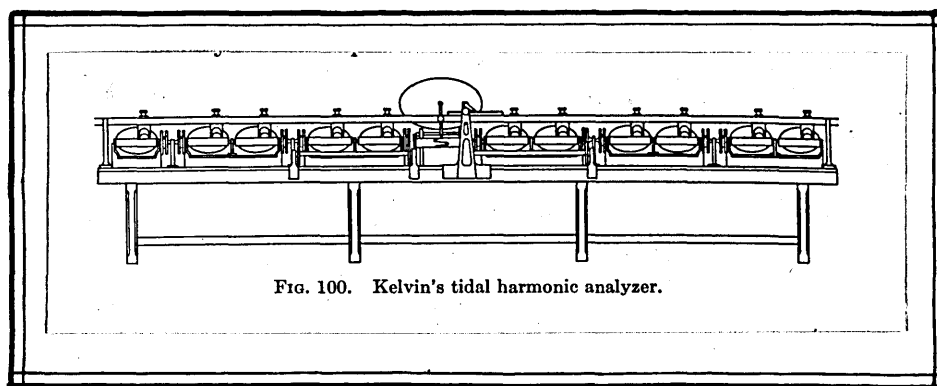


FIG. 100. Kelvin's tidal harmonic analyzer.

FIGURE No. 43.

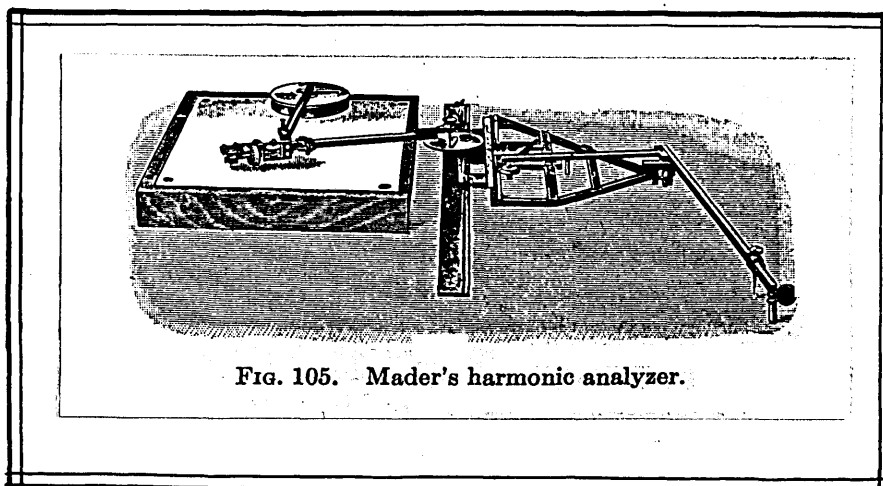


FIG. 105. Mader's harmonic analyzer.

FIGURE No. 44.

ANALYSER DUE TO WIECHERT AND SOMMERFELD.

(Walther Dyck, Katalog Mathematischer und Mathematisch-Physikalischer Modelle, Apparate und Instrumente, Munchen, 1892).

The analyser consists of a cylinder ABCD around which the curve to be analysed is wrapped, and which has two rotating movements. The first rotation is around its central axis and the second is around a vertical center through the point O. The two motions bear a definite relation to each other dependant upon the order of the harmonic being sought. A thread or wire is supported over the top element of the cylinder, EF, arranged so that it rotates with the cylinder around O but always remains immediately over and parallel to the top element. In addition to the cylinder mechanism is a disc S upon which rolls the recording wheel R, carried by the frame J. The disc S rotates at the same rate as the cylinder around its central axis. At the lower end of the frame J is a second thread or wire GH. In order to operate the analyser the cylinder is turned through its compound motion and the frame J maintained in such a position that the line GH is always vertical to the line OK and passes through the intersection of the curve and the line EF. The dimensions are such that when the line GH passes through O the wheel R is at the center of the disc S. Thus the rotation of the wheel R depends upon, first the radius at which it rolls, which is proportional to the projection of the ordinate of the curve upon the vertical line OK, and second upon the rotation of the disc S, which is proportional to ωt of the curve.

The angle POK = $k\omega t$, and OP = y

Thus: $OQ = y \cdot \cos k\omega t$ @ Radius at which R rolls.

The speed or amount of rotation of S = ωt

which is the same as the rotation of the cylinder around its central axis.

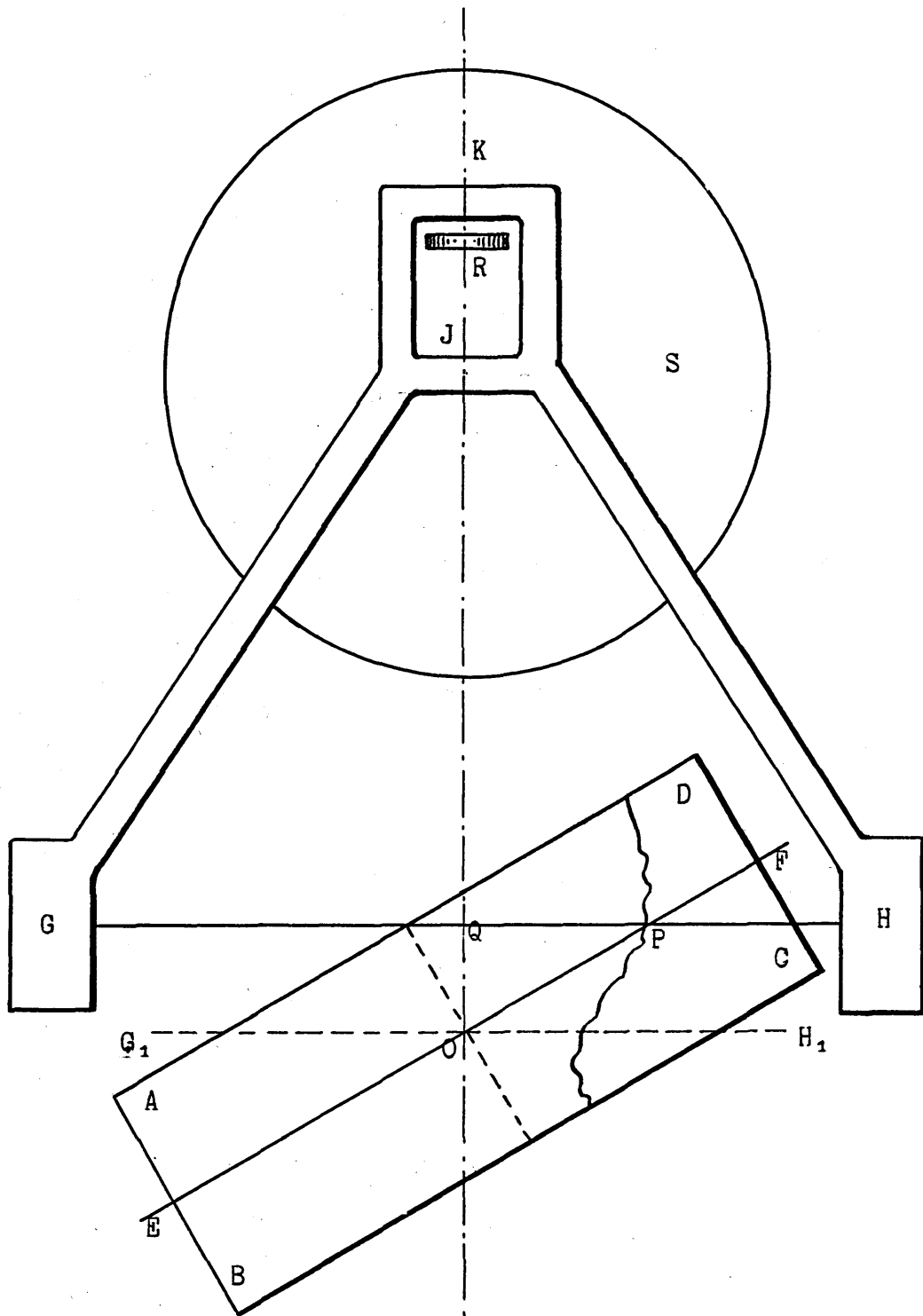
The wheel R is rotated an amount proportional to these two expressions, hence:

$$\text{Rotation of wheel R} = \omega \int_0^t 2\pi y \cdot \cos.k\omega t . dt = \pi b_k$$

Thus for the conditions the value of the k-th cosine coefficient is determined. The cylinder has been assumed to lie with its axis parallel to OK when starting the trace. If the axis is started parallel to G_1H_1 , then by the same reasoning the value of the sine coefficient is determined.

The value of k is the ratio between the revolutions of the cylinder around O and around its central axis. Thus if the cylinder makes two revolutions around O for one trace of the curve, or in other words, one revolution of the cylinder around its central axis, then the coefficients of the second harmonic will be read from the wheel R. A copy of an engraving of the completed machine taken from the University of Konigsberg in 1890. The cylinder is rotated with one hand, while the cage carrying the wheel R is made to follow the curve by means of the hand-wheel in the upper right hand corner.

It is necessary to have the curve of such a length that one full period is equal to the circumference of the cylinder, and the double adjustments makes the action rather slow. The beginning of the trace requires careful adjustment, and must be adjusted in two different positions, one for the sine and one for the cosine components. The gears or other driving means between the two rotations of the cylinder must be altered for each harmonic.



HARMONIC ANALYSER DUE TO WIECHERT AND SOMMERFELD.

FIGURE No. 45.

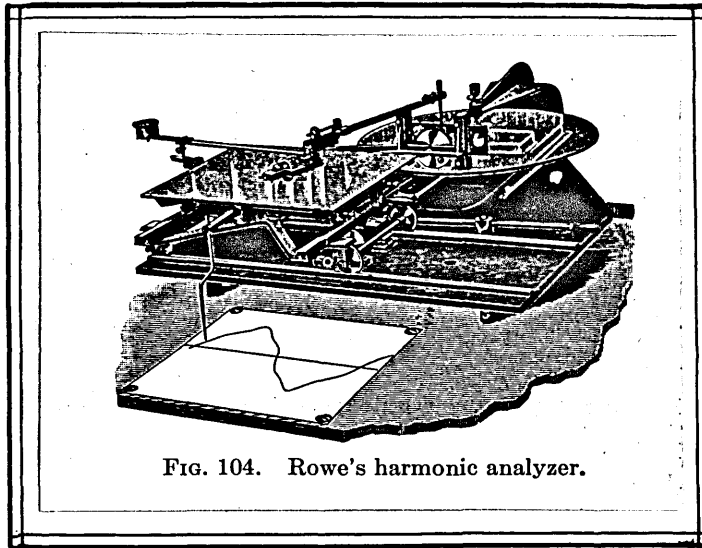


FIGURE No. 46.

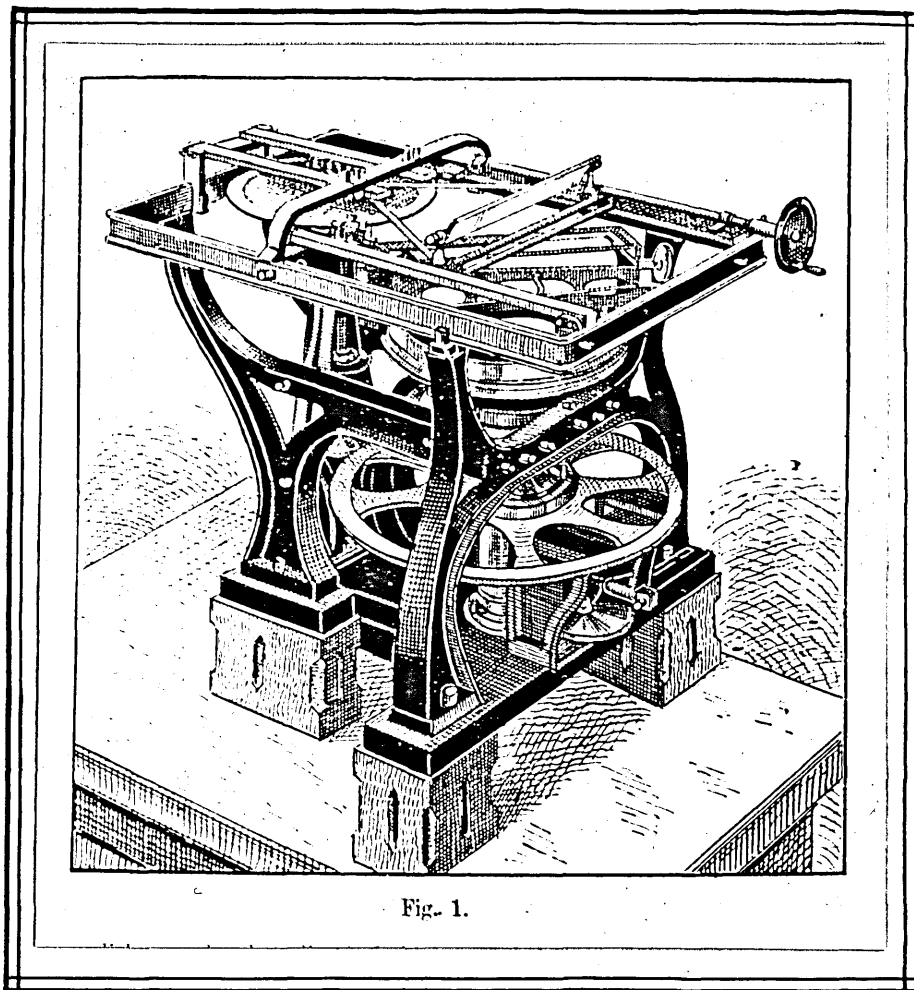


FIGURE No. 47. Analyser of WIECHERT and SOMMERFELD.

ANALYSER DUE TO BASHFORTH.

(F. Bashforth, British Association Report, 1892.)

Bashforth proposed a mechanical framework to perform the analysis in exactly the same manner as the Harrison-Ashworth graphical method, or method of coplanar forces. By referring to the Harrison-Ashworth scheme it will be seen that the ordinates are represented as vectors, and the position angles, or harmonic multiples of them, used as the angles of the vectors.

The Bashforth machine consists of a series of rods or bars, connected to each other by adjustable connections carrying protractors to indicate the angles. The bars are provided with scales indicating the distance from the joint in one connector to that in the next. To operate the curve to be analysed is divided up into any number parts, equal or less than the number of bars, and the ordinates measured. The framework of bars is then adjusted so that the distance between joint-centers on successive bars is equal to the successive ordinates of the curve. For the fundamental the angles between bars are then set the same as the angles between successive ordinates, and the distance between the ends of the framework gives the fundamental amplitude, and its angular relation gives the phase. For the second harmonic the angle at each joint is doubled, and for the third harmonic is trebled etc.

It has the advantage of determining both sine and cosine components with one setting, but the number of settings for an extensive analysis would be very large. It also has the limitations of the schedule analysis methods upon which it is based.

ANALYSER DUE TO HENRICI.

(O. Henrici, Philosophical Magazine, Vol. 38, p. 110, 1894.)

The analyser developed by Prof. Henrici is at present the most accurate and speedy as well as one of the best known of the mechanical harmonic analysers. The manufacture and some of the developments have been carried out by Coradi of Zurich. The present form is the result of trials with two other types, in addition to many improvements in detail of the present type.

Henrici's first type is roughly indicated in figure 48. This was made in 1889 and consisted of a cylinder C around which the curve was wrapped, and a flat board B which carried the planimeter P. The board B ran in guides and was given simple harmonic motion by means of a crank mechanism geared to the cylinder. As the cylinder was rotated the planimeter point was caused to follow the curve, and thus the displacement of the planimeter wheel was determined by the ordinates of the curve and the motion of the board. By proper adjustments this would give the harmonic coefficients one at a time, according to the formulas (2), (3), or by considering the integration of these formulas by parts.

$$k\pi a_k = [y \cdot \sin.k\theta]_0^{2\pi} - \int_0^{2\pi} \sin.k\theta \cdot dy$$

If the curve is continuous the first term becomes zero, and it is so small as to be negligible even if the curve has discontinuities. As the planimeter point has a motion proportional to $\int dy$, and the surface upon which it operates has a motion proportional to $\sin.k\theta$, it is evident that the final reading will evaluate this integral.

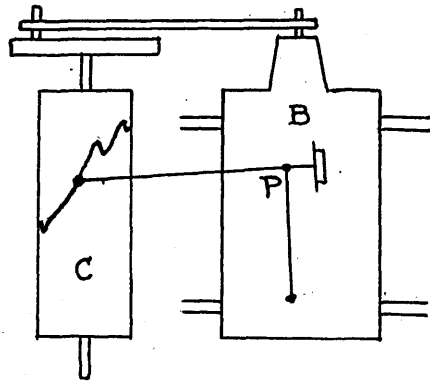


Fig. 48.

The next type is an inversion of the planimeter wheels, and is shown in Fig. 49. The curve is again wrapped around a cylinder C and followed with a stylus T. The stylus, however, now is carried by an arm running in tracks SS so that it only moves parallel to the axis of the cylinder. At e a vertical spindle carries the recording wheels, and now two can be used to give the sine and cosine coefficients of one harmonic with a single trace, the two wheels R_1 and R_2 being mounted at right angles. The vertical spindle is revolved through an angle $k\theta$ for a motion of the cylinder C through an angle θ , k being the order of the harmonic sought. This motion of the vertical spindle is accomplished by means of the jointed arm ABA and the belt b , which is driven by the wheel H, which is in turn driven by gears or friction from the cylinder C through the wheel VV. The gear ratio between VV and H determines the order of the harmonic coefficient read on R_1 and R_2 . The theory is similar to the previous type.

This introduced, however, a sliding motion between the recording wheels R_1 and R_2 and the surface upon which they operated, and introduced considerable error where the value of the harmonic was small and its order high. Therefore the third modification was developed by the introduction of glass spheres to operate the recording wheels and the elimination of sliding contacts. This is shown in figures 51, 52, and 53. The completed machine has five glass spheres carried by a framework t , Fig. the spheres resting upon wheels driven by the rollers carry-

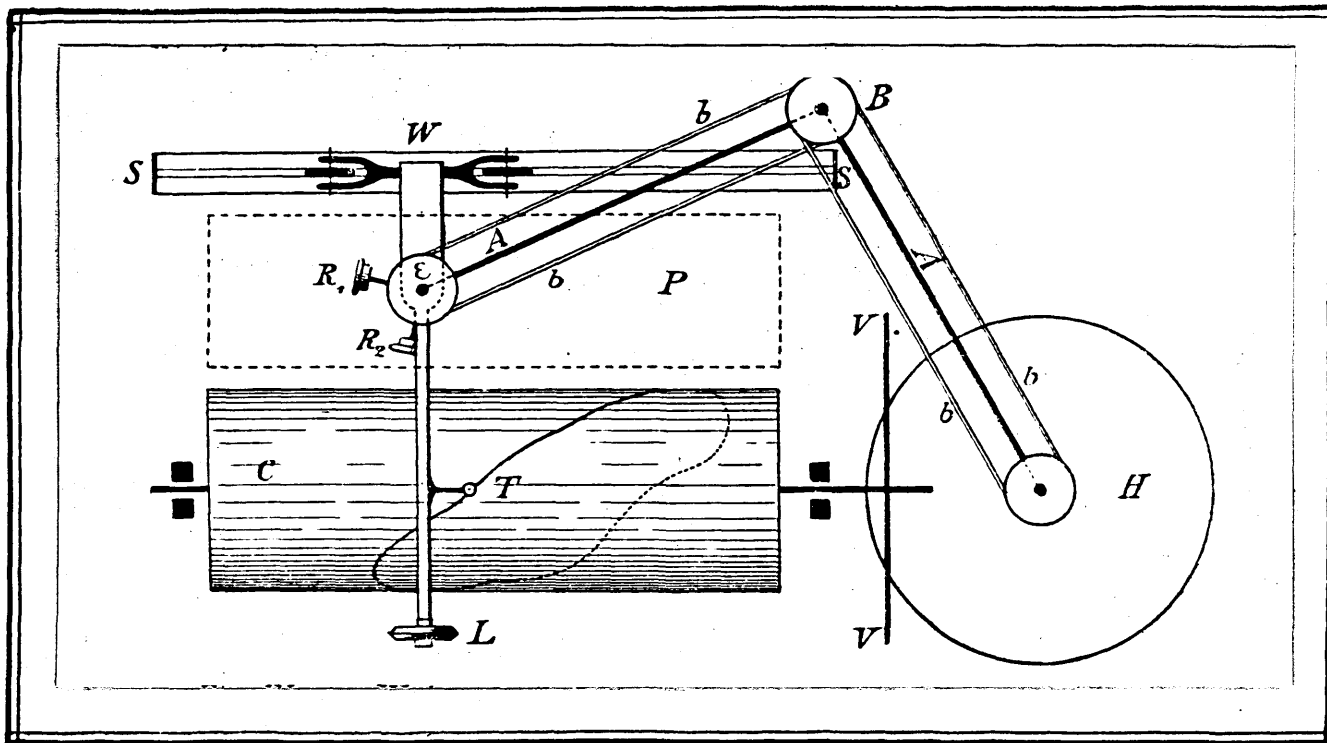


FIGURE No. 49. Early Model of Henrici Analyser.

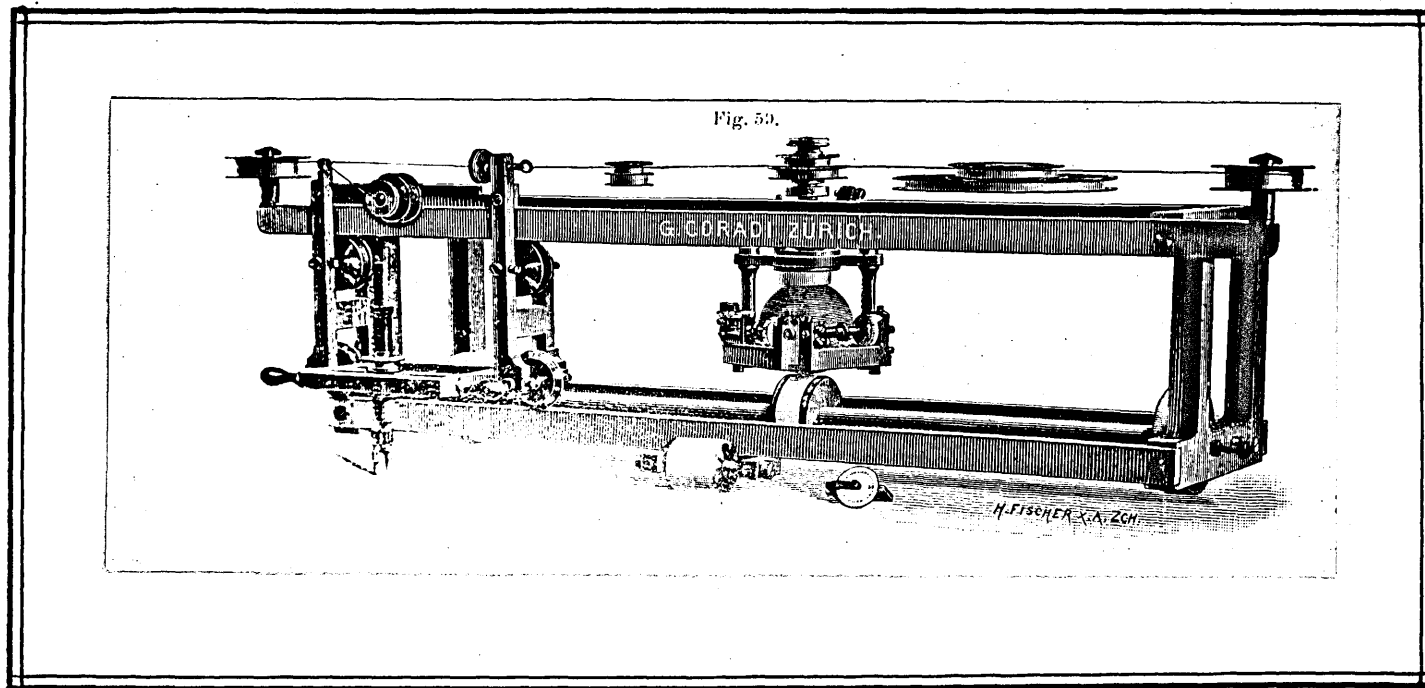
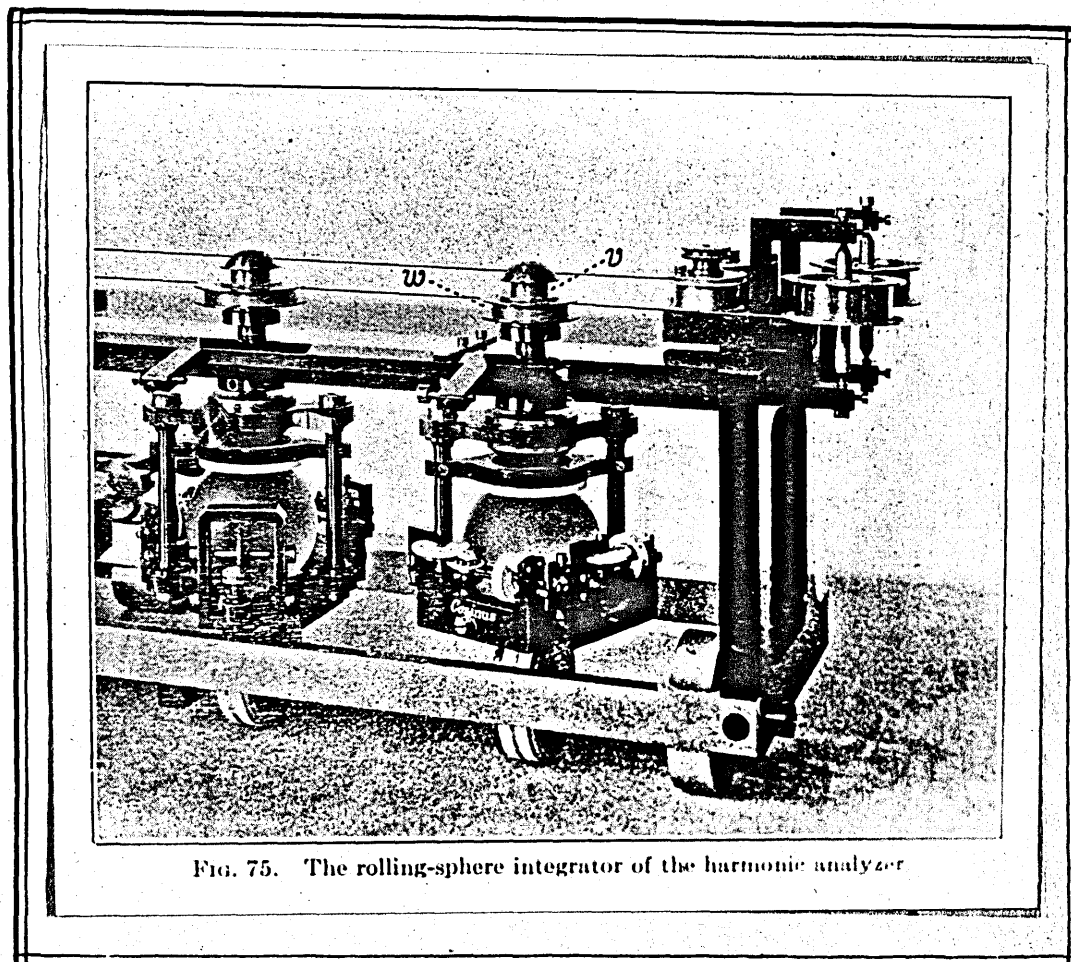
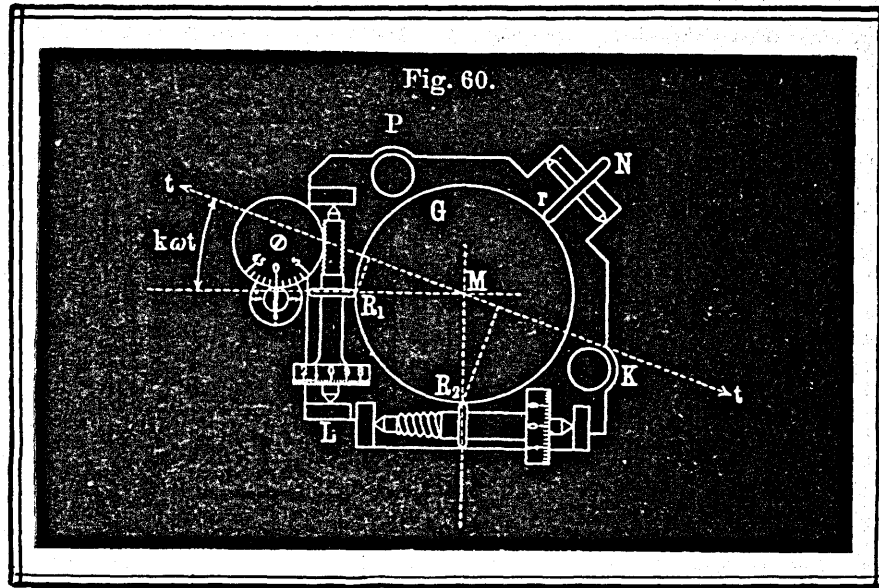


FIGURE No. 50. Single Element Henrici-Coradi Analyser.



FIGURES No. 51 and 52. Details of Henrici Analyser.

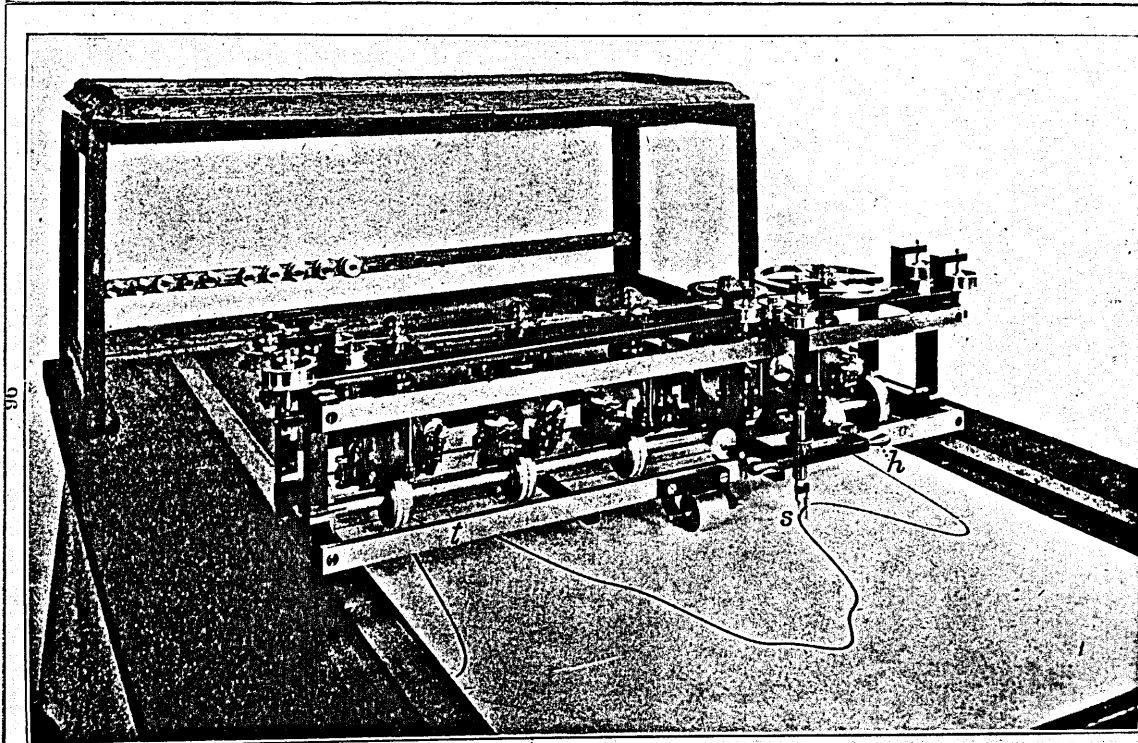


FIG. 74. Henrici's harmonic analyzer.

FIGURE 53. Complete Henrici-Coradi Five Element Analyser.

ing the framework, and running in tracks along the sides of the operating table. The curve is layed flat upon the table instead of being wrapped around a cylinder, and traced with the stylus S, which moves freely along the framework in the direction of the X-axis, but carries the framework with it in the direction of the Y-axis. Thus the spheres are all rotated through an angle proportional to the ordinates of the curve. Around each sphere is a cage capable of separate motion concentric with the spheres, each cage carrying two recording wheels in contact with the spheres at points 90° apart. This is shown in Fig. 51, the sphere being G and the recording wheels R₁ and R₂, the pressure of the wheels being counterbalanced by the idler r at N. Thus if the cages remained stationary the recording wheels would also be given a deflection proportional to the ordinates of the curve. But the cages carrying the recording wheels are connected to the stylus by means of the pulleys, v, w, Fig 52, operated by a wire attached to the stylus carriage. In this way as the curve is traced the recording wheels revolve around the spheres, or actually cause the spheres to revolve upon their contact with the bottom driving wheels. By having the proper sized pulleys the recording wheels will thus operate at a radius proportional to the sine or cosine of the units used in the X-axis of the curve being analysed. Since the spheres are revolving proportional to the ordinates of the curve, the final readings of the integrating wheels will evaluate:

$$\int_0^{2\pi} \sin.k\theta.dy \quad \text{and} \quad \int_0^{2\pi} \cos.k\theta.dy$$

which is the solution necessary to determine the harmonic coefficients according to formulas (2), (3). Each of the five spheres will determine one sine and one cosine coefficient, and thus ten coefficients may be determined for one trace of the curve. By changing pulleys upon the wheel carrying cages any order of harmonics may be determined, the limit being the point at which the order is so high that the pulleys become too small to be operated without error due to slipping of the wire.

A little more rigorously the mathematical analysis becomes:

$$\int y \cdot \sin.k\theta \cdot d\theta = -y \cdot \cos.k\theta/k + 1/k \int \cos.k\theta \cdot dy.$$

And since $A_k = 1/\pi \int_0^{2\pi} y \cdot \sin.k\theta \cdot d\theta$;

Then: $A_k = 1/k\pi \int_0^{2\pi} \cos.k\theta \cdot dy.$

In figure 51 the recording wheels are shown in the starting position and the line MR_1 represents the axis of rotation of the spheres for the deflection due to the curve ordinates. After a small portion of the curve has been traced the wheels will have revolved through an angle marked $k\omega t = k\theta$. The sphere will have rotated about its original axis MR_1 an amount proportional to dy in the same interval. The motion imparted to the recording wheel R_1 will then have been proportional to the sphere deflection dy and the radius at which the wheel rests; or a perpendicular between the line MR_1 and the line tt , which is proportional to $\sin k\omega t = \sin.k\theta$. Thus after the complete trace has been made the wheel R_1 will have a deflection proportional to $\int \sin.k\theta \cdot dy$ and in like manner the wheel R_2 will have a deflection proportional to $\int \cos.k\theta \cdot d\theta$; which, from the reasoning above, is the evaluation of the k -th harmonic cosine and sine coefficients..

It is interesting to note that there are many similarities between this machine and that of Kelvin. If it be considered that the cylinder of the Kelvin machine is replaced by the wheels R_1 and R_2 , and the disc by the linear motion upon the table, then the two mechanisms are almost identical in basic features, although the operation is inverted. That is the elements introducing the dy and $\sin.k\theta$ components are interchanged. The accuracy obtained is very great and checks obtained by synthesizing curves after analysis surprisingly close.

HARMONIC ANALYSER DUE TO SHARP.

(Sharp, Philosophical Magazine, Vol. 38, p. 121, 1894.)

The analyser consists of a carriage FF, rolling upon the three wheels W_2 , W & W_1 . Across the front of the carriage is a slot SS in which the cage G is guided by two small wheels. The pointer P is fixed to the cage G so that it moves proportional to the ordinates of the curve to be analysed. The diameter of the wheels WW is adjusted so that they make the same number of revolutions for one period of the curve as the order of the harmonic being determined. The arm F supports the three discs D_1 , D_2 & D_3 , the first of which is geared 1:1 to the axle of the wheels WW. The second disc acts as a coupling between the first and third, and carries right angled slots, one on one side and the other on the other side, in which slide keys or splines in the adjacent faces of the top and bottom discs. In this way the third disc is constrained to rotate in the same manner as the wheels WW, but may have its center displaced in any direction. The action is similar to a trammel gear or Oldham coupling. The motion of the top disc is further constrained by the wheel R_2 which makes a friction contact with the disc, and is driven through a shaft by the wheel R_1 which is driven by friction upon the surface of the main frame beside the track SS. Thus as the pointer P is caused to trace the curve the top disc is moved horizontally back and forth amounts proportional to the ordinates of the curve, while it rotates amounts proportional to the abscissae of the curve.

The wheel R_2 draws a curve upon the top disc such that each element of length dy makes an angle of \cot with the Y axis, and the result is similar to the graphical coplanar force diagram, except that it is drawn for an infinite number of infinitesimal forces. Projections of this curve upon lines parallel to the X and Y axes then give the integrals determining the harmonic coefficients of the sine and cosine terms. Furthermore the line drawn through the ends of the curve represents the absolute value of the harmonic coefficient, and its angle represents the phase angle of the harmonic. Thus it is not actually necessary to draw the curve but only the first and last positions of the wheel R_2 are necessary. This leads to the interesting fact that it is not necessary to make any preliminary adjustments, as no matter how the machine is set at the start, the two positions of the wheel R_2 will determine the coefficients.

The different orders of harmonics are obtained by either changing the size of the wheels WW or the gearing between them and the bottom one of the three discs. This machine is distinctive in being practically the only method which gives the absolute value and phase angle of the harmonics, the others determining the sine and cosine terms separately and requiring later mathematical combination.

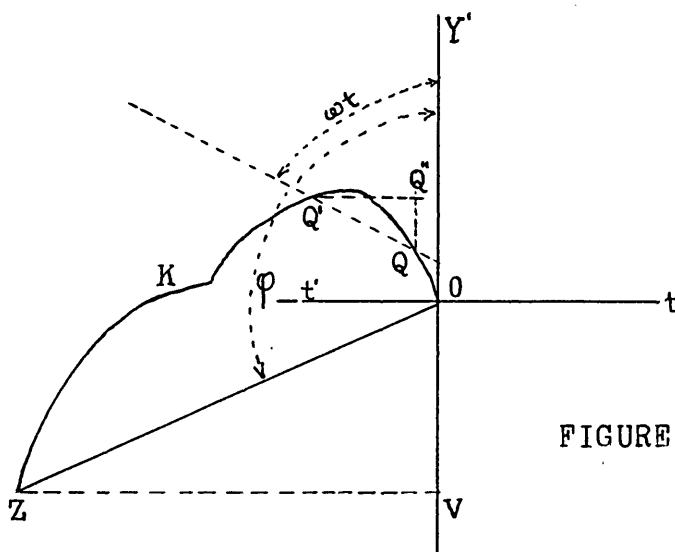


FIGURE No. 54.

Assume that OKZ represents the curve drawn by the wheel R_2 on the top disc D_3 . When the pointer P is at some point of the curve having the coordinates y, t , let the wheel R_2 be in the position Q. The disc D_3 has been turned through an angle ωt . Now let the pointer be moved to some new point P' such that the angle between QQ' and $OY' = \omega t$. Then $QQ' = dy$.

Hence: $QQ'' = \cos \omega t \cdot dy$
 $Q''Q' = \sin \omega t \cdot dy$

Thus the coordinates of Z are:

$$\left. \begin{aligned} OV &= \int_0^{\pi} \cos \omega t \cdot dy = \pi b_1 \\ ZV &= \int_0^{\pi} \sin \omega t \cdot dy = -\pi a_1 \end{aligned} \right\} \text{From Equation (8) \& (9).}$$

Further: $OV \cong OZ \cos \phi = \pi c_1 \cos \gamma_1$
 $ZV \cong OZ \sin \phi = -\pi c_1 \sin \gamma_1$

i.e. $OZ = \pi c_1$
 $\tan \gamma_1 \cong -ZV/OV = -\tan \phi$

Or: $\gamma_1 = \pi - \phi$

γ being the phase angle of the harmonic and the modulus or absolute value being c . The above is carried out for the fundamental, but would be similar in every respect for higher harmonics.

HARMONIC ANALYSER DUE TO SHARPE.

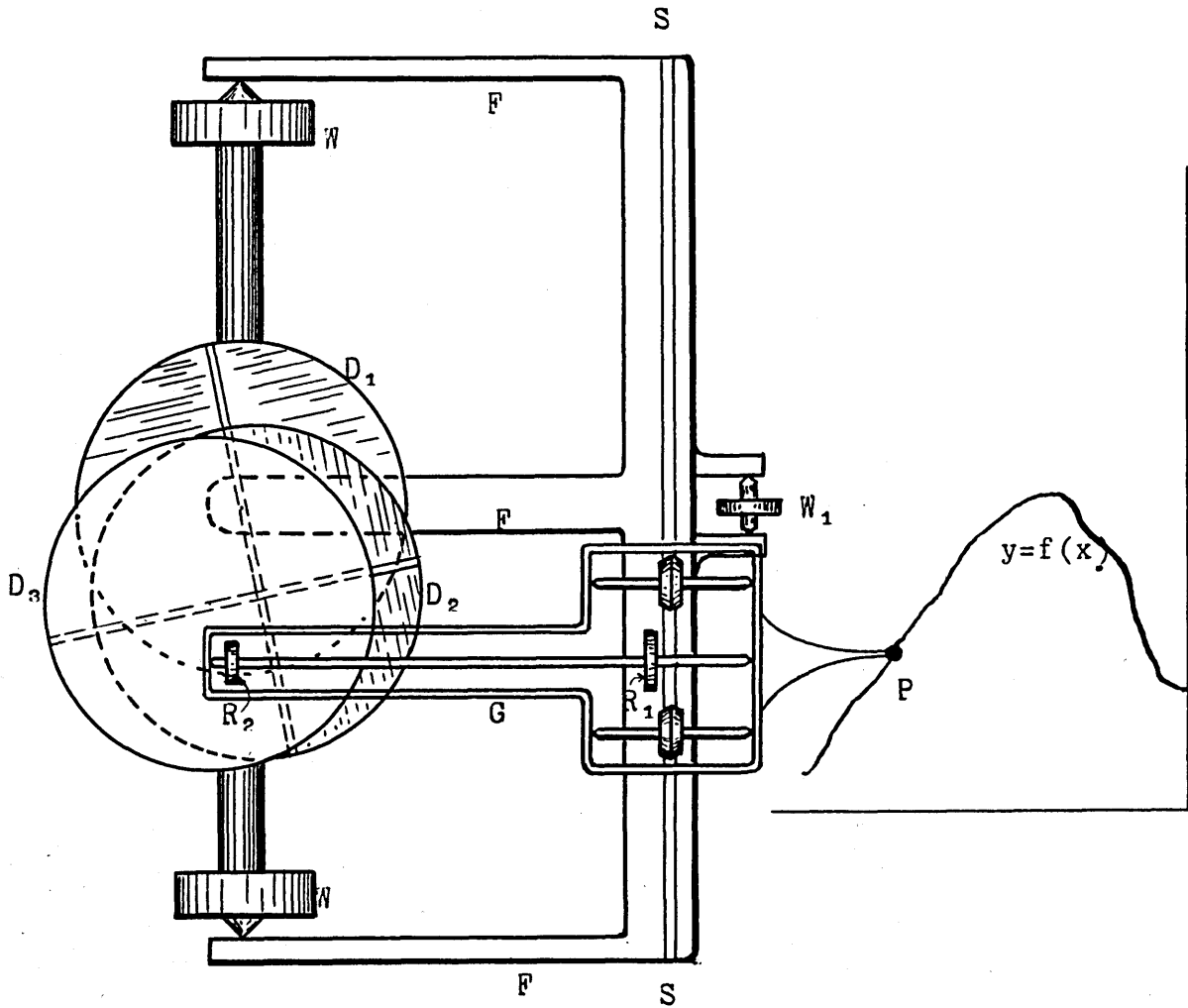


FIGURE No. 55.

ANALYSER DUE TO YULE

(G.U.Yule, Philosophical Magazine, Vol.39, p.367, 1895)

The first machine proposed by Yule was designed with the idea of great simplicity and cheapness. It is indicated by Fig. 56; and consisted of a board B, on which a carriage A could slide in the Y-axis direction. The member A carried a stylus T, and also a scale of sines at S, with small indentations opposite the different points of the scale. A planimeter then had its fixed point mounted in B, and its tracer point placed in one of the indentations of the sine scale. In operation the curve is divided up into a number of equal spaces, and placed upon a second board C. P is then placed at zero; T placed at the point where the curve crosses the axis, and the planimeter adjusted to zero. The curve is then slid along and T adjusted to the ordinate y_1 , when P is moved to the angle at which y_1 occurs. T is then shifted to y_2 and P to angle corresponding to y_2 etc. The final reading of the planimeter gives the area of the polygon:

$$y_1(\sin x - \sin 0) + y_2(\sin 2x - \sin x) + y_3(\sin 3x - \sin 2x) + \dots$$

Which approximates the integral: $\int_0^{2\pi} y \cdot d(\sin \theta)$
 which is one form of the integral evaluating the harmonic coefficient of the fundamental sine term. For the higher harmonics the same procedure is followed except that the pointer P is placed in every other scale step for the second harmonic, every third for the third harmonic etc. For the cosine term coefficients the point P is started at 90° instead of 0° . The greater the number of ordinates used the greater will be the closeness of the approximation.

The final machine practically makes this procedure automatic and continuous and so gives the actual integral instead of its approximation. It is indicated in Fig. 57. It consists of three pieces: A rolling rule L with a rack on the lower edge. A wheel P with teeth to fit the rack, and an

arm in the end of which the third part, an ordinary polar planimeter, rests its tracer point. The curve is fastened to any convenient board and the rolling rule placed parallel to its X-axis. The wheel P has a hole in its center which is placed at the zero point of the curve with the arm pointing to the left and in line with the X-axis. The planimeter is set to zero and the board inclined so that the rule L will rest against the wheel P. The wheel is then caused to trace the curve with its center, the planimeter following the gyrations of the end of the arm PD. The trace is carried back to the starting point along the X-axis, and with some types of curves the planimeter will then give the reading of one of the coefficients. With curves involving a constant term the planimeter must be caused to trace the curve backwards without any wheel, which eliminates the constant term, and gives the coefficient sought as the remainder.

The cosine terms are determined by having the arm DP perpendicular to the X-axis at the start, and the sine terms from the setting used above. The order of the harmonic is determined by the number of revolutions of the wheel per wave length of curve. Thus one revolution for complete trace of one wave length will give the fundamental components etc.

Let the coordinates of D be η, ξ .

$$\begin{aligned}\xi &= t - r \cdot \cos.k\omega t & \text{where } r &= PD \\ \eta &= y - r \cdot \sin.k\omega t\end{aligned}$$

For trace from P to R, planimeter reading = $+\int_{t=0}^{t=\tau} \eta \cdot d\xi$

$$= \int_0^\tau (y - r \cdot \sin.k\omega t)(dt + k\omega r \cdot \sin.k\omega t \cdot dt)$$

And back again along the X-axis gives:

$$\text{Planimeter reading} = \int_\tau^0 (-r \cdot \sin.k\omega t)(dt + k\omega r \cdot \sin.k\omega t \cdot dt)$$

Thus the total planimeter reading will be :

$$\int_0^{\tau} y \cdot dt + k\omega r \int_0^{\tau} y \cdot \sin k\omega t \cdot dt$$

The first part of this equation is the area of the curve, and the second part will be recognized as the evaluation of the k-th sine term coefficient of the Fourier Series. Therefore if the planimeter now be run over the curve in the ordinary way but with opposite sense it will subtract the first term and leave $k\omega r$ times the coefficient sought. If r is designed so that it is equal to $1/k\omega$, then the final planimeter reading will be the harmonic coefficient direct, times π .

If a complete period of the curve has been traced, and it is an alternating current wave free from even harmonics, then the value $\int_0^{\tau} y \cdot dt$ will be zero, and it is unnecessary to perform the second trace of the curve with the planimeter.

If even harmonics are present however, the term $\int_0^{\tau} y \cdot dt$ will not be zero and must be subtracted as above. With waves where even harmonics are absent the trace of a half cycle is sufficient to give the harmonic coefficient, but in this case the second trace of the curve must be made, since the planimeter has not followed both halves of the period and so $\int_0^{\tau} y \cdot dt$ will not be zero.

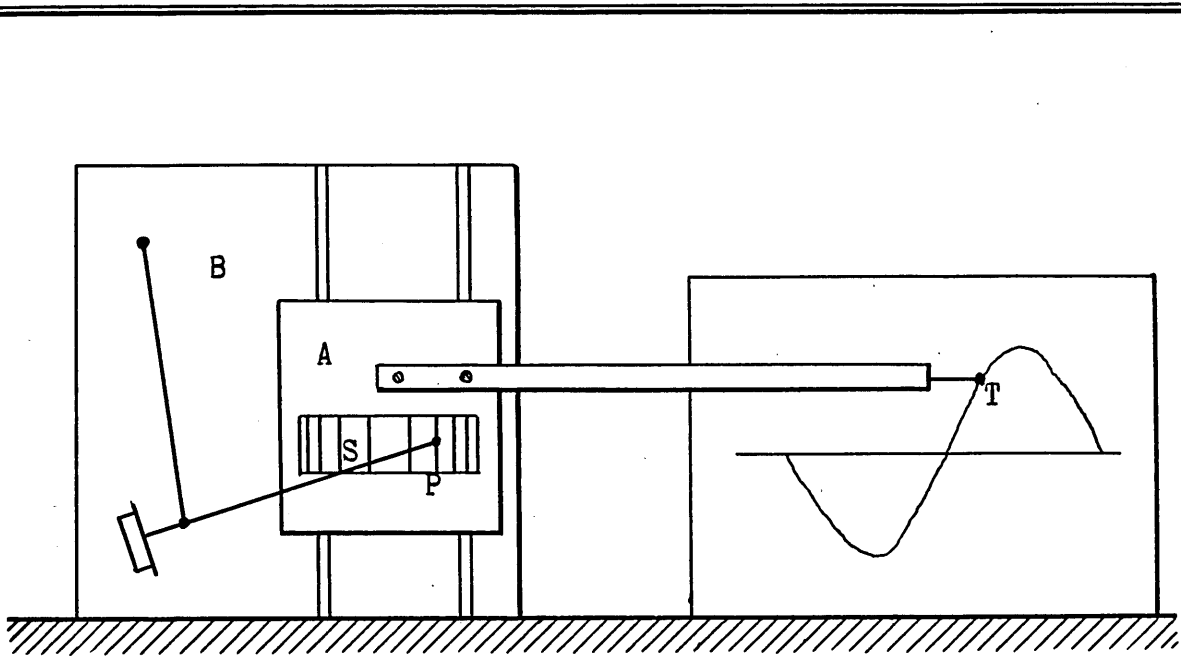
If the planimeter reading be f_k , then in general:

$$f_k = b_0 \tau + \pi k r \cdot a_k$$

Or, if $b_0 \tau$ is eliminated;

$$a_k = f_k / \pi k r$$

The same things hold for the cosine terms, the arm of the wheel being started at right angles to the position for the sine term coefficients, as already stated.



FIRST ANALYSER PROPOSED BY G.U.YULE.

FIGURE No. 56.

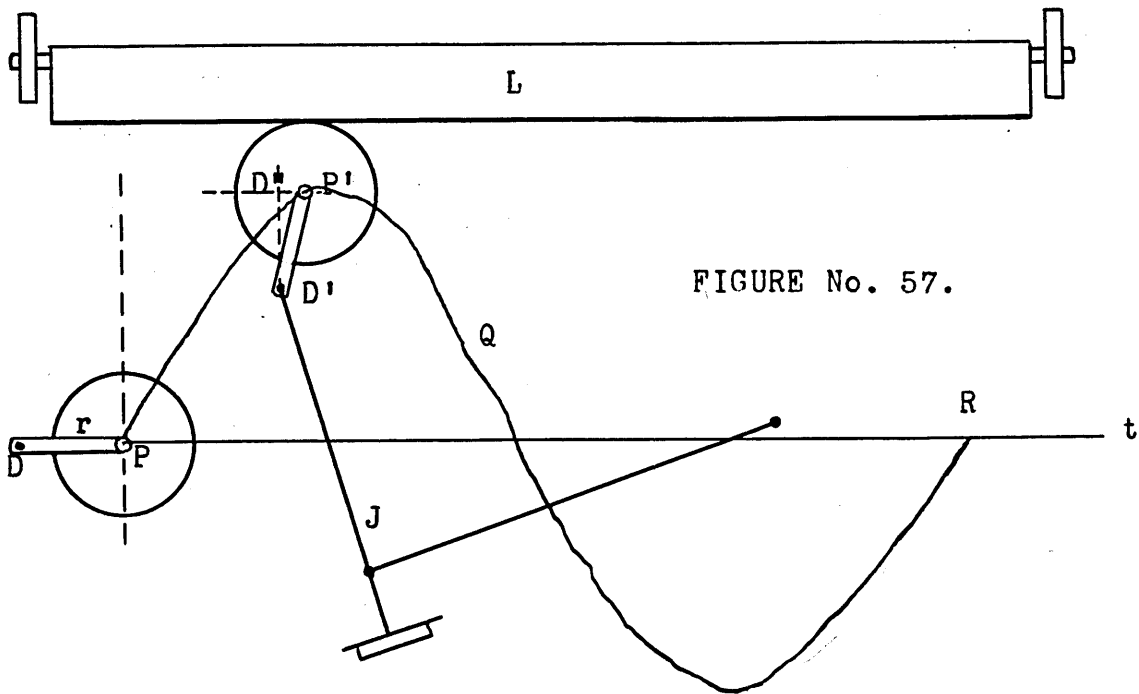


FIGURE No. 57.

SECOND ANALYSER PROPOSED BY G.U.YULE.

ANALYSER DUE TO LE CONTE.

(A. Galle, Mathematische Instrumente, B.G. Teubner, p. 137, 1912).

The Analyser of Le Conte is shown in Fig. 58. It does not seem to be possible to determine its actual date of construction. It does not seem to have been mentioned previous to 1898. In action it is very similar to the first analyser proposed by O. Henrici. The curve is placed upon the base of the instrument and traced by means of the stylus T. This stylus is carried by a platen GF mounted upon a set of tracks held by the plate LN, which in turn rides upon tracks at right angles to the first. Thus the platen GF may move in any direction. Attached to this platen is the fixed point of a polar planimeter at Q, and its wheel R also rolls on the same surface. The other end of the planimeter is carried by a slider S driven with simple harmonic motion by means of a gear meshing with a rack (not shown) which rack is attached to the plate LN. Thus as the curve is traced one end of the planimeter is given a motion proportional to y and the other end proportional to sine or cosine of $k\theta$. Arrangements are made so that various gears may be introduced to operate S, and so give various harmonics. The ultimate deflection of the planimeter wheel R will thus evaluate the integrals $\int y \sin k\theta$ and $\int y \cos k\theta$ which determine the values of the harmonic coefficients.

For the higher harmonics the gear must turn many times for one trace of the curve and therefore it is not possible to operate the analyser smoothly from the stylus. Thus a handle is provided which operates the gear through a worm and must be turned by one hand while the stylus is moved to left or right so that it follows the curve with the other hand. This is one of the simplest machines to construct and is accurate within the limits of the accuracy of the planimeter, but will only determine one coefficient at a time and is thus slow, and the accurate tracing of the curve a little difficult.

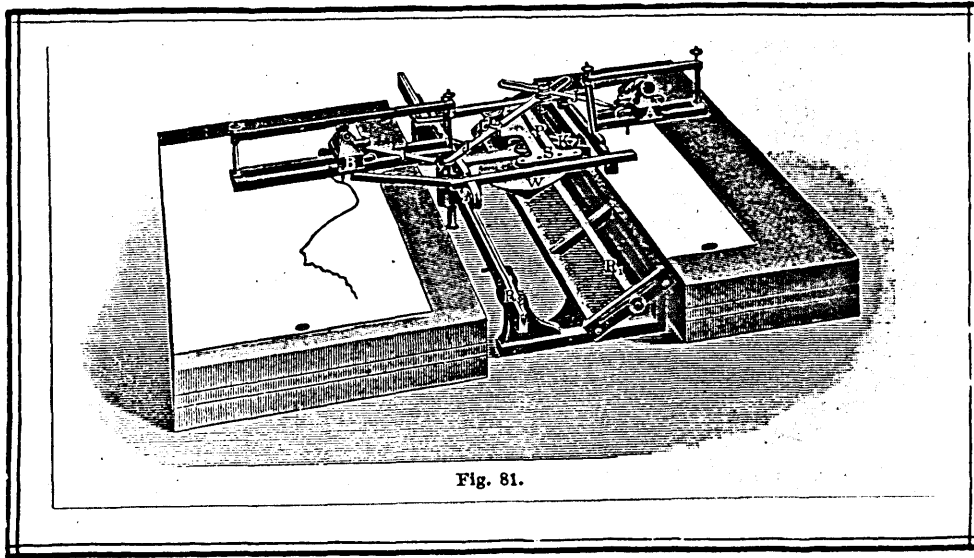
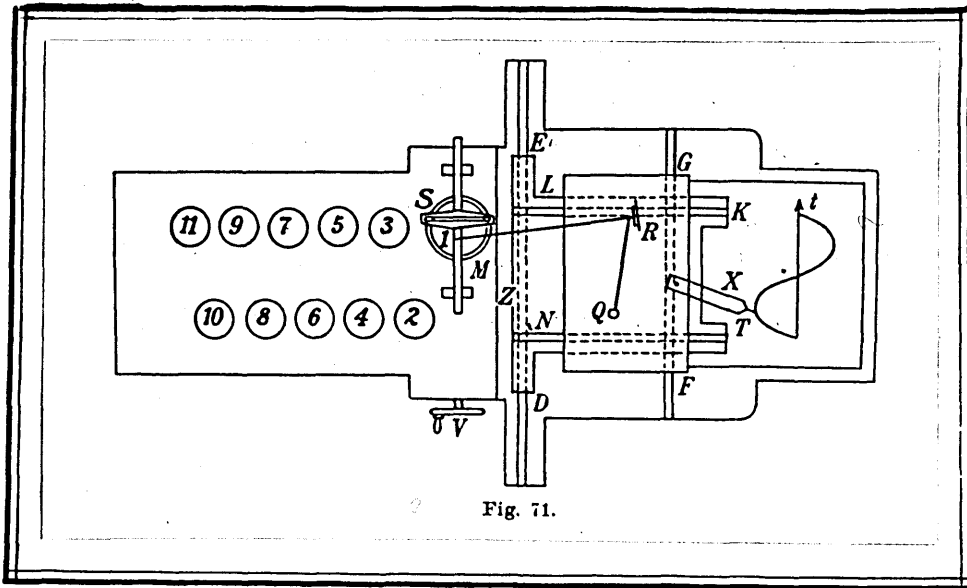


FIGURE No. 58. Analyser of Le Conte.

FIGURE No. 59. Analyser-Synthesizer of Terada.

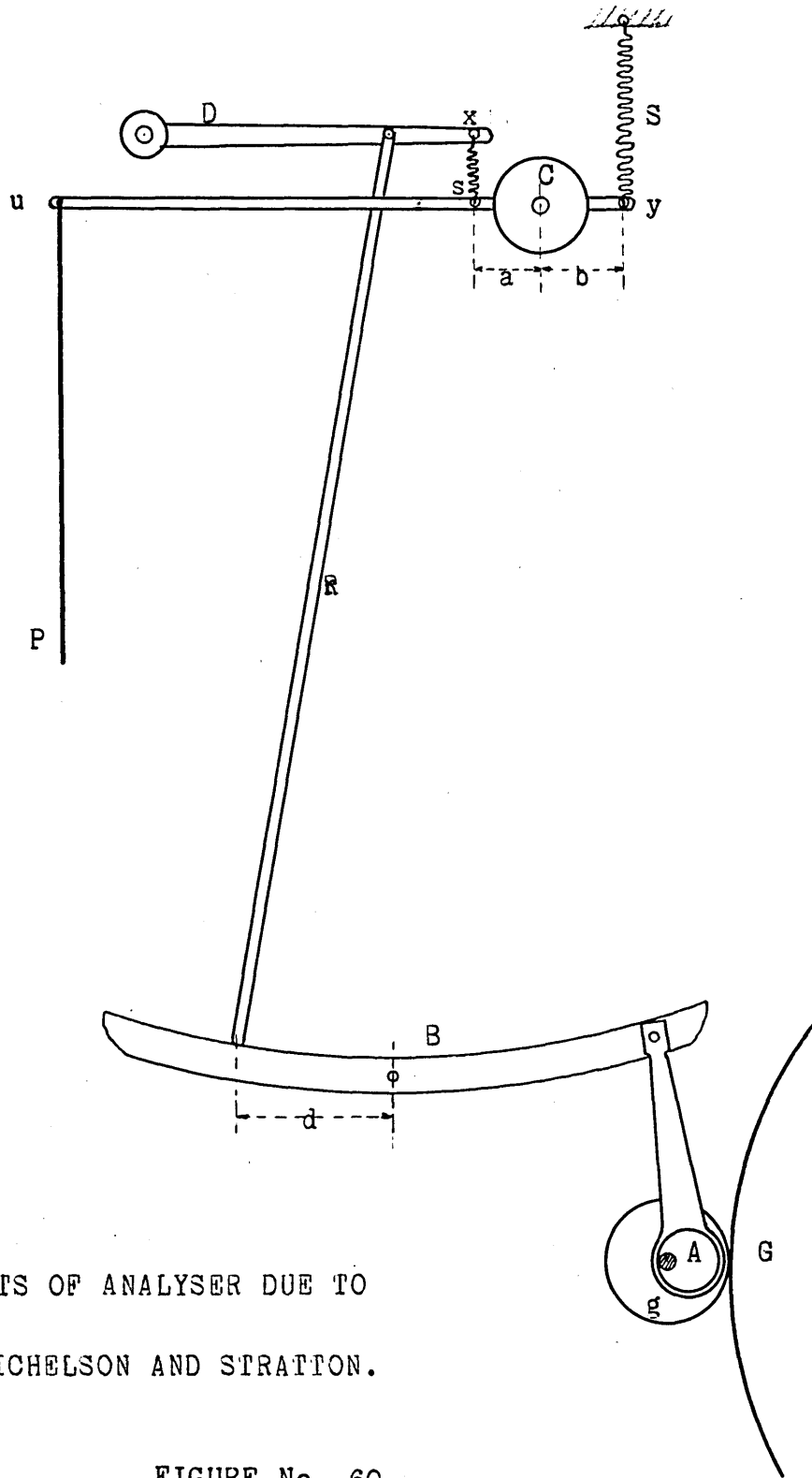
ANALYSER DUE TO MICHELSON AND STRATTON.

(A.A. Michelson and S.W. Stratton, Philosophical Magazine, Volume 45, p. 85, 1898.)

This analyser is remarkable in that it may be used as either analyser or synthesiser, and is practically the only successful synthesiser differing in principle from that originally built by Lord Kelvin.

Fig. 60 shows the elements of the mechanism. A large gear G drives another gear g , the ratio being dependant upon the order of the harmonic which the element represents. The gear g carries an eccentric A which drives the rocker arm B with simple harmonic motion. This motion is imparted by means of the radius rod R to the arm D , carrying at its extremity x a small spring s . This spring is in turn connected to a small arm of radius a , projecting from the cylinder C . The parts so far enumerated are duplicated for each harmonic component, the regular machine having eighty such elements, the gear ratios G/g being successive integers, 1, 2, 3, 4, etc.. The cylinder C has one arm of radius b connected at y through a heavy spring S to a fixed point on the frame. It will thus be seen that the cylinder C will be displaced an amount proportional to the difference between the tension of the heavy spring S and the sum of the tensions upon all (80) the little springs, s, s, s , etc. The displacement if the cylinder is imparted to a pen by means of the radius bar u and connection P . This pen draws upon a flat paper which is given translational motion by gearing to the driving gear G . If the various parts of the mechanism are then properly set the curve drawn upon the paper will be either an analysis or synthesis of the curve being investigated, depending upon the manner in which the original settings were made.

This may be shown theoretically as follows:-



ELEMENTS OF ANALYSER DUE TO
MICHELSON AND STRATTON.

FIGURE No. 60.

Let a =lever arm of small springs s .
 b =lever arm of large spring S .
 l_0 =natural length of small springs s .
 L_0 =natural length of large spring S .
 $l+x$ =stretched length of small springs.
 $L+y$ =stretched length of large spring.
 e =constant of small springs.
 E =constant of large springs.
 n =number of small springs.
 p =force due to one of small springs.
 P =force due to large spring.

Then: $p = (e/l_0)[l+x - (a/b)y]$

$$P = E/L_0[L+y]$$

$$a \sum p = bP \quad \text{Whence:} \quad y = \sum x / [n(l/L + a/b)]$$

From this it follows that the resultant motion is proportional to the algebraic sum of the components. The order of accuracy will depend upon the accuracy with which the springs follow Hooke's Law.

The radius rod R is adjustable at any distance from the fulcrum of B , and thus when the machine is operating the small springs will be given an increase in length proportional to d and the motion of A , and thus B . As the motion of B may be $\sin\theta$ or $\cos\theta$ or $\sin(\theta+\phi)$, depending upon the way in which the gears g and G are meshed (which can be adjusted), the motion of x will be $d \cdot \sin(\theta+\phi)$, and the motion of the pen will be:

$$\sum_k^n d_k \sin(k\theta + \phi_k)$$

Where k has successive integral values from 1 to n . It will at once be seen that this is a Fourier's Series of n terms; where the value of d is the coefficient of the term.

Therefore in order to operate as a synthesiser each arm R

is set along the rocker arm B an amount proportional to the coefficient of the particular term represented by that element, and the gear g is adjusted to the proper phase relation with respect to G or the fundamental. The pen will then draw the curve represented by the series. A number of such curves are shown in the section under WAVE SHAPE AND CONVERGENCE.

As an analyser the process is reversed. It has been seen that it is necessary to evaluate equations of the form:

$$b_k = 2/\pi \int_0^{2\pi} f(x) \cos.kx.d x.$$

If n is the number of elements in the analyser, α the distance between any two elements, and the distance d is made proportional to $f(n\alpha)$, the pen will then draw the curve:

$$\sum_0^m f(n\alpha) \cos.n\theta = \sum_0^m f(x) \cos(m/\pi)\theta x$$

which is proportional to b_k if $k = m\theta/\pi.\theta$ in this case being the angular deflection of gear G . Thus to obtain the analysis the lower ends of rods R are moved along B to points such that d is proportional to ordinates read from the curve to be analysed at values of x chosen so that there will be the same number of ordinates read as it is intended to use elements on the analyser. By operating the analyser with the gears all set in phase, the curve obtained is a continuous function of k , and the ordinates of this curve at integral values of k give the harmonic coefficients desired. To obtain these values the distance corresponding to $\theta = \pi$ on the curve is divided up into m equal parts, and the coefficients read as the ordinates erected at these points. Where m is the number of elements used on the analyser. A separate curve must be drawn for the sine terms and for the cosine terms, the gears being arranged all in phase with the time variable of the curve for the first case, and all 90° out of phase for the second case.

Fig. 6) shows three analyses made in this way. Curve 25 is

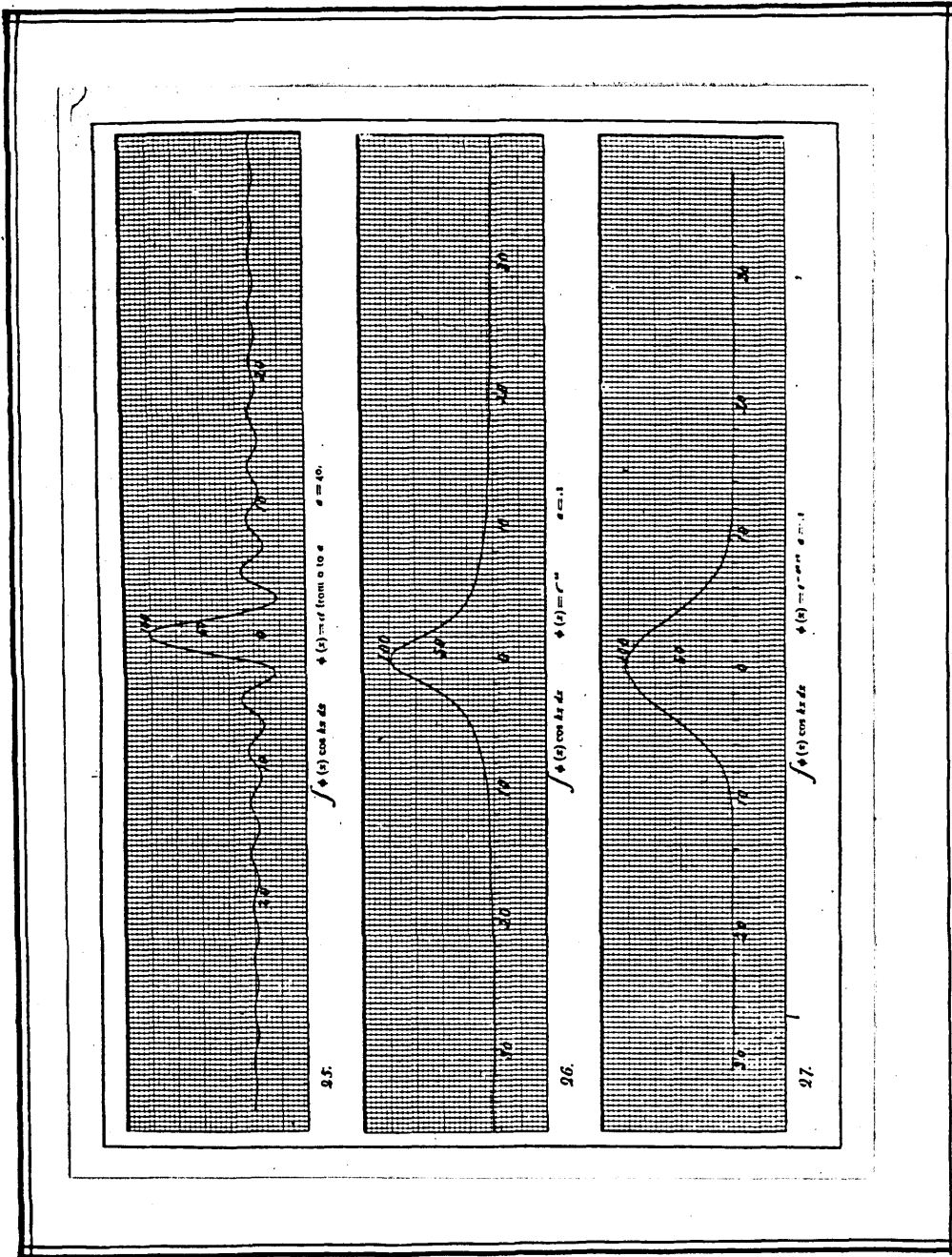


FIGURE No. 61. Analyses by Michelson and Stratton.

the approximate value of $\int \varphi(x) \cos kx dx$ when $\varphi(x) = a$ constant from 0 to a and is zero for all other values. This is one of the most difficult forms to evaluate as will be seen by reference to section on WAVE SHAPES AND CONVERGENCE, Fig. . The exact integral is $\sin ka/k$, and the accuracy of the approximation is shown in the following table, which gives the observed and calculated values of the first 20 coefficients for $a=40$

n	Observed	Calculated	Δ
0	100.0	100.0	0.0
1	65.0	64.0	1.0
2	0.0	0.0	0.0
3	-20.0	-21.0	1.0
4	0.0	0.0	0.0
5	12.5	13.0	-0.5
6	-1.5	0.0	-1.5
7	-9.0	-9.0	0.0
8	00.0	0.0	0.0
9	6.0	7.0	-1.0
10	-2.0	0.0	-2.0
11	-6.0	-6.0	0.0
12	0.0	0.0	0.0
13	4.0	5.0	-1.0
14	-2.0	0.0	-2.0
15	-4.0	-4.5	0.5
16	0.5	0.0	0.5
17	3.5	4.0	-0.5
18	-1.0	0.0	-1.0
19	-3.5	-3.0	-0.5
20	0.0	0.0	0.0

The average error is only 0.65 percent of the value of the greatest term.

Curve 27 gives the analysis of $\varphi(x) = e^{-a^2 x^2}$

Curve 26 gives the analysis of $\varphi(x) = e^{-ax}$

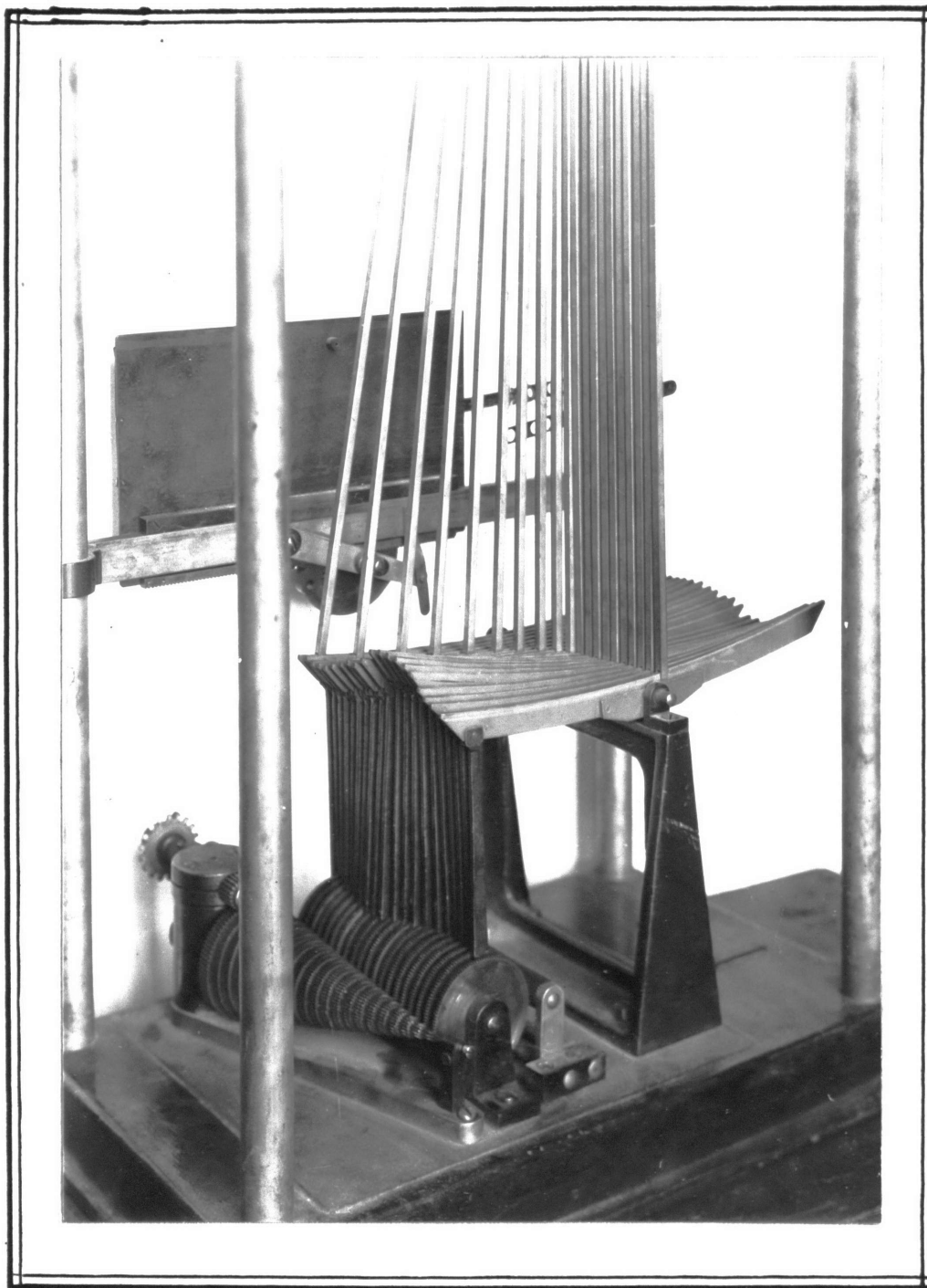


FIGURE No. 62. Details of Michelson and Stratton
Analyser-Synthesizer.

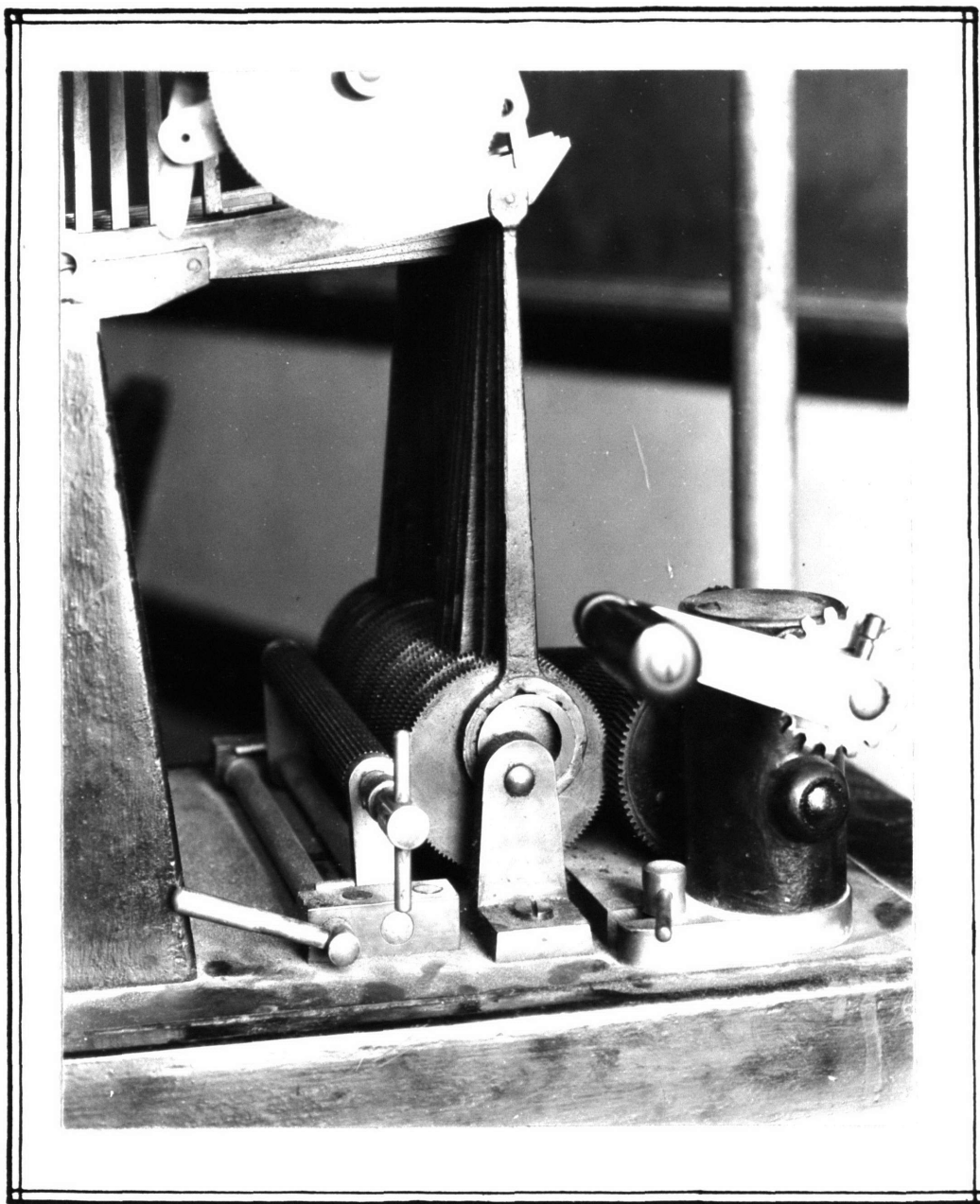


FIGURE No. 63. Details of Michelson and Stratton
Analyser-Synthesizer.

Figs. 64 and 65 give the general appearance of the analyser, one being the eighty element type and the other a smaller and thus more approximate one. Michelson and Stratton state that after the experience with building the eighty element machine they believe it would be perfectly possible to construct one with any number of elements; several hundred or even a thousand. They also suggest that its use is not limited to the summation of trigonometric series, since the rocker arms may be operated by cams or templates instead of eccentrics, and so enable any function to be studied. In the first experimental model the sinusoidal motion was obtained from metal templates cut out in sine waves, and worked satisfactorily, but of course was not as convenient or accurate as the method later developed.

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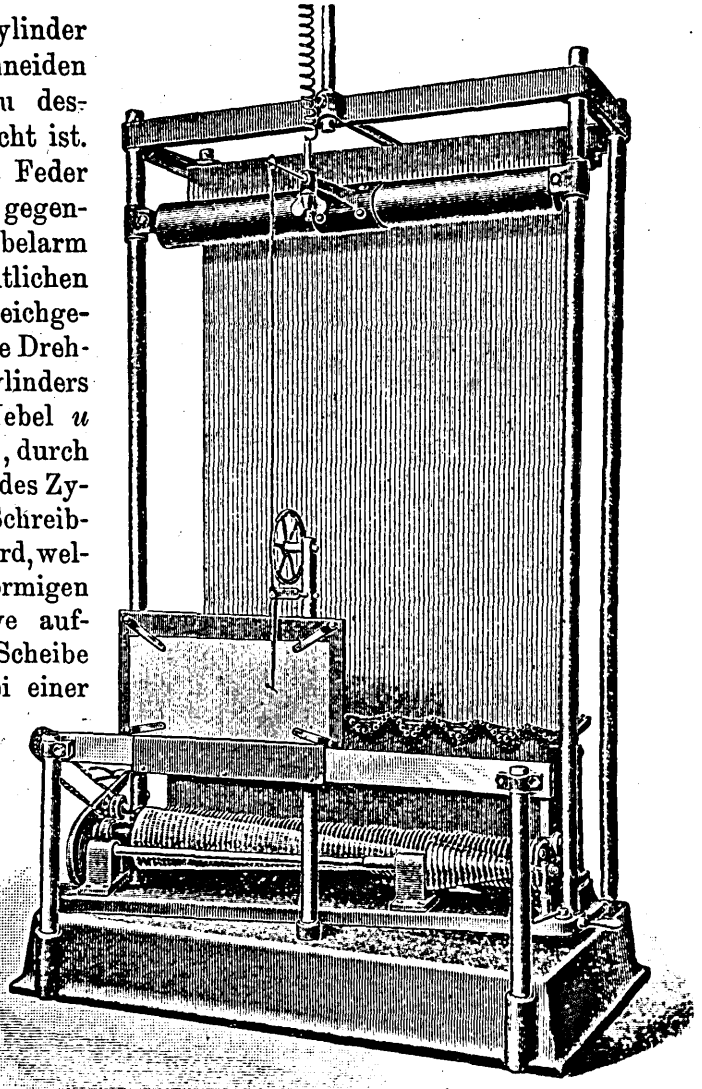


FIGURE No. 64. 80 Element Michelson and Stratton
 Analyser-Synthesizer.

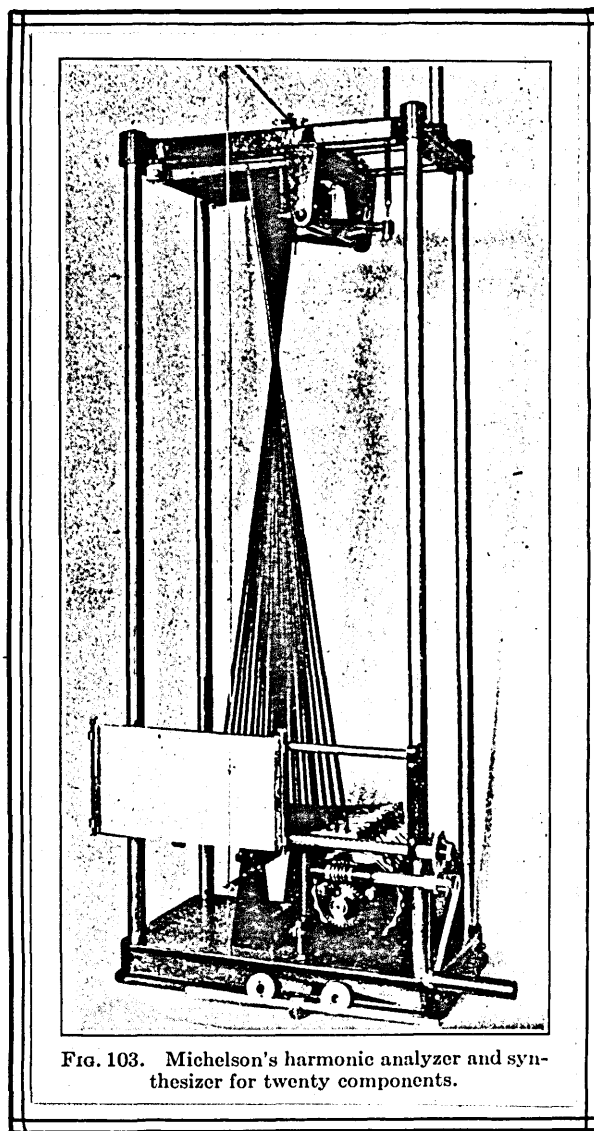


FIGURE No. 65. 20 Element Michelson and Stratton Analyser-Synthesizer with Cam Wheels instead of Eccentrics.

ANALYSER DUE TO ROWE.

(G.H.Rowe, Harmonic Analyser, Electrical World, Vol. 45, p. 587, 1905)

The construction of the analyser is shown in Figs. 66 and 67, while its general appearance is shown in Fig. 46. The operation is very similar to that of Le Conte, one end of the planimeter being given harmonic motion of the proper frequency while the board upon which the planimeter wheel travels follows the ordinates of the curve.

The instrument consists of two platforms 1 and 11, moving at right angles to each other. On the upper platform is placed the planimeter wheel 2, and pole 4, the tracing point 48 being given a simple harmonic motion by means of the pin and slit arrangement shown in Fig. 67 at 45 etc. Every point of the upper table is made to follow the curve 13, 14, 8, 15, 16 by means of the handles 12, and the cross hairs 8. Motion is transmitted from table 11 by means of string 21 attached to stud 23 on lower platform, making complete turn around pulley 24 and thence to weight 25. The pulley may be changed to vary the range of the instrument. From disc 29 motion is communicated by friction to the wheel 36, which in turn gives the point 49 simple harmonic motion. The number of complete oscillations of the point 49 is determined by the diameter of the pulley 24, the diameter of the wheel 36 and the distance of wheel 36 from the center of disc 29. The proper distance of the point of contact of the wheel and disc is in general:

$$r = n\pi d' / 2l = \text{constant} / \text{Wave Length.}$$

n = order of harmonic, $d = \text{and } d'$ = the radii of wheels 36 and 24 and l the wave length of the curve to be analysed. Rowe states that about 2 hours were required for analysing an A.C. wave up to the fifteenth harmonic.

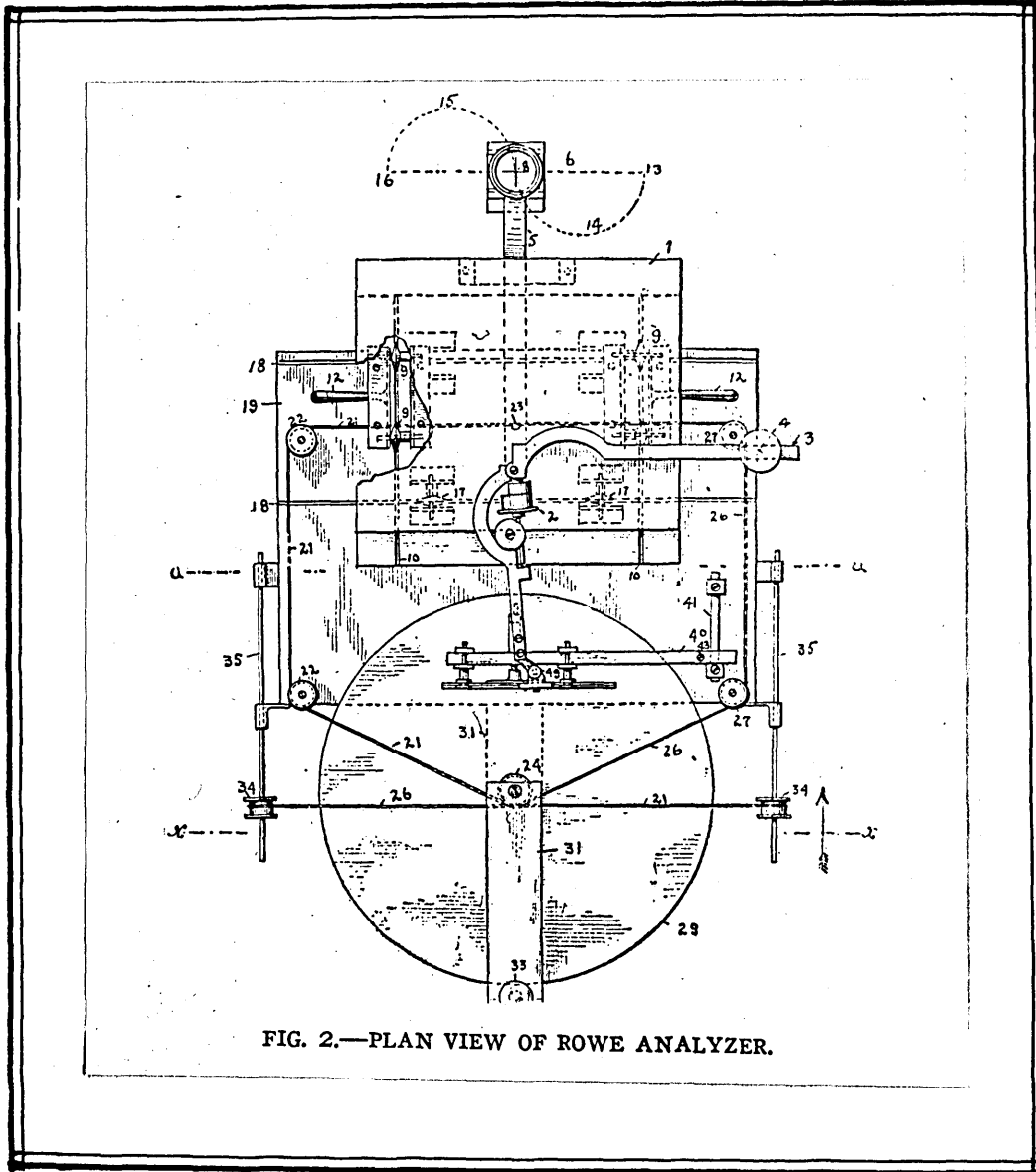


FIG. 2.—PLAN VIEW OF ROWE ANALYZER.

FIGURE No. 66. Rowe Harmonic Analyser.

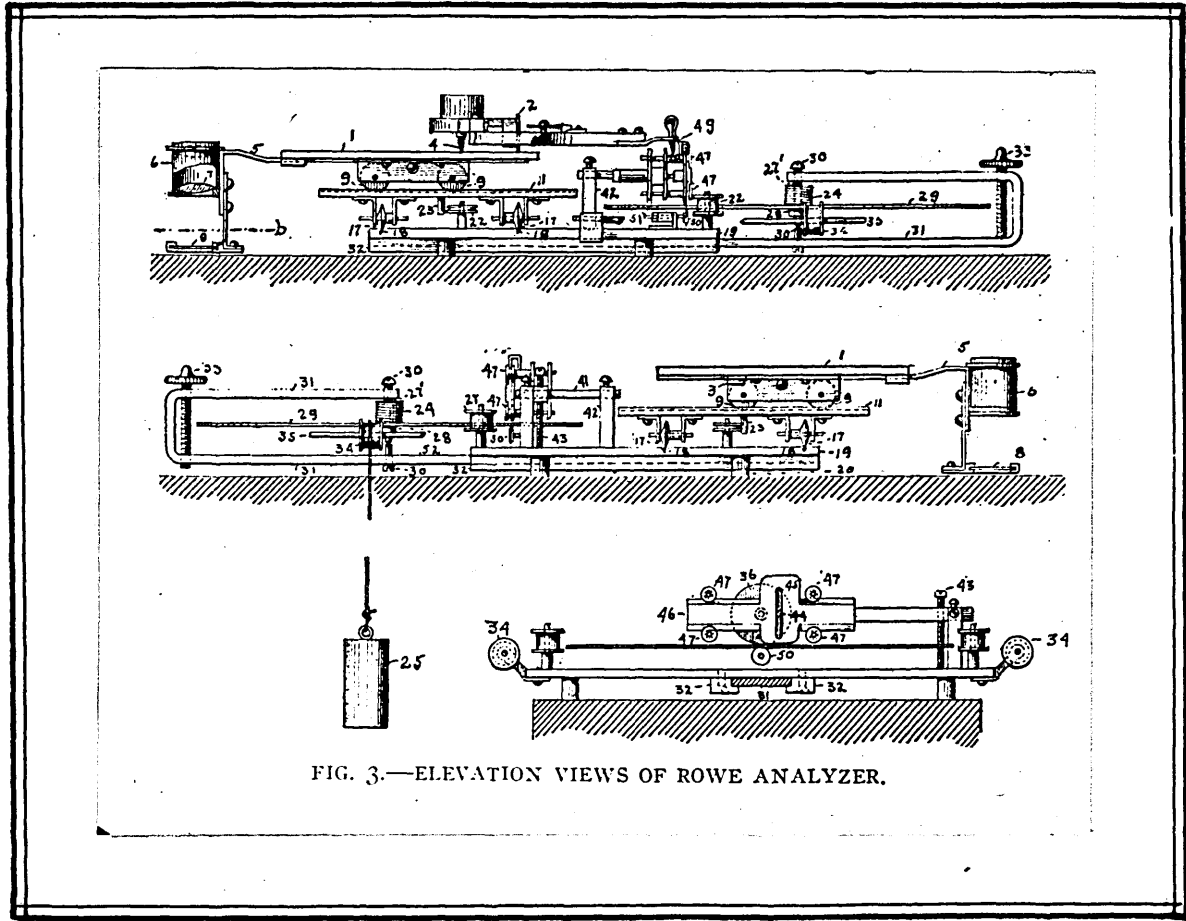


FIG. 3.—ELEVATION VIEWS OF ROWE ANALYZER.

FIGURE 67. Elevation drawings of Rowe Analyzer.

ANALYSER DUE TO TERADA.

(E.Orlich,Aufnahme und Analyse von Wechselstromkurven,p.109,
Braunschweig,1906.)

It is not certain just when this device was first published but it does not appear to have been mentioned before the above date.It is very simple in theory,but rather indirect in action,especially considering the amount of mechanical linkages necessary.

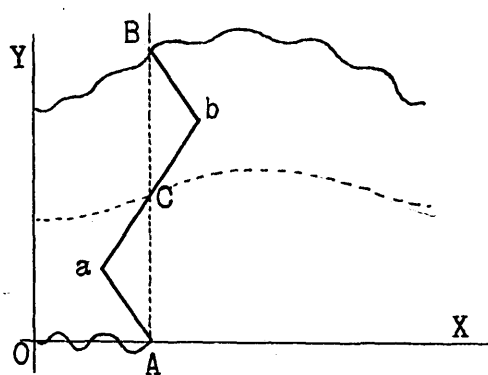


FIGURE 68.

It consists primarily of a three armed link ACB;jointed at a and b,pivoted at C and with a stylus at B and a pen at A. The three points A,C and B are constrained so that they can only move in a direction parallel to the Y axis of the curve being analysed.

The curve is placed so that it can be traced by B and a derived curve is drawn by A,the point C being arranged to follow a pure sine wave of any desired wave length.If the point C follows a wave similar to the fundamental of the curve being analysed,then the curve drawn by A will contain no fundamental but only harmonics.This curve is then placed where the original curve was before,and a second trace of the derived curve made with the point C following a sine wave of the lowest harmonic frequency.The second derived curve drawn by A will then be the original curve with the fundamental and first higher harmonic subtracted.This is repeated until A draws practically a straight line,when the various curves may be inspected for the different harmonics,and the ordinates and wave lengths,or periods,measured.It may also be used as a synthesiser by reversing the process and tracing a curve at A with the point C following the frequency to be added,when B will draw the sum.

By sufficient repetitions any number of harmonics may be combined. The chief use of the analyser would appear to be the separation of one high frequency term such as tooth ripple in alternators.

Mathematically the curve to be analysed is expressed as:

$$y = a_0 + a_1 \sin \omega t + \sum_{k=2}^{\infty} a_k \sin k \omega t$$

Then C must follow a curve given by : $y' = a_0/2 + a_1/2 \cdot \sin \omega t$

Then A will draw a curve whose ordinates $y'' = 2y' - y$, or:

$$y'' = -\sum_{k=2}^{\infty} a_k \sin k \omega t$$

For greater convenience the lengths AC and CB are made adjustable, so that if:

$$BC:CA = 1:\gamma$$

Then: $y' = (\gamma/\gamma+1)(a_0 + a_1 \sin \omega t)$

And: $\gamma(y - y') = (y' - y'')$

Whence: $y'' = (\gamma+1)y' - \gamma y = -\gamma \sum_{k=2}^{\infty} a_k \sin k \omega t$

The actual machine is shown in Fig. 59, Page 1. The two points A and B run in guides over the platens for holding the paper. The sinusoidal motion is given to the point C by means of a cone W which is rotated by friction against the track R_1 . The track R_1 may be adjusted to give any desired number of rotations of the cone for one trace of curve. The only advantage that this machine seems to have, to offset the many obvious disadvantages, is that a definite base of one period of the curve being analysed is not necessary.

ANALYSER DUE TO MADER.

(Mader, Elektrotech. Zeit. Vol. 36, 1909; Also A. Schreiber, Phys. Zeit., Vol. 11, p. 354, 1910; Also Horsburgh, Modern Instruments of Calculation, p. 231, 1914.)

The appearance of Mader's machine is shown in Fig. Page 44, and the schematic arrangement in Fig. 69. The construction is based on Clifford's graphical method. The integrals evaluated are of the form of (2) and (3).

The instrument consists of two carriages, one over the other. The lower one is constrained to move in a straight line parallel to the Y-axis, and carries an angle lever PFQ. F is fixed to the lower carriage and so only moves in the Y direction. The curve being analyzed is traced with a stylus at P. Q drives the upper carriage through a slot QK, and thus operates the rack MN fixed along the edge of the upper carriage. Meshing with the rack is a gear wheel W, pivoted on the lower carriage, and thus its rotary displacement is a measure of the linear displacement between the two carriages. The gear is provided with two depressions at equal radii and 90° apart, in one of which is placed the tracing point of any ordinary form of planimeter. Then by tracing the curve the planimeter traces an area which gives the value of the coefficient desired, one of the depressions in W giving the sine terms and the other the cosine terms. The order of the harmonic is determined by the size of the gear wheel W. It is difficult to see at once why the planimeter should give the desired results; except that the vertical displacements depend largely on the ordinates, while the abscissae enter as a sine or cosine function due to the angularity of the rod PFQ.

In operation the middle point of the curve to be analysed is placed so that it is under the bottom of line AF, and the Y-axis parallel to AF. The length of the arm PF is adjusted

so that when the tracing point P is at O the radius at the end of which the planimeter rests is parallel to the X-axis.

If the coordinates of P be x, y, z :

$$\begin{aligned} x &= a - m \cdot \cos \psi & \text{Where: } a &= \text{length OA} & m &= \text{length FP} \\ y &= z - m \cdot \sin \psi & z &= \text{length FA} & \psi &= \angle \text{ between FP and X-axis.} \end{aligned}$$

If (ξ, η) be the coordinates of T and $(-c, \eta_0)$ the initial coordinates of C', the center of gear W, and z_0 initial value of z :

$$\begin{aligned} \xi &= -(c + r \cdot \cos \phi) & \text{Where: } r &= \text{length C'T.} \\ \eta &= \eta_0 + r \cdot \sin \phi + z - z_0 & \phi &= \text{angle turned through by gear W.} \\ & & l &= \text{length of arm FQ.} \\ & & \psi_0 &= \text{initial value of } \psi \\ l(\cos \psi - \cos \psi_0) &= R\phi & R &= \text{radius of gear W.} \end{aligned}$$

$$\text{Thus: } x - x_0 = -m(\cos \psi - \cos \psi_0)$$

$$\text{And: } x = -m(\cos \psi - \cos \psi_0) \quad \text{Since } x_0 = 0$$

$$\text{Whence: } R\phi = -lx/m$$

$$\begin{aligned} \text{The area traced by T is } \int (\eta - \eta_0) d\xi &= \int (r \cdot \sin \phi + z - z_0) d\xi \\ &= \int (r \cdot \sin \phi + z) d\xi \quad \text{Since } \int z_0 d\xi = 0 \text{ for closed curve.} \end{aligned}$$

Substituting from above the last equation becomes:

$$\begin{aligned} \int (r \cdot \sin \phi + z) r \cdot \sin \phi \cdot d\phi &= \int \left\{ r \cdot \sin\left(-\frac{lx}{Rm}\right) + y + m \cdot \sin \psi \right\} r \cdot \sin\left(-\frac{lx}{Rm}\right) \left(-\frac{l}{Rm}\right) dx \\ &= \frac{rl}{Rm} \int y \cdot \sin\left(\frac{lx}{Rm}\right) dx - \frac{rl}{Rm} \int [r \cdot \sin\left(\frac{lx}{Rm}\right) - m \cdot \sin \psi] \sin\left(\frac{lx}{Rm}\right) dx \end{aligned}$$

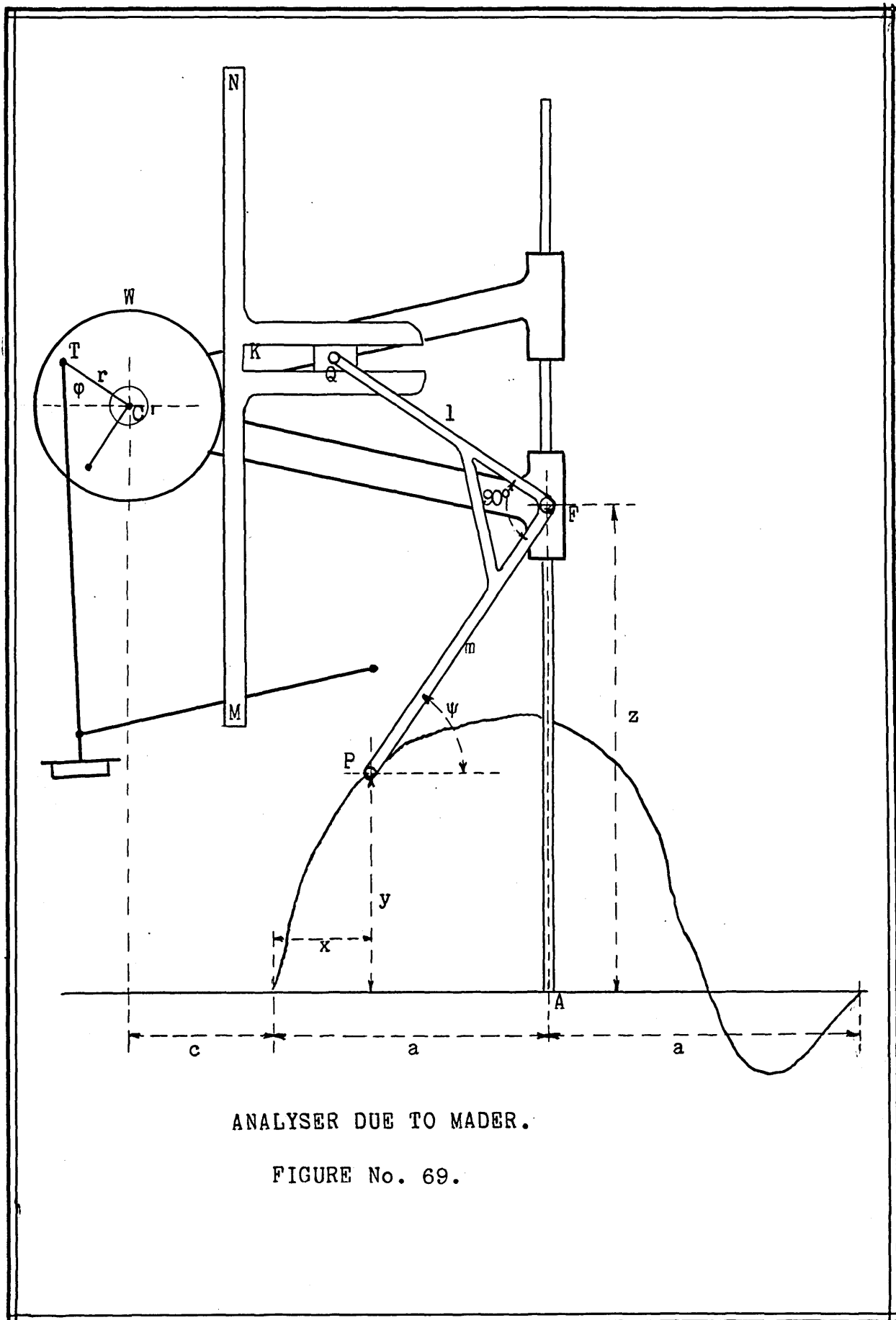
Since ψ can be expressed as a function of x only, the second \int vanishes when taken around a closed curve. Thus area traced by T:

$$= \frac{rl}{Rm} \int y \cdot \sin\left(\frac{lx}{Rm}\right) dx = \frac{rl}{Rm} \int_0^{2\pi} y \cdot \sin\left(\frac{lx}{Rm}\right) dx$$

Since $y=0$ along the X-axis. Note that it is necessary to trace curve as a closed curve.

Thus if the radius of the disc be designed so that $l/Rm = n\pi/a$ the planimeter records the value of the n -th sine term coefficient. By similar reasoning it may be shown that if the position of the radius of the gear W in which the planimeter is placed is started at 90° to its former position, then the reading of the planimeter will give the n -th cosine term coefficient.

It will be noted that the theory is similar to that of the Yule and Bush analysers, but the arrangement of the double carriage eliminates the area of the curve from the final integral and so the necessity for a second trace of the curve to subtract its area is eliminated.



ANALYSER DUE TO BOUCHEROT.

(Morin. Les Appareils d'Integration. p179, 1913.)

(E.M. Horsburgh, Modern Instruments of Calculation, p.239, 1914.)

The exact date of origination of this machine is not known. It appears to have been first mentioned in 1913. In operation it is almost identical with that of Yule, although the mechanism differs slightly. It is probable that Yule was the originator of this form of analyser.

It consists of a vertical rod or frame CD parallel to the Y-axis upon which the mechanism slides. This mechanism involves a rack ST; in which meshes the gear W; whose center slides along the rod AB. Fixed to the gear is an arm PQ, of length l . A second arm has one end pivoted to PQ at Q, and the other end sliding upon the rod AB at R. The tracer point of a planimeter rests at R, the pole being fixed to the table upon which the apparatus is placed. In operation the center of the gear W, i.e. the point P, is caused to trace the curve, which gives the point R a vertical motion equal to the ordinates and a horizontal motion depending upon the trigonometrical functions of the angle through which W turns. The order of the harmonic determined is equal to the number of revolutions made by W for one trace of a complete period. The curve must be traced as a closed loop, i.e. the final trace made back along the X-axis.

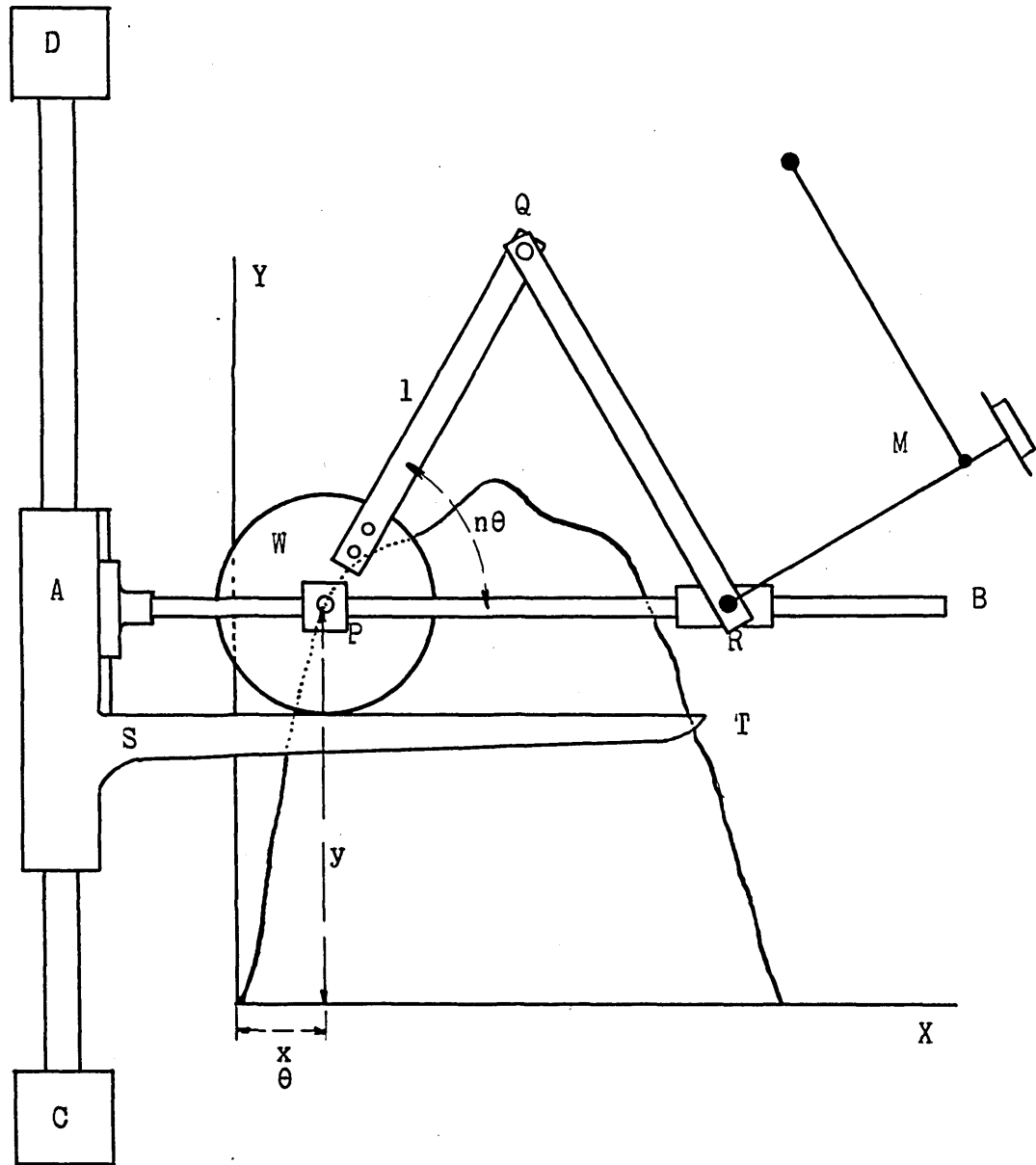
As the coordinates of R are $(\theta + 2l \cdot \cos. n\theta, y)$, the area traced by the planimeter is :

$$\int_0^{2\pi} y \cdot d(\theta + 2l \cdot \cos. n\theta) =$$

$$= \int_0^{2\pi} y \cdot d\theta - 2l n \int_0^{2\pi} y \cdot \sin. n\theta \cdot d\theta$$

If the mean ordinate is zero then the planimeter gives a reading proportional to the coefficient sought; from equation (2); (3).

If the mean ordinate is not zero the area of the curve must be obtained by planimeter and the above integral corrected.



ANALYSER DUE TO BOUCHEROT.

FIGURE No. 70.

*Elec. Engg
Theriac case*

*Reference to
pt II*



ANALYSER DUE TO CHUBB..

(L.W.Chubb, Electric Journal, Vol.11, p..91; 1914..)

This analyser is the most recently developed commercial machine and is manufactured and sold by the Westinghouse Electric and Mfg.Co., of East Pittsburgh, Pa. It was developed particularly for use in connection with polar oscillograms, which the Westinghouse Co. has used for some time in their research division as having certain peculiarities particularly suitable to their requirements. (See Electric Journal Vol.11, pt.263.)

An example of a polar oscillogram is shown in Fig. 71. For use in the analyser a print is made of this, stapled to a piece of Bristol Board, and a template of the curve to be analysed cut out by hand. One of these is shown in Fig. 73. The template is then placed upon the platen of the analyser as shown in Fig. 72. Details of the actual machine are shown in Figs. 74 and 75. The platen upon which the template is mounted is arranged by the gearing so that when the crank is turned it not only revolves about its center, but also traverses back and forth with simple harmonic motion. A bar B has a small wheel at E which rests against the template and is caused to follow its contour during the revolution by means of a small spring not shown. A planimeter is carried by the end of the bar B, its pole resting upon the table of the analyser. The tracing point of the planimeter is thus given a motion horizontally proportional to the radius of the template and thus the curve ordinate. Its motion vertically is proportional to the simple harmonic motion of translation of the whole template. This will therefore be proportional to the sine or cosine of some angle. If the gears driving the device are adjusted so that the platen makes k oscillations back and forth for one complete rotation, then the planimeter will

$$\text{read: } \int y \cdot \begin{bmatrix} \cos \\ \sin \end{bmatrix} k\theta \cdot d\theta$$

which evaluates the coefficient of the k-th harmonic.

More fully: Let:

θ =angular position of template in radians.

R =crank pin radius, driving carriage of platen=1/2 platen travel vertically, or from front to back.

x and y = coordinates of planimeter tracing point.

n =number of oscillations of platen carriage for one turn of the template.

S_a =Area of curve traced by the planimeter.

Then:

$$x=f(\theta)=a_1 \sin\theta + a_2 \sin 2\theta + \dots + b_1 \cos\theta + b_2 \cos 2\theta + \dots$$

$$y=R \sin[n\theta - \pi/2] = -R \cos n\theta$$

$$dy/d\theta = nR \cdot \sin n\theta$$

$$S_a = \int_0^{2\pi} x \cdot dy = \int_0^{2\pi} x \cdot nR \cdot \sin n\theta \cdot d\theta$$

When the value of x is substituted the resulting terms of the form :

$$(nRa_k) \int \sin k\theta \sin n\theta \cdot d\theta$$

$$(nRb_k) \int \cos k\theta \sin n\theta \cdot d\theta$$

are all zero except when $k=n$, and the equation reduces to :

$$S_a = nRa_n \int_0^{2\pi} \sin^2 n\theta \cdot d\theta = nRa_n \pi$$

Or: $a_n = S_a / \pi Rn$

Which gives the value of the sine coefficient of the n-th harmonic. To determine the cosine coefficient the platen is started in a different position. Fig. 72 shows the position of the platen for the ~~above~~^{cosine} derivation; the starting point being in the middle of its translational motion. For the ~~cosine~~^{sine} components it is started at the extreme forward position, when similar theory shows that :

$$b_n = S_b / \pi Rn$$

Since the parts must move slowly for accuracy the crank drives the mechanism through a worm reduction, and a good deal of cranking is required to determine the higher harmonics. The Westinghouse Co. have therefore arranged their own machine to be motor driven, a contact on the platen starting a small motor which drives the crank shaft through gears or chain, and the contact is turned off just before completion of the analysis. The operator thus adjusts the machine and starts it, when the motor picks up and he may go away and do other work. Upon return to the analysis the last adjustment is made by hand in a moment, the planimeter read, and then the gears changed and adjustments made for starting another component.

The machine may be made to draw curves as well by placing a pencil where the planimeter point rested for analysis. Thus if the template is the magnetizing current of a transformer, the harmonic motion of the carriage may be considered as the values of flux and hysteresis curves of the iron drawn directly from the oscillogram. Also by using a spiral template, shown in foreground of Fig. 75, the pencil will draw harmonic waves of any frequency for which the gears are set, and the various components of an analysis may be plotted in cartesian form, with the proper phase relations and amplitudes by adjusting the various parts of the machine.

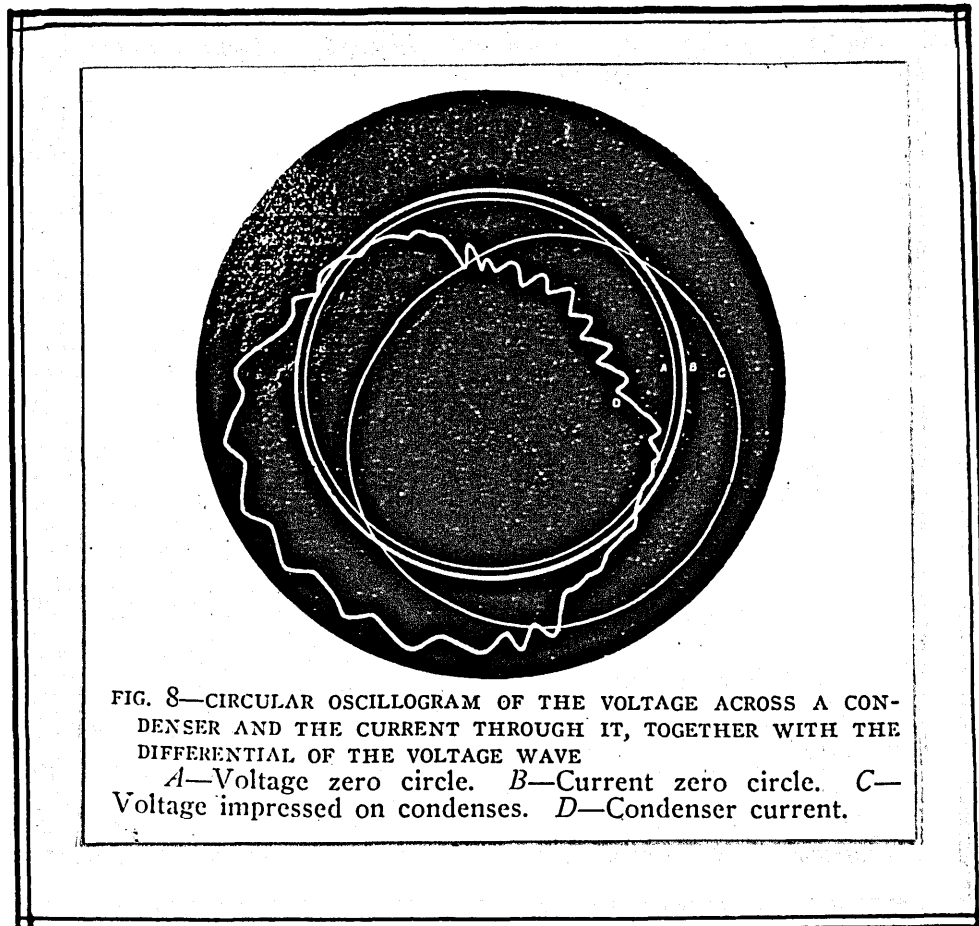
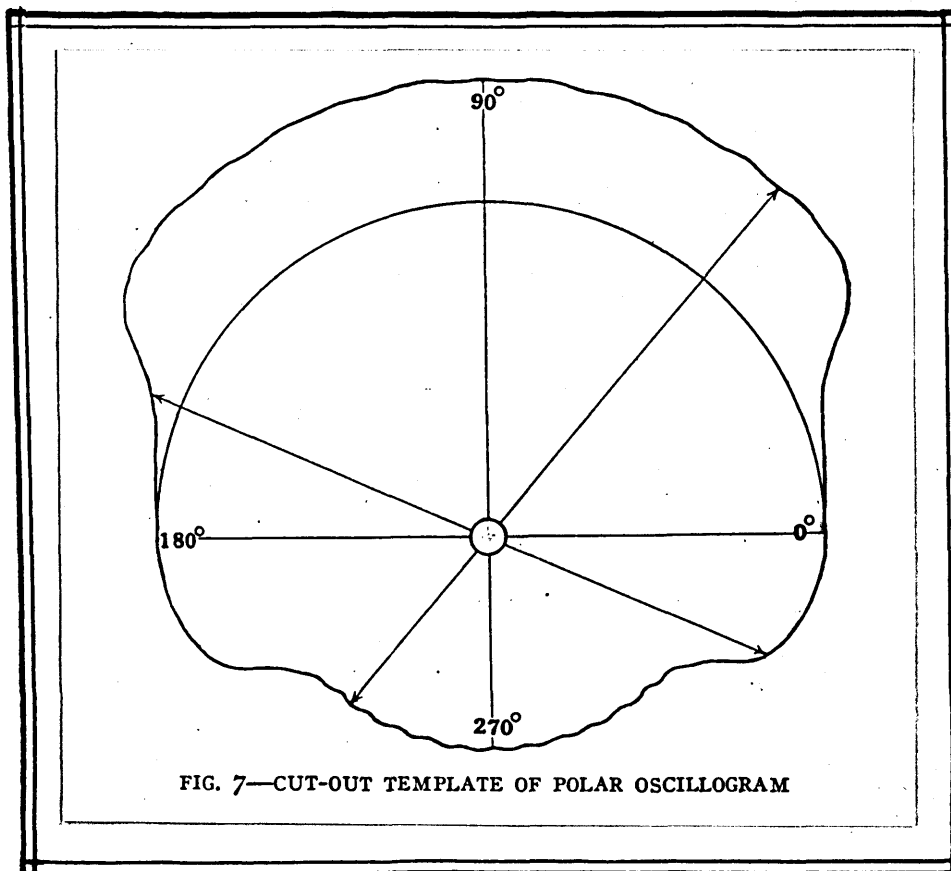
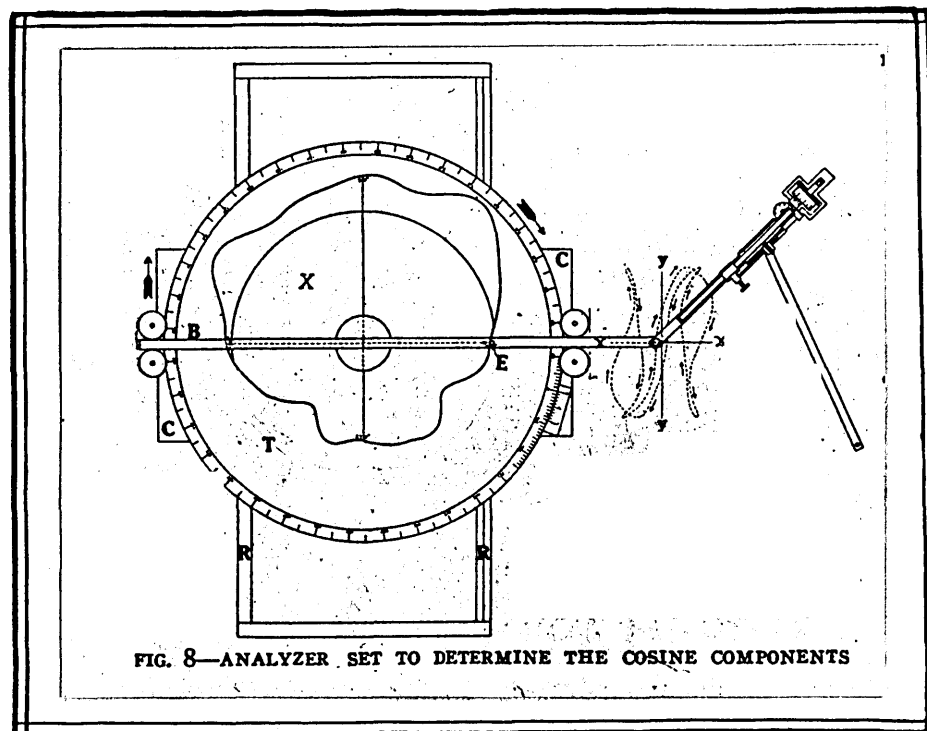


FIGURE No. 71. Polar Oscillogram.



FIGURES 72 and 73. Details of Operation of Chubb-Westinghouse Polar Analyser.

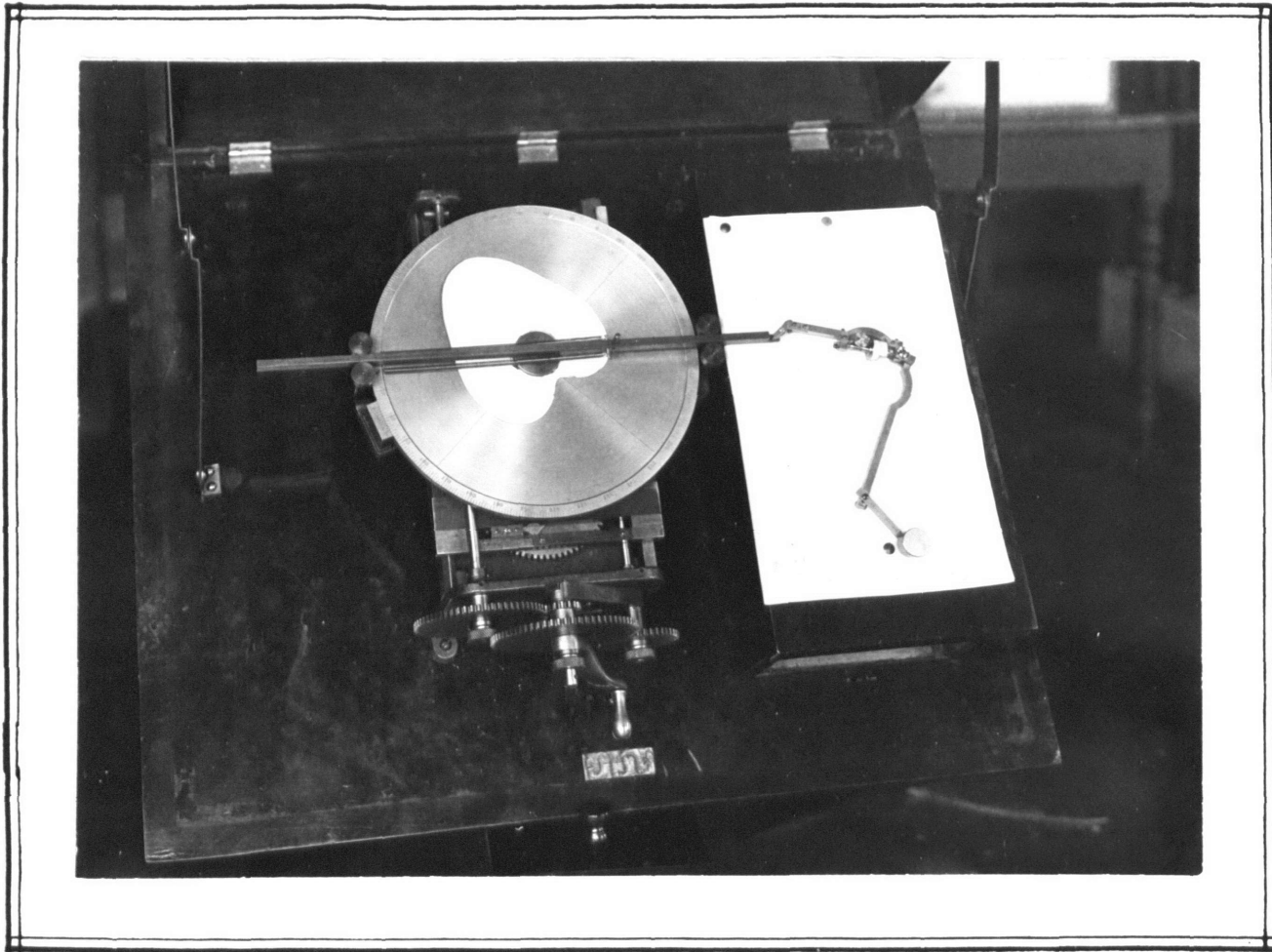


FIGURE No. 74. Chubb-Westinghouse Analyser in Operation.

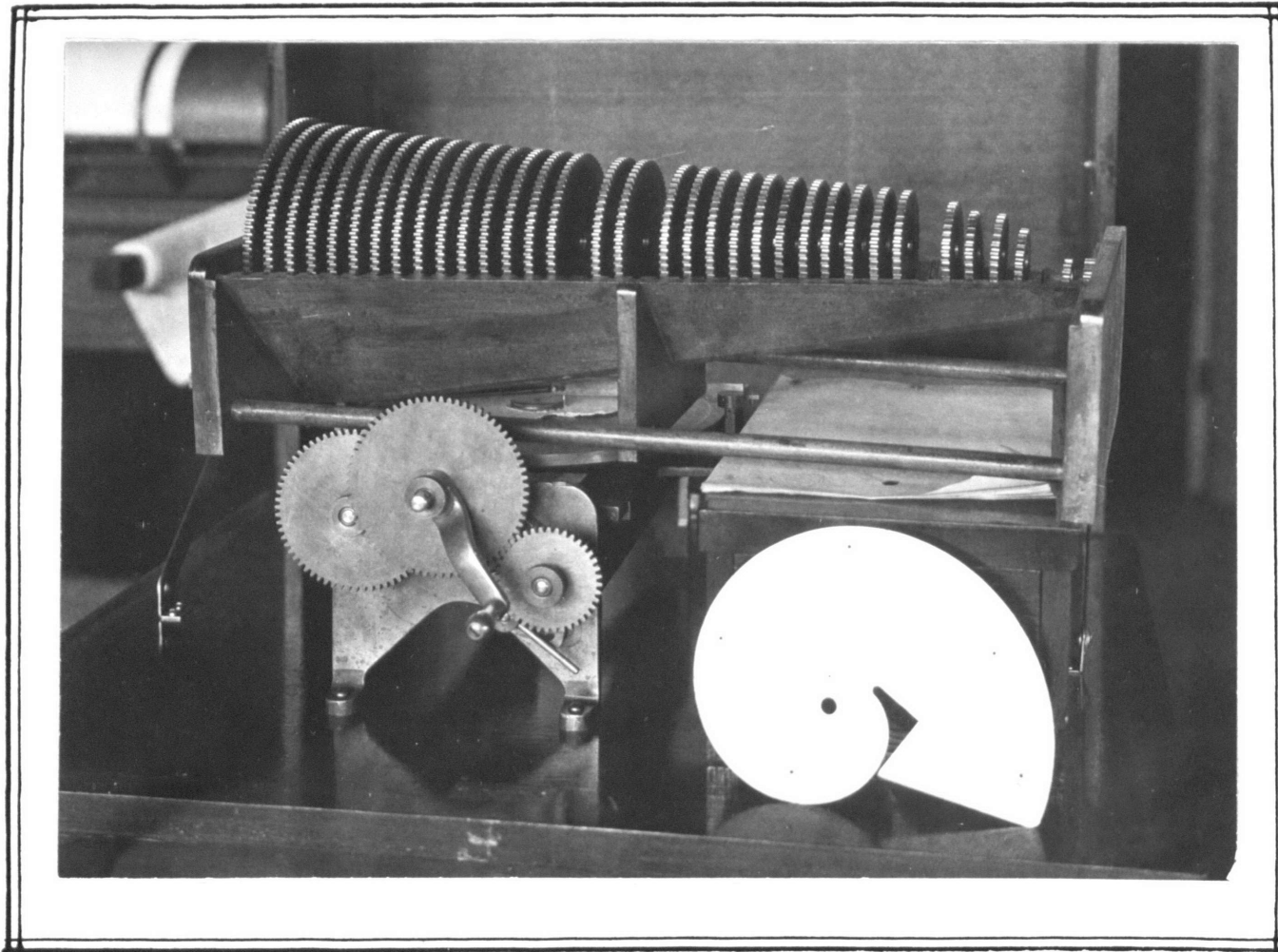


FIGURE No. 75. Chubb-Westinghouse Analyser Showing Gears.

ANALYSER DUE TO BUSH.

(V. Bush, Jour. Amer. Inst. Electrical Engrs. Vol. 39, p. 903, 1920).

This analyser utilizes a planimeter which is caused to evaluate a derived area in a manner somewhat similar to that of Yule and Boucherot, but the mechanism required has been so much simplified, and the construction is so simple, that it is worthy of special comment as being a definite step towards the ideal analyser.

Figs. 76 and 77 illustrate its operation; Fig. 79 shows the actual analyser arranged for operation, and Fig. 78 shows the series of discs required for the determination of various harmonic components.

The mechanism consists of a celluloid disc with a grooved edge, divided in degrees, and provided with an indentation for carrying the tracing point of a planimeter. A string is passed once around the disc, and the ends fastened to the two ends of the table upon which it is placed. The curve is then traced with the center of the disc, returning along the X-axis to the origin. The string must be nearly parallel to the X-axis, and must be fastened at points far enough away so that it will stretch enough to move from the X-axis to the maximum ordinate. The points of fastening should also be approximately equi-distant from the mid-ordinate so that there will not be unequal stretch of the string on the two sides thus giving an angular twist to the celluloid disc. From this it will be seen that the tracer point of the planimeter follows a curve whose ordinates are proportional to the ordinates of the original curve modified by the projection of the radius of the disc. The disc will move vertically without rotating, but will rotate when moved horizontally. The order of the harmonic determined will depend upon the number of revolutions of the disc for the trace of one period of the curve.

If the center of the disc traces the curve OEDO, the planimeter will trace some curve FHGKF. If the coordinates of any point in the two curves are (x, y) and (u, v) , respectively, then:

$$u = x + (d/2)\cos[2x/d + \beta] \quad v = y + (d/2)\sin[2x/d + \beta]$$

The area measured by the planimeter will be: $\int v \cdot du$

Along the return path DO, $y=0$, and so the area will be:

$$\int v \cdot du = \int [(d/2)\sin(2x/d + \beta)\{dx - \sin(2x/d + \beta)dx\}] - \\ - \int \{y + d/2 \cdot \sin(2x/d + \beta)\}\{dx - \sin(2x/d + \beta)dx\}$$

$$\text{Whence: } \int v \cdot du = -\int y \cdot dx + \int y \cdot \sin(2x/d + \beta)dx$$

If the disc be started so that $\beta=0$ and the dimensions arranged so that $2/d=\omega$, then the integral reduces to:

$$\int v \cdot du = -\int y \cdot dx + \int y \cdot \sin(\omega x)dx.$$

the second term of which is the coefficient of the sine component of the harmonic corresponding to ω . (From (3).)

Further if the length of the period of the curve is P , then the diameter of the disc should be $P/n\pi$, in which case the second term of the integral read by the planimeter will give the sine coefficient of the n -th harmonic. If the disc be started with $\beta=90^\circ$ then:

$$\int v \cdot du = -\int y \cdot dx + \int y \cdot \sin(\omega x + 90^\circ)dx = \\ = -\int y \cdot dx + \int y \cdot \cos(\omega x)dx$$

from which it is seen that the second term gives the cosine coefficient.

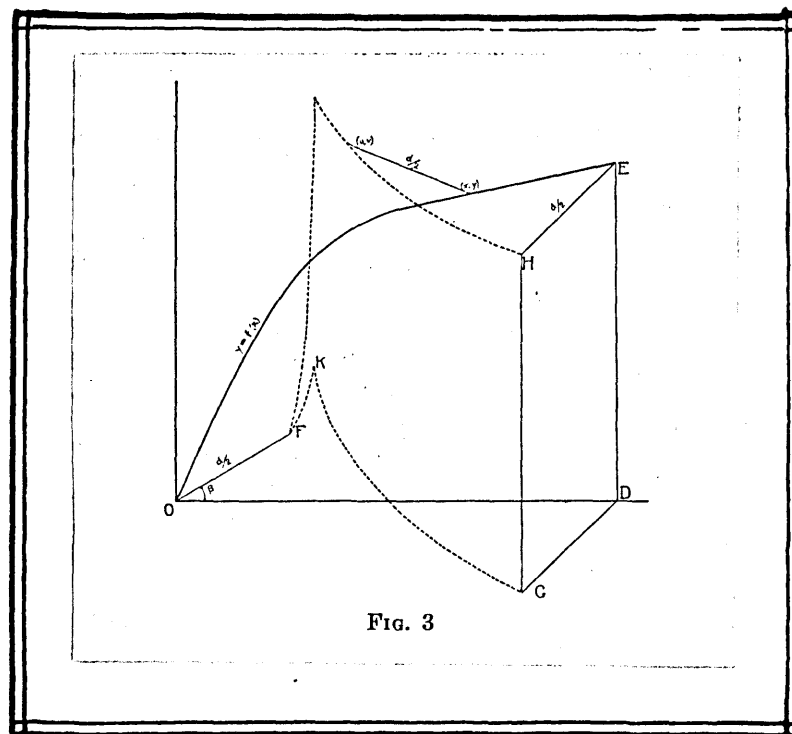
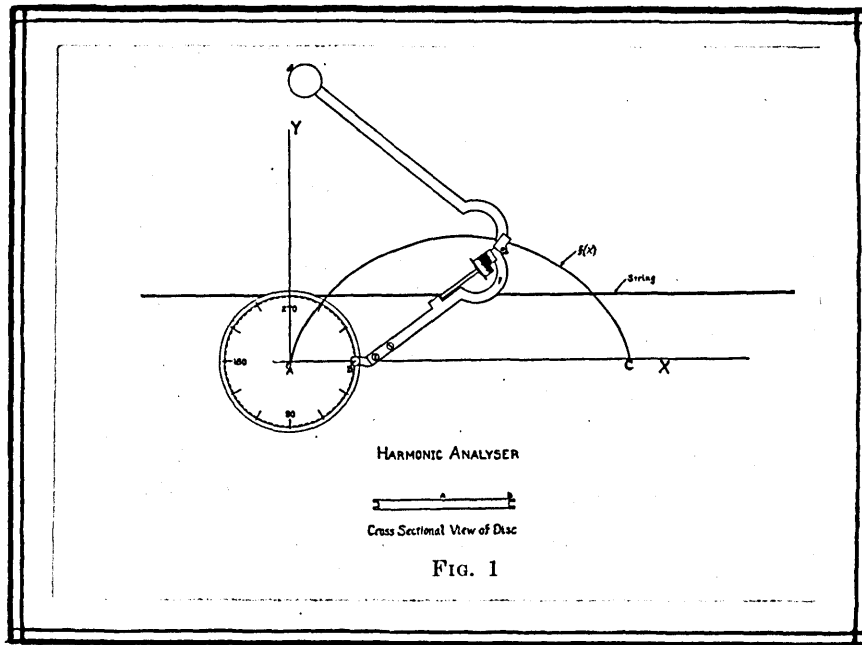
In order to be useful the first term $\int y \cdot dx$ must be eliminated. This may be accomplished in several ways. If the curve be irregular and contains even harmonics as well as odd it is usually necessary to determine the constant term, which corresponds to the mean ordinate. If a line be drawn through the mean ordinate parallel to the X-axis be used for the return trace of the

disc, then the areas on each side of this line are equal and the value of $\int y \cdot dx$ becomes zero, since the planimeter has taken one as positive and the other as negative.

If the value of the mean ordinate has not been determined the area of the curve may be measured separately and subtracted from the planimeter reading. If many harmonics are determined the same value will suffice for all. If only one harmonic is investigated it may be easier to remove the planimeter from the disc and without changing the setting retrace the curve with the planimeter tracer point in the opposite direction to that previously followed. Then the final reading will have the area of the curve subtracted. This will always be necessary if the curve being investigated is not a periodic one.

If the curve is one in which the even harmonics and constant term are absent, then it is only necessary to trace one half period of curve, but the area $\int y \cdot dx$ must then be subtracted by one of the last mentioned methods. If the whole period is traced, however, then $\int y \cdot dx$ becomes zero, and the planimeter reading gives the coefficient direct.

One disc may be used to obtain different harmonics from different curves. Thus the disc which would give the fundamental with a curve having the length of period P ; would give the second harmonic for a curve of length $2P$ etc. The limitations of the method are due to the size of the discs. A disc for the fundamental with a curve of 16" period length would be 5.09" in diameter. The diameter for the 11th harmonic would then be 0.463", which is probably about as small as could be used with accuracy, while a curve larger than 16" base could not conveniently be used. Also if the curves were obtained by oscillograph records, for instance, an enlarged print would be necessary, which is an added loss of time and increase in expense.



FIGURES No. 76 and 77. Details of Bush Analyser.

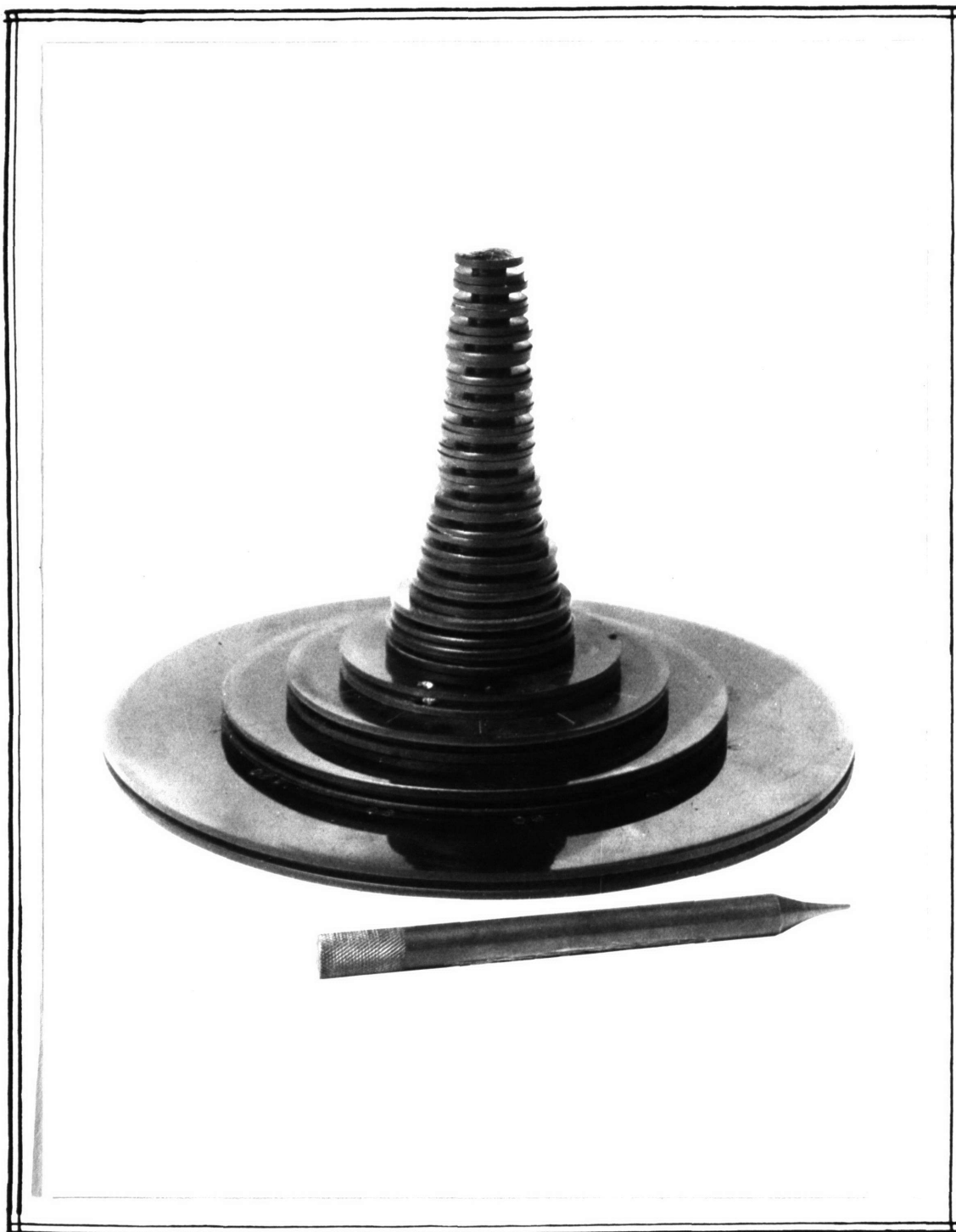


FIGURE No. 78. Set of Celluloid Discs for Bush Analyser.

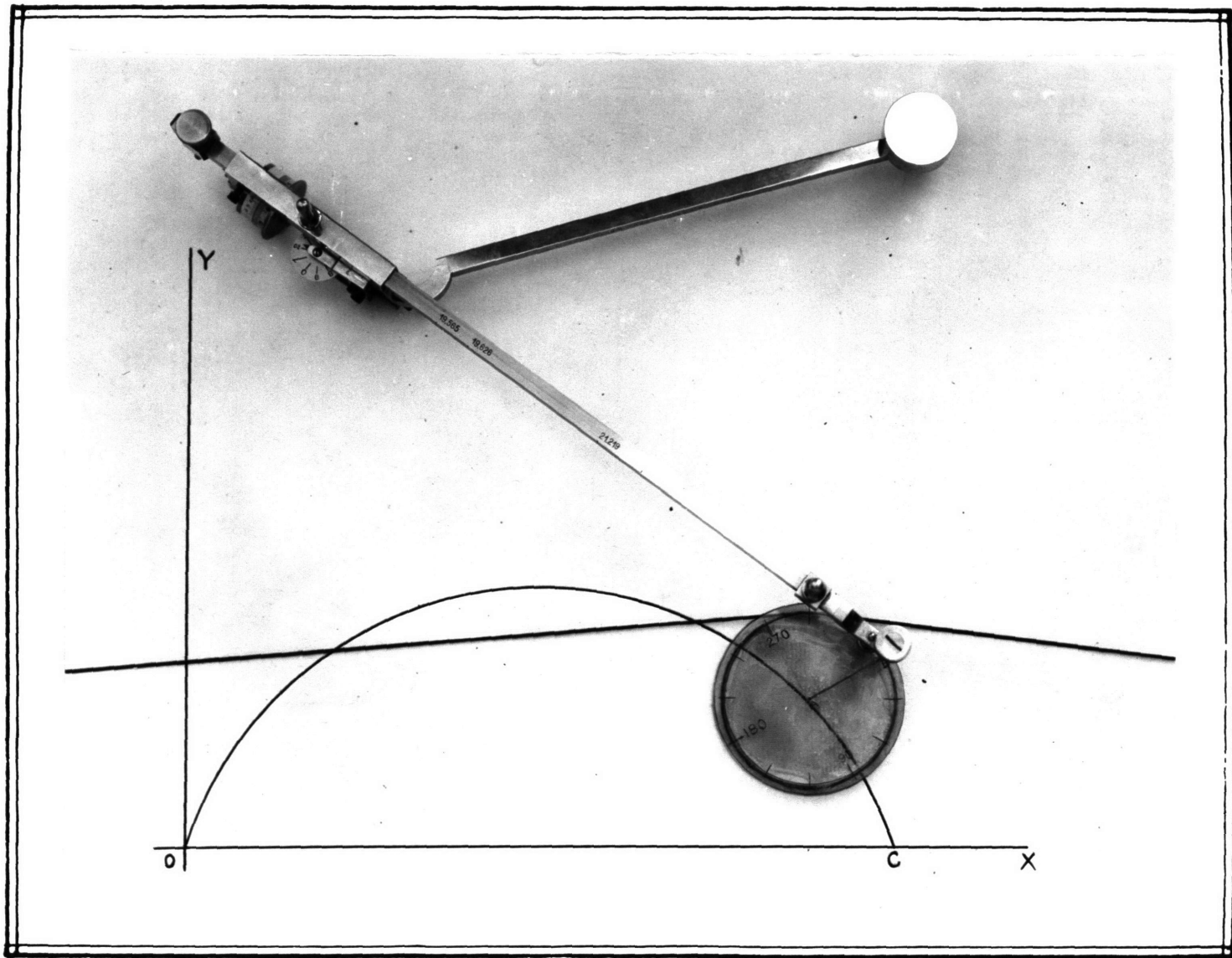


FIGURE No. 79. Bush Analyser in Operation.

ANALYSER DUE TO DELLENBAUGH.

(F.S.Dellenbaugh Jr., Jour.A.I.E.E., February, 1921).

Since one of the great difficulties in harmonic analysis of more than a few components is the neccessity for repeated traces of the curve, and the contingent possibility of introducing different errors each time, an attempt was made to make use of the simplicity of altering the combinations of electric circuits, by means of which all the harmonics could be obtained from one setting of the machine.

The theory of the schedule method was taken as a basis for the machine, as in formulas (6) & (7). The fundamental circuit is shown in Fig. 80. Current is passed through a number of slide wires in parallel. The sliding contacts are connected to resistances, which are multiplied with a common return and the circuit closed through a milli-ammeter and one of the slide wire busses. If the resistances are large compared to that of the slide wires the voltage at any point of contact will be proportional to its distance from the end of the slide wire. Therefore if the sliders are set at distances proportional to y_r read from the curve to be analysed, the voltage impressed across the resistances will also be proportional to y_r . If the resistances be made proportional to $1/\sin.k\theta_r$ then the current flowing will be proportional to:

$$y_r \sin.k\theta_r \quad \text{in each resistance,}$$

and the current in the milli-ammeter will be proportional to:

$$y_1 \sin.k\theta_1 + y_2 \sin.k\theta_2 + y_3 \sin.k\theta_3$$

By suitable resistances the cosines instead of the sines may be introduced into the equation. It will be seen that this is of the same form as the solution of Fourier's Series referred to above. Therefore by the choice of a suitable number of slide wires the analysis of any number of harmonics may be made.

In order to determine various harmonics the settings of the sliders on the slide wires remains the same for all but the resistances must be altered. This can be easily accomplished by means of multiple or gang swithhes, taps being provided upon the resistances at suitable points.

The error introduced by the assumption that current drawn from the slide wires does not alter the voltage distribution may easily be calculated and is given by the expression:

$$\text{Error} = [(1 - x)/x](Rx/rl)$$

Where l is the length of the slide wire, R the resistance of the slide wire, r the resistance in series with the slider, and x the position of contact from one end. The error will be maximum when x is one half of l, and if the value of r is 500 times the value of R, the error will be only 0.1%, which is entirely negligible in work of this nature.

Fig. 80 shows the arrangement of such a device for the determination of the first, third and fifth harmonic coefficients. The resistance of the slide wires is taken as two ohms. Therefore the minimum resistance in series with the slider must be 1000 ohms. The resistances must be inversely proportional to the sines and cosines of the ordinate position angles. The ordinates occur every 30 degrees, therefore the smallest sine or cosine will be .50 and the largest will be 1.00, neglecting those that are zero. The resistances must be inversely proportional to these, the smallest has been fixed by the allowable error at 1000 ohms, and so the largest will be 2000 ohms, with an intermediate tap at 1155 ohms on some of them to correspond to the sine of 60°. When the sine or cosine is zero the resistance will be infinity, and thus the circuit is left open.

Some of the sines or cosines will, however, be negative, and in order to make the machine take this into account either negative voltage or negative resistance must be used. Obviously the former is resorted to, and each slide wire is made double, and folded back from the common bus for the milli-ammeter return circuit. Two sliders moving in unison are provided. Thus the voltage from one slide wire may be considered negative and the other positive with respect to the mid bus. The procedure is to divide the curve up into six equal spaces along the X-axis and read the five resulting ordinates. These ordinates are then set on the slide wires by means of convenient scales and the sliders adjusted to the same values. The comparative values of the coefficients are then read on the milli-ammeter corresponding to the position of the dial switch. The switches S, S, S, etc. are mechanically connected and thrown to the right for the fundamental and to the left for the 5th harmonic components.

It is still not possible to analyse a curve which has negative ordinates within the half wave, but this could easily be done if reversing switches were included between the slide-wires and the feeder busses.

A rough model of this five coefficient analyser was made and tested. It worked very well, and upon the basis of this a larger one capable of analysing through the 11th harmonic, odd harmonics only, was constructed. The principles are exactly the same, but the circuits of necessity more complex.

Fig. 81 shows the connections of this analyser. The resistances are indicated by the heavy lines and the values of the resistance taps tabulated alongside. The gang switches are indicated by the many black dots, which represent gang switch contacts, the handles being indicated on the right. When the handle is turned in the direction indicated for the proper harmonic, all the contacts in that row are connected together and also connected to the common wire leading to the positive meter connection.

The actual construction of the machine is shown in Figs. 83 and 84, while the machine set up for operation is shown in Fig. 86. The slide wires are wrapped around wooden discs, one edge of which projects through the face plate for manipulation. Scales graduated from 0 to 100 are glued in place beside the slide wires. An opening beside the manipulating flange provided with an index mark gives the proper setting of the slide wires. A double set of phosphor bronze contacts are mounted upon the under side of the face plate and bear upon the slide wires. The resistances are mounted on spools in the front part of the box, one of them being raised up out of place in Fig. 84. The brushes are all connected to one end of the resistance coils, and the various taps are connected to the gang switches, which may be seen with their connections in the front part of the last mentioned figure. The connections to the slide wires are made by flexible cables which connect to bus-bars fastened to the under side of the face plate. The current found most convenient in the slide wires was .20 amperes each, and the meter used in the indicating circuit had a maximum scale reading of 20 milli-amperes. The size of the wires will easily allow larger currents to be used, so that the machine may be fitted to whatever meter equipment happens to be at hand.

The complete analysis up to the limits of the machine may be made in 3.5 minutes after having obtained the ordinates from the curve. The latter can be done very rapidly by using proportional dividers, or a transparent template if the curves are of the same length. The accuracy is reasonable, tests showing a maximum error of 3.7%, and the average error of about 2 to 3% or less. The errors are calculated in percent of the fundamental component, since if a harmonic is itself a small percent of the fundamental, the error as a percent of the harmonic may be very large.

In using the machine a simple check will determine whether all the connection are in order. The settings for a pure sine wave are marked in red on the slide wire scales. The discs are then set to these points and the various switches manipulated. All readings except the fundamental should be zero. As all the resistances used in the fundamental connection are used again in some other connection this test includes all units involved.

If the same scale units are used in measuring the ordinates and in setting the slide wires, then the results read on the meter will be in the same numerical form as those obtained with the schedule method, provided that the meter or battery current is adjusted to give a reading of 100 for the sine wave setting mentioned in the last paragraph.

The various values of resistance required for the different settings and the results of several tests upon the machine compared with schedule analyses are given in Figs. 82 to 89. The machine is of course limited in every way that the mathematical schedule is limited, but does not appear to introduce any additional limitations, and the errors are not much increased as long as the harmonic is a moderately large component. If the harmonic is very small the error seems to be somewhat increased. The error will also be largest in the last harmonic determined, since the number of ordinates is then only just sufficient to determine the value of the coefficient, and a small error in the readings from the curve or the settings of the slide wires will introduce large errors in the results.

(NOTE: In Fig. 80 the resistance connection for the right hand Y_1 resistance should be at 1155 ohms instead of 2000 ohms as shown.)

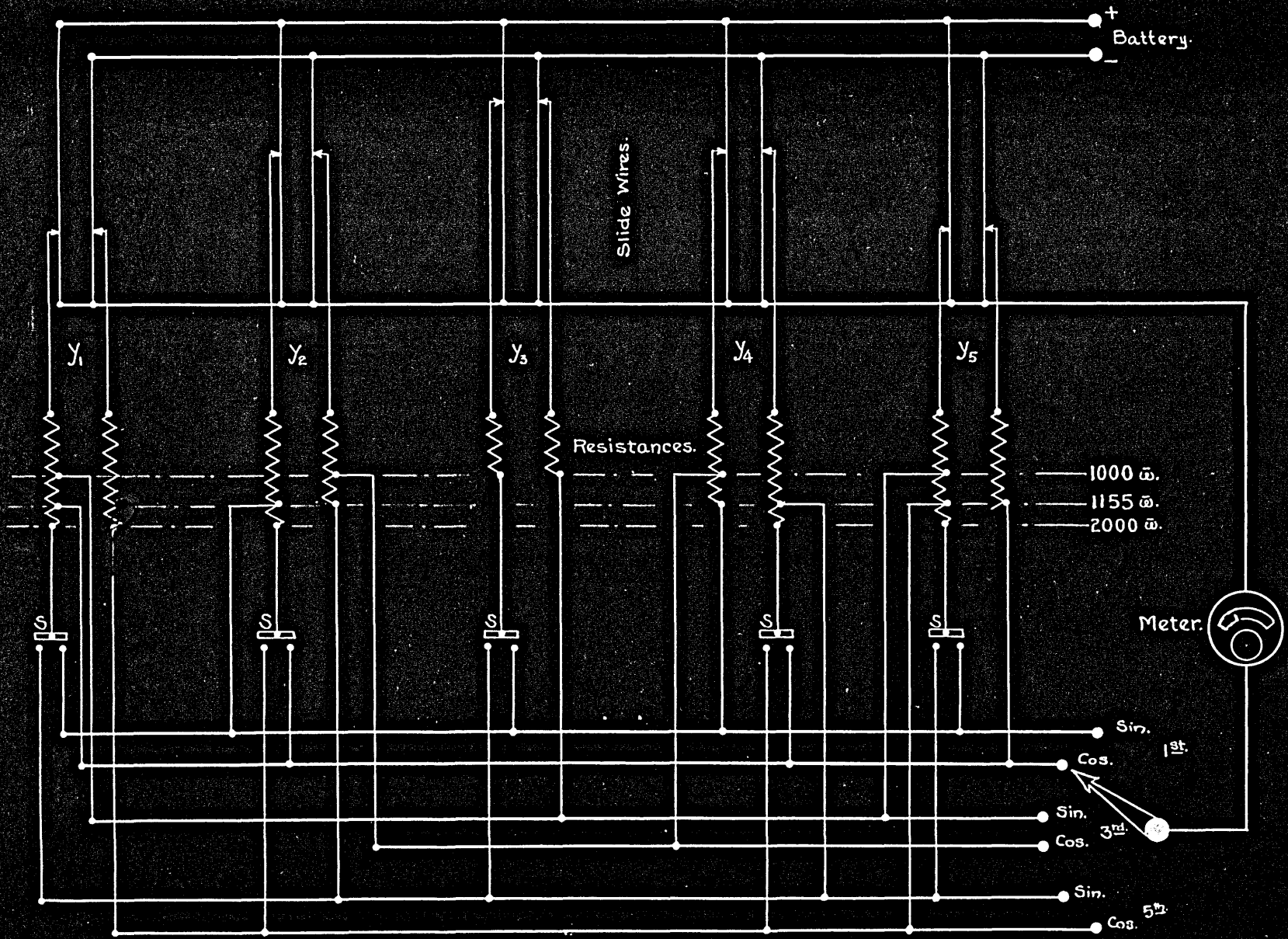


FIG. 80.

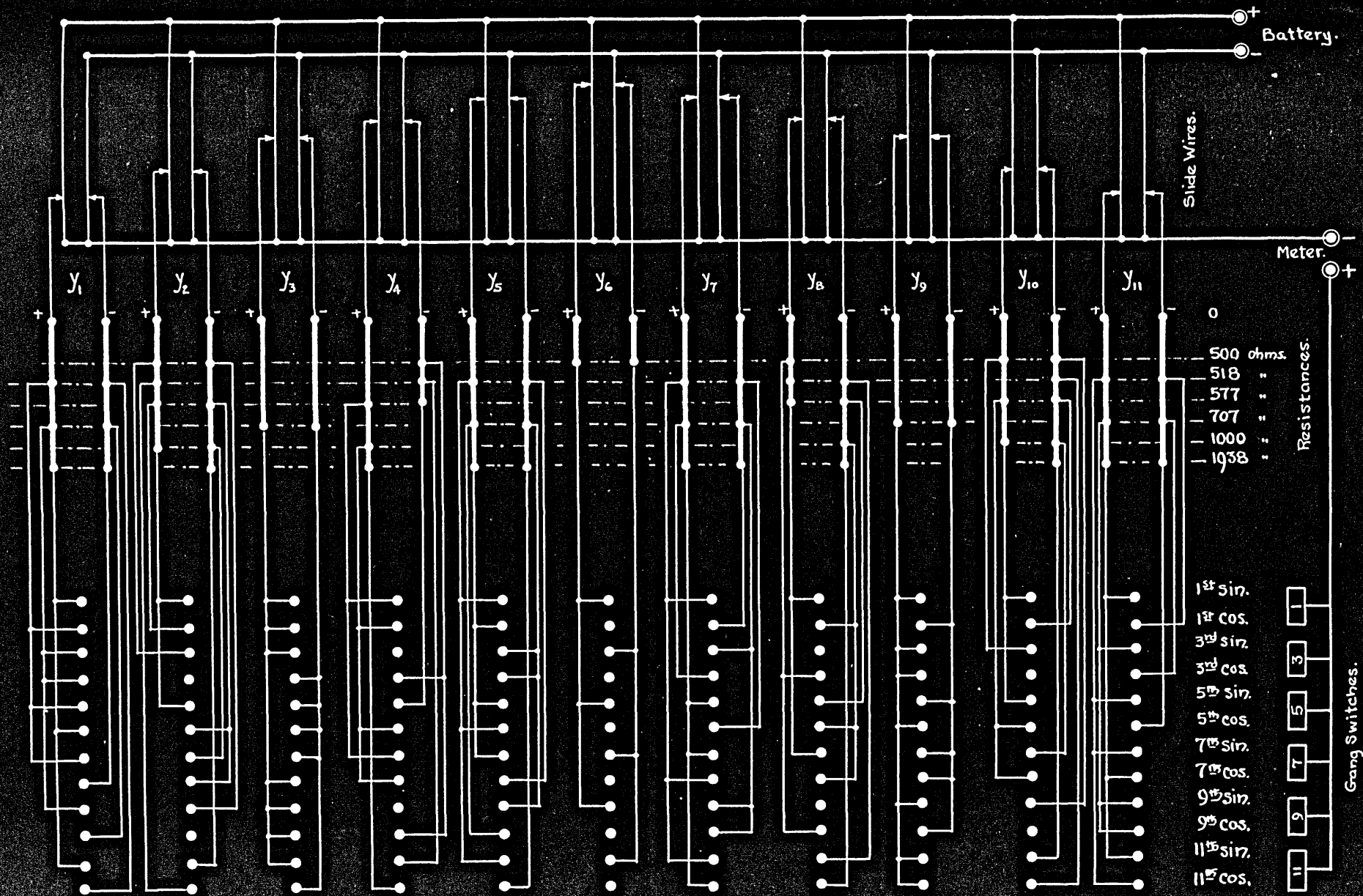


FIG. 81.

TABLE II
Connections of Resistances for Different Components in 11-Ordinate Harmonic Analyzer.

Order of Harmonic												
1		3		5		7		9		11		
Component:	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
Top Coils												
Plus Connection												
1	1938	518	707	707	518	1938	518	707	1938
2	1000	577	500	Inf.	1000	Inf.	518
3	707	707	707	707	707	707	707
4	577	1000	Inf.	1000	577	1000	Inf.	1938
5	518	1938	1938	518	1938	707	518
6	500	Inf.	Inf.	500	Inf.	Inf.	500	Inf.	Inf.
7	518	707	1938	1938	518	518	1938
8	577	Inf.	500	577	Inf.	500
9	707	707	707	707	707	707
10	1000	500	Inf.	1000	577	577	Inf.
11	1938	707	518	1938	707	1938	518
Bottom Coils												
Minus Connection												
1	1938	707	518
2	Inf.	577	1000	577	500	Inf.	1938
3	707	707	707	707	707
4	Inf.	500	577	Inf.	500	518
5	707	707	518	707	1938
6	Inf.	500	Inf.	500	Inf.	Inf.	500	Inf.
7	707	707	707	707
8	1938	707	518	707
9	1000	Inf.	1000	Inf.	518
10	707	707	707	707
11	577	1000	500	Inf.	1938	518
11	707	1938

FIGURE No. 82.

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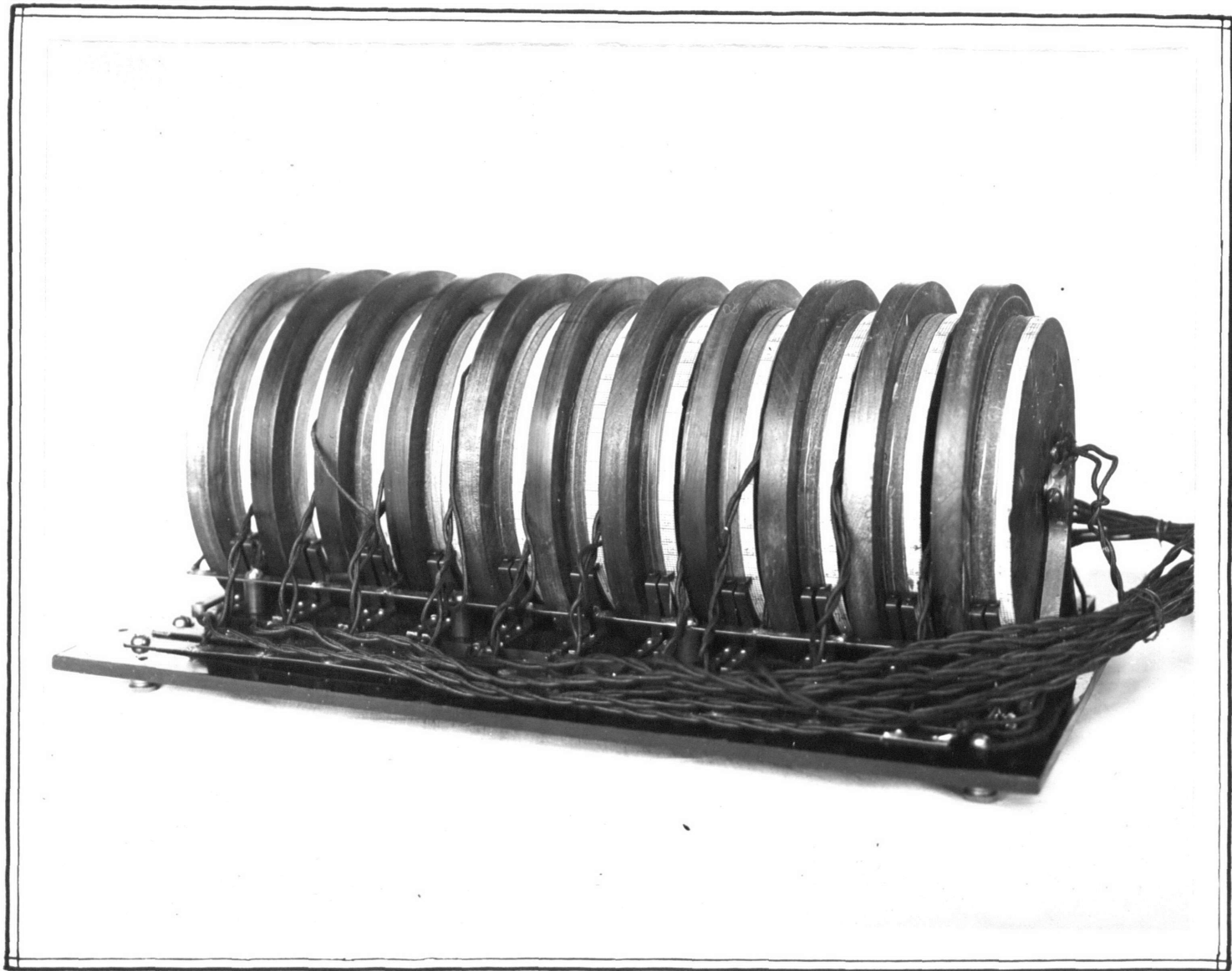


FIGURE No. 83. Slide Wires, Discs and Contacts of Dellenbaugh Analyser

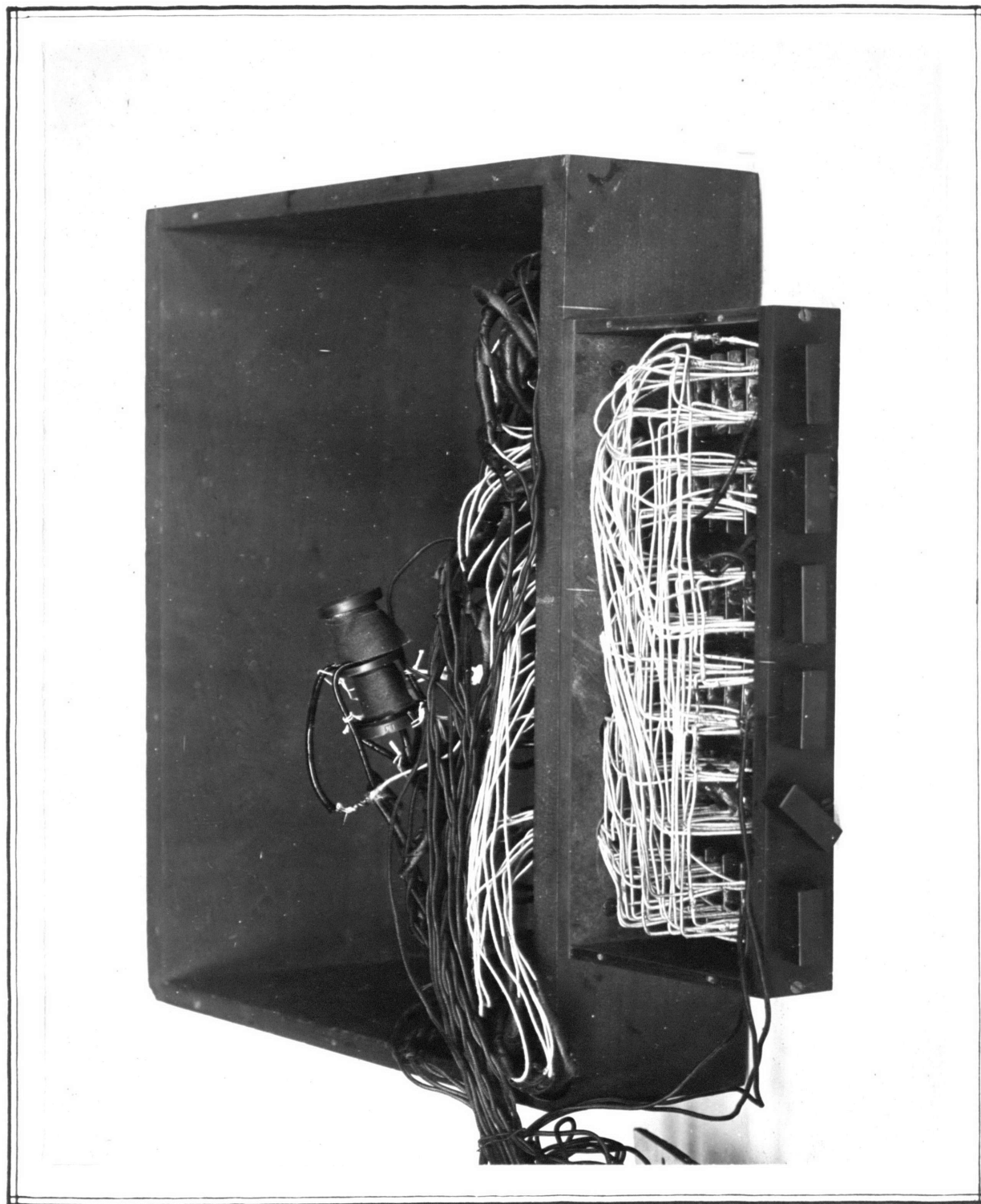


FIGURE No. 84. Details of Gang Switches and Connections.
Dellenbaugh Analyser.



FIGURE No. 85. Dellenbaugh Analyser Complete.

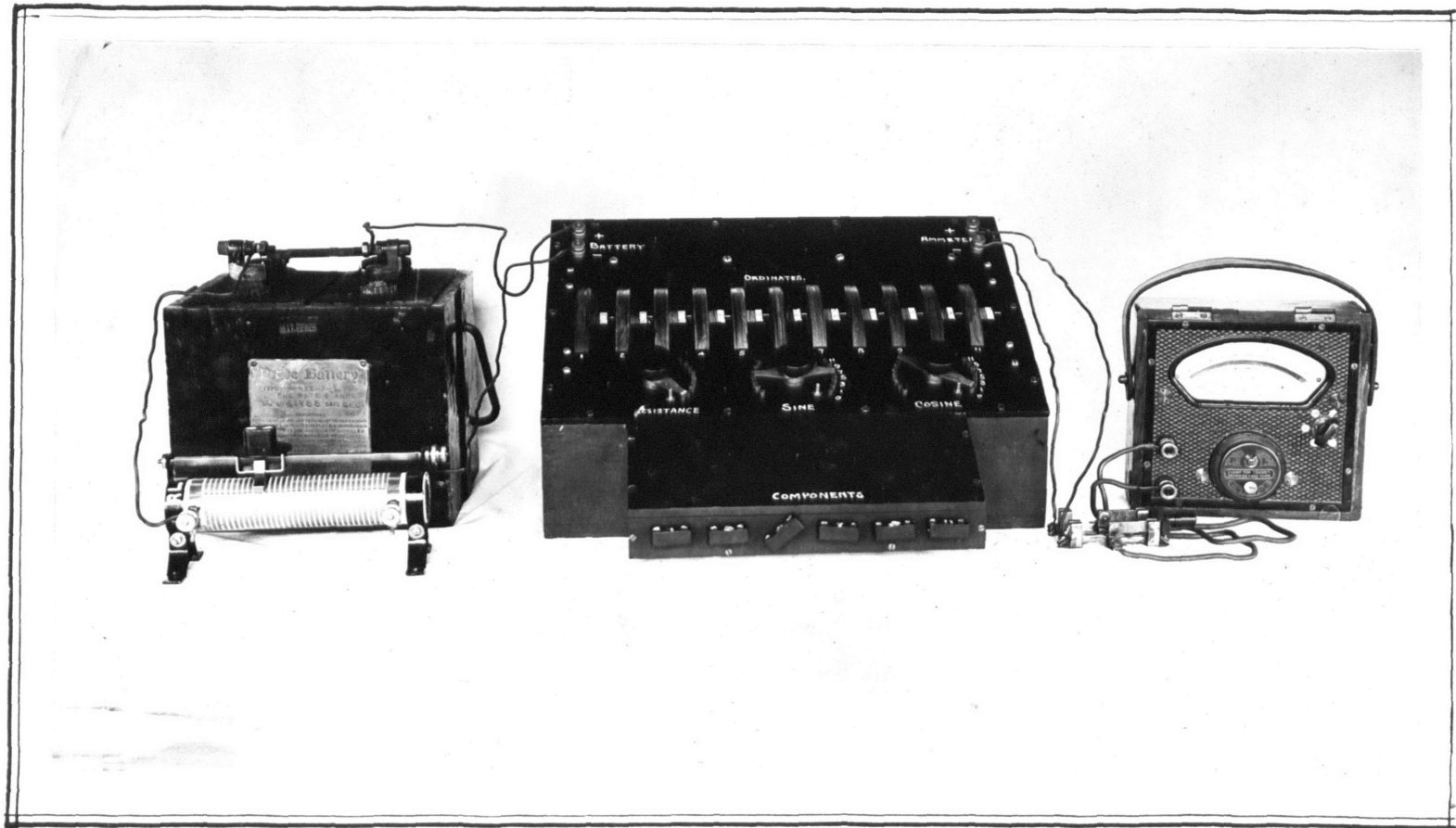


FIGURE No. 86. Dellenbaugh Analyser set up for use.

TABLE III

Curve of Equation: $y = \sin \theta + 0.5 \sin 3 \theta$

Ordinates:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
0.612	1.0	1.06	0.866	0.613	0.50	0.613	0.866	1.06	1.0	0.612

Harmonics	Electric analyser		Calculated		Error in per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 1.0	0	+ 0.9986	0	0	0
3rd.	+ 0.5	0	+ 0.4996	0	0	0
5th.	0	0	+ 0.0016	0	0	0
7th.	0	0	+ 0.0016	0	0	0
9th.	0	0	+ 0.0003	0	0	0
11th.	+ 0.02	0	- 0.0013	0	+ 2.0	0

TABLE IV

Curve of Equation: $y = \sin \theta + 0.8 \sin 3 \theta + 0.6 \sin 5 \theta + 0.4 \sin 7 \theta + 0.2 \sin 9 \theta + 0.1 \sin 11 \theta$

Ordinates:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
0.978	0.587	0.388	0.297	0.307	0.250	0.307	0.297	0.388	0.587	0.978

Harmonics	Electric Analyser		Calculated		Error per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 1.0	0	+ 0.9987	0	0	0
3rd.	+ 0.80	0	+ 0.8081	0	0	0
5th.	+ 0.59	0	+ 0.6040	0	- 1.0	0
7th.	+ 0.38	0	+ 0.3884	0	- 2.0	0
9th.	+ 0.195	0	+ 0.1920	0	- 0.5	0
11th.	+ 0.12	+ 0.005	+ 0.0990	0	+ 2.0	+ 0.5

TABLE V

Curve of Equation: $y = \sin \theta - 0.8 \sin 3 \theta + 0.6 \sin 5 \theta - 0.4 \sin 7 \theta + 0.2 \sin 9 \theta - 0.1 \sin 11 \theta$

Ordinates:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
0.0015	0.013	0.035	0.048	0.623	1.55	0.623	0.048	0.035	0.013	0.0015

(Actual values twice these values)

Harmonics	Electric analyser		Calculated		Error in per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 1.0	0	+ 0.9998	0	0	0
3rd.	- 0.79	0	- 0.7980	0	- 1.0	0
5th.	+ 0.58	0	+ 0.5890	0	- 2.0	0
7th.	- 0.39	0	- 0.4030	0	- 1.0	0
9th.	+ 0.22	0	+ 0.2120	0	+ 2.0	0
11th.	- 0.12	0	- 0.1200	0	+ 2.0	0

TABLE VI

Curve of Equation: $y = \sin \theta + 0.2 \cos \theta + 0.5 \sin 3 \theta + 0.1 \cos 3 \theta + 0.1 \sin 9 \theta + 0.2 \cos 9 \theta$

Ordinates:

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}
0.806	1.073	1.342	0.666	1.130	0.60	0.042	1.066	0.919	0.727	0.560

Harmonics	Electric analyser		Calculated		Error in per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 1.0	+ 0.18	+ 1.014	+ 0.189	0	+ 2.0
3rd.	+ 0.49	- 0.10	+ 0.489	- 0.098	- 1.0	0
5th.	+ 0.018	+ 0.01	+ 0.004	+ 0.003	+ 1.8	+ 1.0
7th.	+ 0.023	- 0.028	+ 0.004	- 0.002	+ 2.3	+ 2.8
9th.	+ 0.013	+ 0.21	+ 0.090	+ 0.195	+ 3.0	+ 1.0
11th.	+ 0.011	- 0.023	+ 0.014	- 0.004	+ 1.1	+ 2.3

FIGURE No. 87.

TABLE IX

Rectangular Wave:
 $y = \sin \theta + 1/3 \sin 3 \theta + 1/5 \sin 5 \theta + 1/7 \sin 7 \theta + \dots$

Ordinates are all equal.

Harmonic	Electric analyser		Calculated		Error in per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 1.0	0	+ 1.000	0	0	0
3rd.	+ 0.325	0	+ 0.318	0	+ 0.3	0
5th.	+ 0.178	0	+ 0.172	0	+ 0.6	0
7th.	+ 0.099	0	+ 0.101	0	- 0.2	0
9th.	+ 0.058	0	+ 0.055	0	+ 0.3	0
11th.	+ 0.055	+ 0.014	+ 0.018	0	+ 3.7	+ 1.4

NOTE. Errors in above table are calculated against the calculated solution of the wave, since the fact that higher harmonics are neglected introduces large errors in the schedule method and the error figures are intended as a criterion of electric machine accuracy.

TABLE X

Triangular Wave: $y = 4/\pi (\sin \theta - 1/3^2 \sin 3 \theta + 1/5^2 \sin 5 \theta - \dots)$

Ordinates:

$U_1 \quad U_2 \quad U_3 \quad U_4^m \quad U_5 \quad U_6 \quad U_7 \quad U_8 \quad U_9 \quad U_{10} \quad U_{11}$
 0.262 0.524 0.785 1.05 1.31 1.57 1.31 1.05 0.785 0.524 0.262

Harmonic	Electric analyser		Calculated		Correct	Error per cent fundamental	
	Sin	Cos	Sin	Cos	Sin	Sin	Cos
1st.	+ 1.27	0	+ 1.2805	0	+ 1.2732	- 0.3	0
3rd.	- 0.143	0	- 0.1489	0	- 0.1415	+ 1.2	0
5th.	+ 0.058	0	+ 0.0588	0	+ 0.0509	+ 0.8	0
7th.	- 0.034	0	- 0.0346	0	- 0.0259	+ 0.8	0
9th.	+ 0.025	0	+ 0.0266	0	+ 0.0157	+ 0.9	0
11th.	- 0.018	- 0.001	- 0.0222	0	- 0.0105	+ 0.7	+ 0.1

NOTE. It will be noticed that the errors in the calculated values are of the same magnitude as those in the electric machine. The percentages given in the error column are calculated against the correct coefficients and therefore are chargeable in a large measure to the method rather than the electric device for interpreting it.

FIGURE No. 88.

TABLE VII

Curve of magnetizing current in transformer

Ordinates:

$U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5 \quad U_6 \quad U_7 \quad U_8 \quad U_9 \quad U_{10} \quad U_{11}$
 0.30 0.95 1.03 0.90 0.75 0.60 0.55 0.50 0.42 0.35 0.20

Harmonics	Electric analyser		Calculated		Error in per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 0.812	+ 0.22	+ 0.8121	+ 0.2165	0	0
3rd.	+ 0.199	- 0.160	+ 0.1933	- 0.1492	+ 0.74	+ 1.23
5th.	- 0.027	- 0.088	- 0.0280	- 0.0886	- 0.12	0
7th.	- 0.041	- 0.020	- 0.0407	- 0.0179	0	+ 2.45
9th.	- 0.041	+ 0.017	- 0.0401	+ 0.0158	+ 0.11	+ 0.12
11th.	+ 0.019	+ 0.038	+ 0.0086	+ 0.0233	+ 3.32	+ 1.85

TABLE VIII

Curve of Sinusoid plus discontinuous peak.

$U_1 \quad U_2 \quad U_3 \quad U_4 \quad U_5 \quad U_6 \quad U_7 \quad U_8 \quad U_9 \quad U_{10} \quad U_{11}$
 0.60 0.725 0.775 0.775 0.725 0.65 0.75 1.07 1.25 1.15 0.725

Harmonics	Electric analyser		Calculated		Error in per cent fundamental	
	Sin.	Cos.	Sin.	Cos.	Sin.	Cos.
1st.	+ 1.065	- 0.165	+ 1.064	- 0.1630	0	+ 0.19
3rd.	+ 0.425	+ 0.098	+ 0.407	+ 0.0940	+ 1.7	+ 0.38
5th.	+ 0.037	+ 0.085	+ 0.037	+ 0.0833	0	+ 0.15
7th.	+ 0.044	- 0.012	+ 0.040	- 0.0098	+ 0.37	+ 0.19
9th.	+ 0.025	+ 0.005	+ 0.0009	+ 0.0050	+ 1.88	0
11th.	+ 0.037	- 0.020	+ 0.0024	- 0.0088	+ 3.30	+ 1.3

FIGURE No. 89.

ANALYSER DUE TO WOODBURY.

(D.O. Woodbury, Thesis, Mass. Inst. Technology. 1921)

The analyser devised by Woodbury is extremely simple in construction and operation, while it may be used with great speed, and the accuracy is about the same as obtained by schedule methods.

The theory is a direct application of the Fischer-Hinnen method of selected ordinates. The analyser is shown in Fig. 95. The curve to be analysed is mounted upon the cylindrical segment and held in place by adjustable bands. Cylindrical form is not necessary but was adopted owing to constructional advantages in having the parts rotate about centers instead of sliding on tracks. In front of the curve a stylus is carried upon two bars, which swing in the direction of the curve ordinates. The stylus can slide along the bars in the direction of the curve abscissae. The outside bar has notches cut in it at points equally spaced and arranged so that it gives $2n$ ordinates for a complete period of the curve, in accordance with the Fischer-Hinnen rules. Thus the machine can only be used for the analysis of curves with the same length of period, but can easily be modified for other lengths of curves by providing special bars, the bars being easily exchangeable.

In order to allow many harmonics to be determined with a single bar, groups of notches are provided, for $n=3, 5, 7$ etc, the value desired being brought into use by rotating the bar, which has a knob on the right with slots arranged to lock it in the desired position, the slots being numbered with the order of the harmonic which will be determined. A spring operated finger on the stylus carriage catches in the notches and is arranged so that it drops

in place with a definite click, but will slide out again with a little extra pressure upon the stylus.

At the left of the machine is an iron disc carrying a vernier registering with a scale. The arm carrying the stylus and division bar also has a small electro-magnet energised by a small battery through a switch in the thumb-grip on the left of the stylus and division-bar carriage. When the magnet is energised the plate and vernier are carried with it. When it is not energised a brake prevents motion of the plate. A still further adjustment is provided so that the position of the stylus with respect to the notching finger may be varied.

To operate the curve is placed anywhere on the platen and the X-axis adjusted parallel to the stylus tracks. The division-bar is set to the harmonic desired, and the notching finger placed in the first slot. The stylus is then adjusted to the zero point of the curve and locked to the notching finger. The stylus is then slid along until the finger drops into the first notch. With the stylus on the X-axis the thumb contact is pressed and the carriage raised until stylus coincides with the curve. This moves the vernier an amount equal to the ordinate. The thumb contact is then released; the stylus pushed to the next notch, and registered with the curve. The thumb contact is then closed and the carriage lowered until the stylus is again at the X-axis. This subtracts the second ordinate from the scale. This process is continued until the end of the curve is reached. The scale reading given by the vernier will then represent the sum of the ordinates taken alternately plus and minus exactly as in the Fischer-Hinnen method. It is then divided by the number of ordinates measured, and the result is the sum of the harmonic coefficients as for Fischer-Hinnen method.

The machine shown in Fig. 95 has a capacity of all odd

harmonics up to and including the 21st. The fundamental third and fifth components can be determined in about five minutes. It is not necessary to print the oscillogram since by placing paper under the film the analysis can be made directly from the negative. The divisions on the bar correspond to a half wave length of slightly over six inches.

The other Figures show earlier models upon the same principle. Fig. 92 to 94 are identical with the machine just described, being merely an earlier model along similar lines. Figs. 90 to 91 show the same principle slightly modified in application. The curve is wrapped around the cylinder, which rotates and the ordinate averaging device remains stationary. Instead of a notching device, the 2n divisions were marked on templates which were mounted on the cylinder and could be adjusted slightly to allow for small changes in the length of period. The addition of ordinates was accomplished by a mechanical clutch instead of a magnetic one.

The description has only dealt with sine components. It will be remembered that the ordinates for the cosine components lie midway between those for the sine components. Therefore the notching bar has a single extra notch to the left of zero spaced one half the distance of the other notches in that row. Then for the cosine components the first adjustment of the stylus to the zero point of the wave is made with the notching finger in this extra notch, and the subsequent procedure carried out exactly as before.



FIGURE No. 90. Early Model Woodbury Analyser.

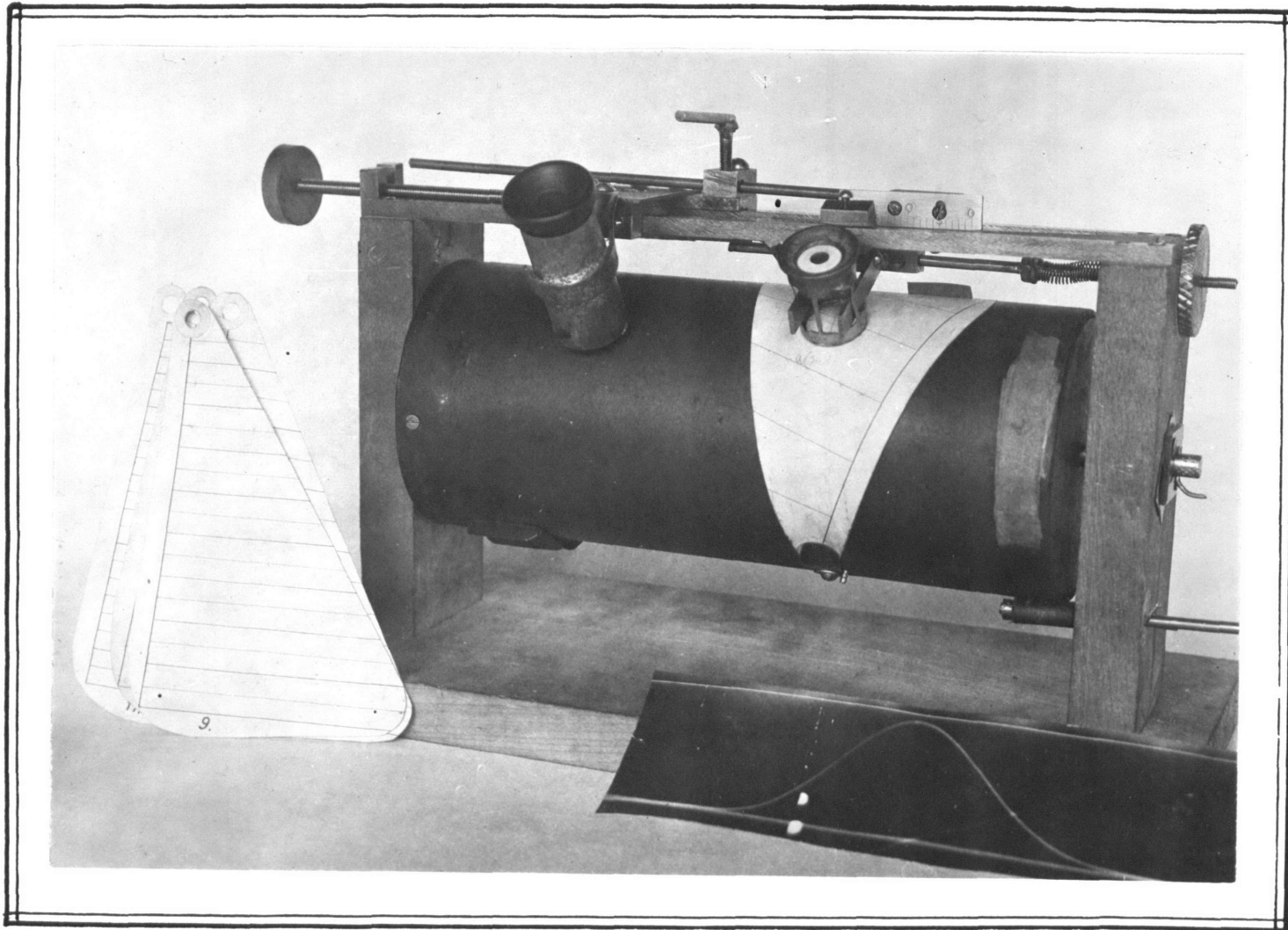
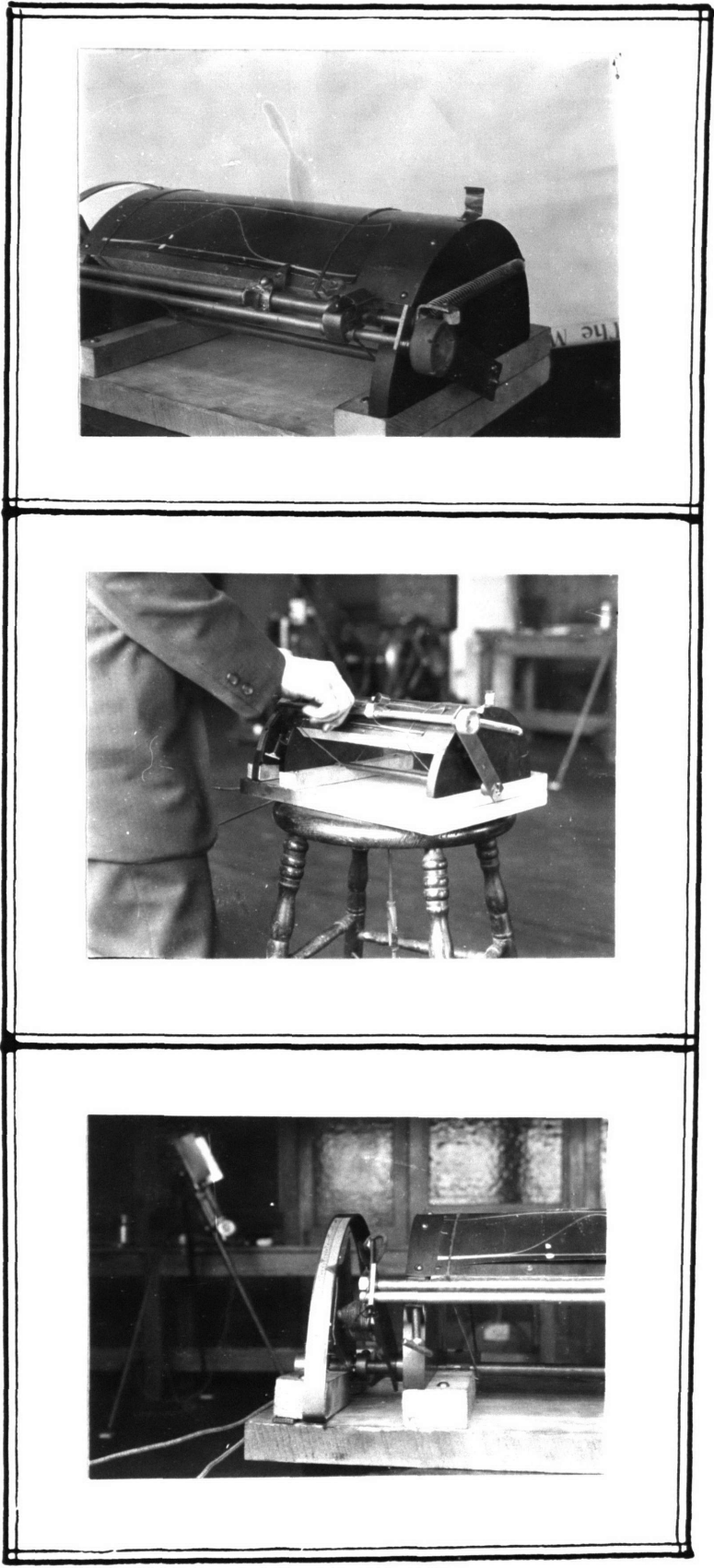


FIGURE No. 91. Early Model Woolbury Analyser Showing Templates.



FIGURES 92,93,94. Showing Second Model Woodbury Analyser.

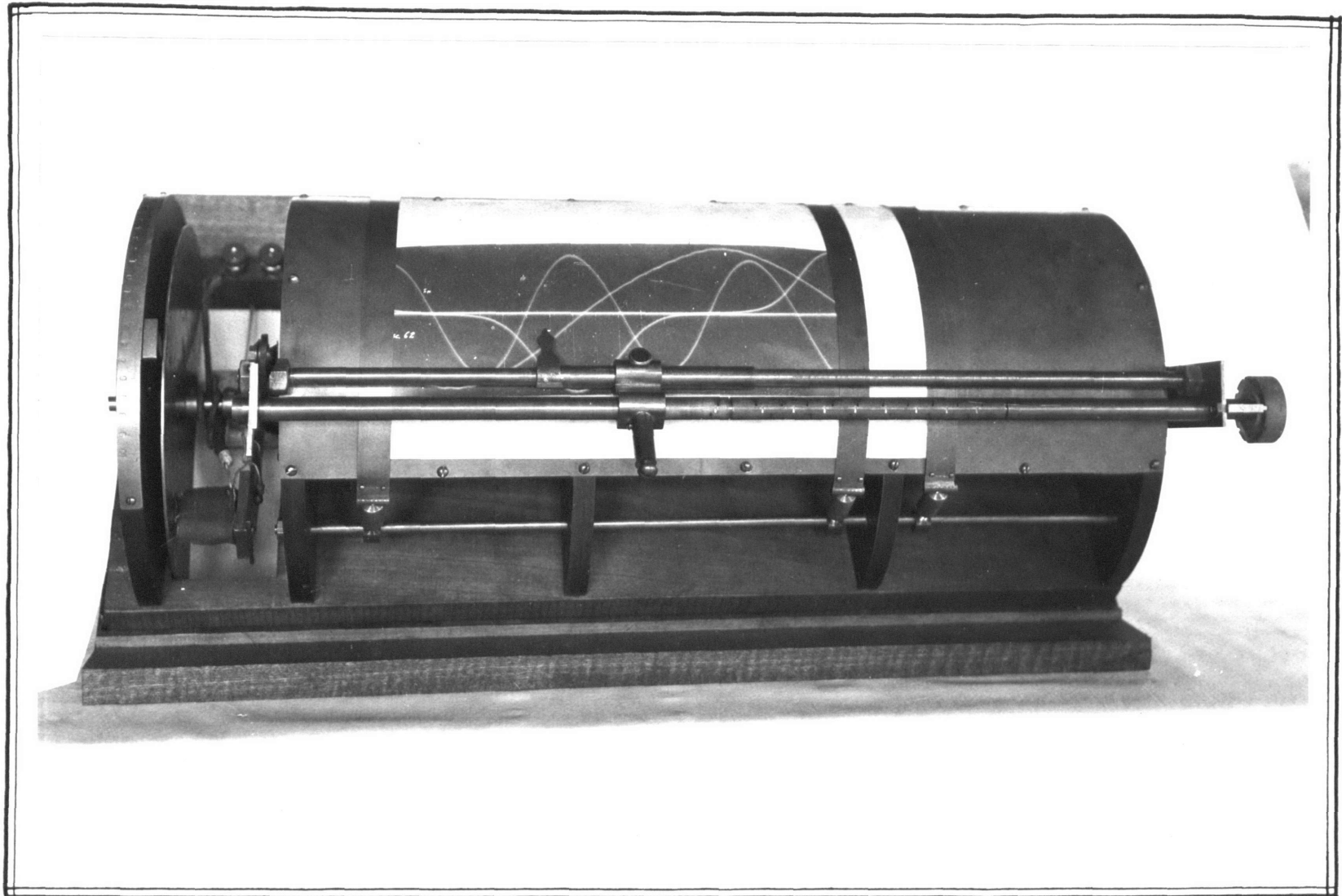


FIGURE No. 95. Final Model Woodbury Analyset.

H A R M O N I C S Y N T H E S I Z E R S .

HARMONIC SYNTHESIZERS.

Harmonic Synthesis has not received the attention that has been given its opposite, analysis, partly because the analysis of an unknown wave is a more frequent and difficult problem than the addition of various harmonic waves, and partly because the original synthesizer devised by Lord Kelvin is so eminently satisfactory that little improvement has been necessary.

The chief use to which synthesizers have been put is the predication of tides, and it was for this purpose that Lord Kelvin designed the original one used by the British Admiralty. Modifications of this type have been made from time to time, the general design being identical. There are several in use in this country for tidal predictions, and D.C. Miller has used a similar and very compact model for the synthesis of sound waves after analysis to prove the accuracy of the analysis.

The original Kelvin machine and some of its more recent adaptations are shown in Figs. 96 to 99 inclusive. In addition to the Kelvin type there are the synthesizers of Michelson and Stratton, Terada, and Prof. Laws suggests a method of synthesis in connection with his dynamometer type of analyser, by means of harmonically driven sliders on drop wires, obtaining a synthesised electric wave. This has never been tried apparently and would appear impractical.

The construction of the Kelvin machine is very simple. There are a number of wheels driven from one master gear, the gear ratios being chosen so that the wheels revolve at rates of 1, 2, 3, etc times the master gear. Each wheel thus represents a harmonic. As many wheels are provided as it is desired to include harmonic terms in the synthesis. Kelvins original machine would combine 12 components, the Michelson and Stratton machine will combine 80 components and the D.C. Miller synthesizer will combine 32 components. Each wheel may be adjusted in phase relation with any other wheel, and carries a pulley which may be adjusted to any radius desired. The radius then represents the amplitude of the component, and the phase setting its phase relation. The motion of the pulleys is combined by means of a wire which runs from one to the other in such a way that the amount of motion of the pulleys takes up or lets out wire against a spring or weight. In the original Kelvin machine the pulleys were not directly on the wheels, but attached to cranks by a wire, so that alternate pulleys were on different levels

and the wire carried back and forth between them. In later machines the wire was run around pulleys directly on the wheels, and in the machines of the U.S. Government the cranks on the wheels fitted into slotted bars carrying pulleys on the ends of the bars, and the motion was thus transmitted with a minimum amount of wire.

This mechanism can be clearly seen in the figures, and it will be easily seen from this that as the wheels revolve, if one end of the wire is made fast to the frame of the machine and the other end given a slight tension by means of spring or weight, the motion of the free end will be the sum of the projections of the wheel cranks upon one axis. Since the cranks are set to a length equal to the harmonic component amplitudes, the motion of the free end of the wire will give the function represented by the Fourier Series for which the machine was set. In order to get the time axis the paper upon which the curve is plotted is rolled over rollers geared to the driving wheel of the synthesizer. The end of the wire is fitted with a pen, arranged to slide in guides, and then by means of a hand crank the whole mechanism is operated and may be made to draw any series desired, within its capacity. A number of such curves as drawn by the Michelson and Stratton synthesizer are shown in the section on Wave Forms and Convergence.

The only serious error in this type of machine is the stretch of the wire. Since the wire must be fairly long, and since it runs over one pulley after another, the errors will be cumulative and may become large compared to some of the smaller harmonics. It was for this reason that Michelson and Stratton devised the synthesizer which has no wires, but operates by using the deflection of springs to perform the summations. If the springs conform to Hooke's law closer than the stretch of the wire in the Kelvin type then the Michelson and Stratton machine is more accurate. If the springs do not elongate proportionally to the force applied, then it is probable that the Kelvin type would be more accurate. The criterion therefore appears to be the quality of springs or wire obtainable. As the mechanism of the Michelson and Stratton machine has been fully gone into under analysers it will not be repeated.

The Terada machine has also been treated under analysers, and requires so many laborious tracings as a synthesizer that it cannot be compared to the other two types.

A special adaptation of the Kelvin Synthesizer by J.R.Milne, (Royal Society of Edinburgh, p.208, 1906), is worthy of mention. He desired to draw synthetic harmonic curves and change the phases and magnitudes of the components quickly and at will while the curves were being drawn. The object of this was to draw a large number of curves with different characteristics in an attempt to provide a standardized set of harmonic curves to be used in connection with the study of tides. It was also suspected that gradual changes in the harmonics occurred in the case of special tidal conditions.

As with the Kelvin machine the pen was driven by a wire, led alternately over pulleys on the harmonic wheels and stationary guide pulleys. The distance between the stationary and movable pulleys was made large enough so that the eccentricity of the connecting wire did not distort the harmonic motion. The machine was driven by a motor and each pair of wheels connected to the drive shaft by means of two cone pulleys and belt. Thus by sliding the belt along the cones the speed of any pair of wheels could be varied at will in small increments.

The harmonic wheels were made in duplicate pairs, connected by a crown wheel in the same manner as the differential of an automobile. The wire was led over pulleys on both wheels and so recorded the sum of two harmonic motions of equal amplitude and frequency. The phase relation could be altered from 0 to 180° by moving the crown wheel through 90°. Thus the amplitude of the resultant harmonic will vary from zero to twice the amplitude set on one harmonic wheel. There will be an attendant shift in phase of the resultant harmonic, which was corrected by a later model not fully described.

From these adjustments it will be seen that both the frequency and amplitude of any harmonic may be altered at will while the machine is in motion, but in order to control the phase relation of the harmonic further additions are necessary.

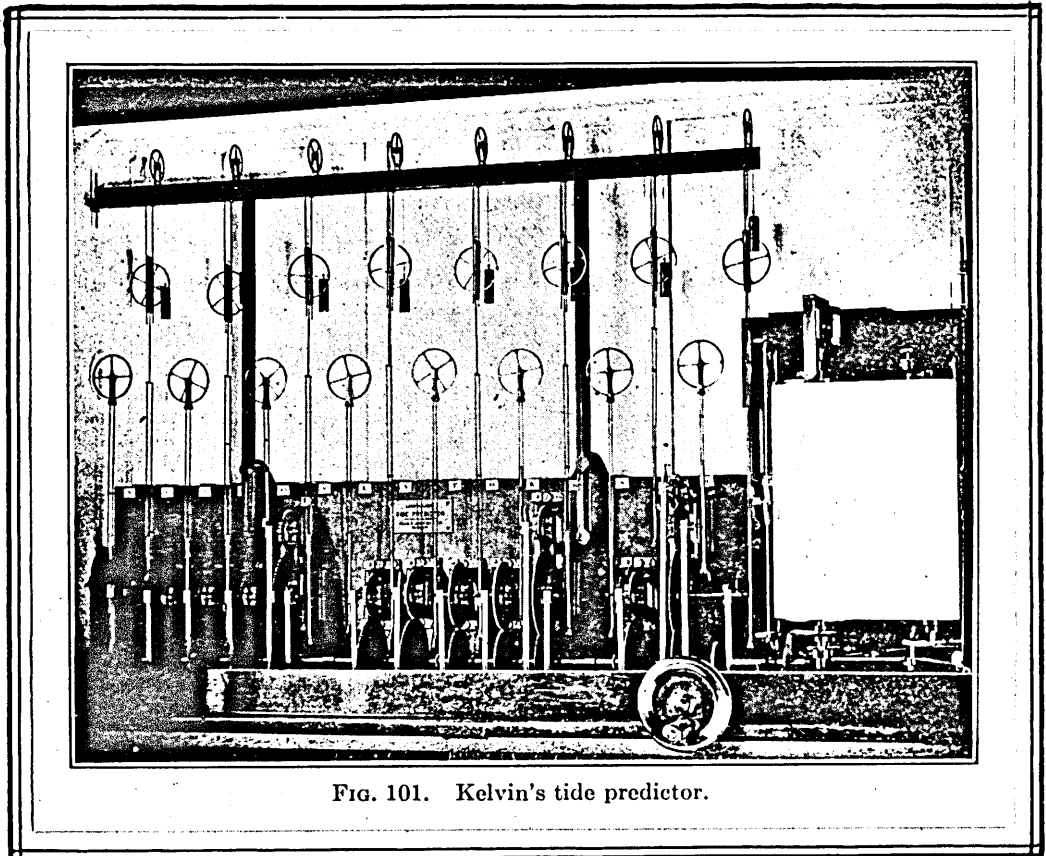
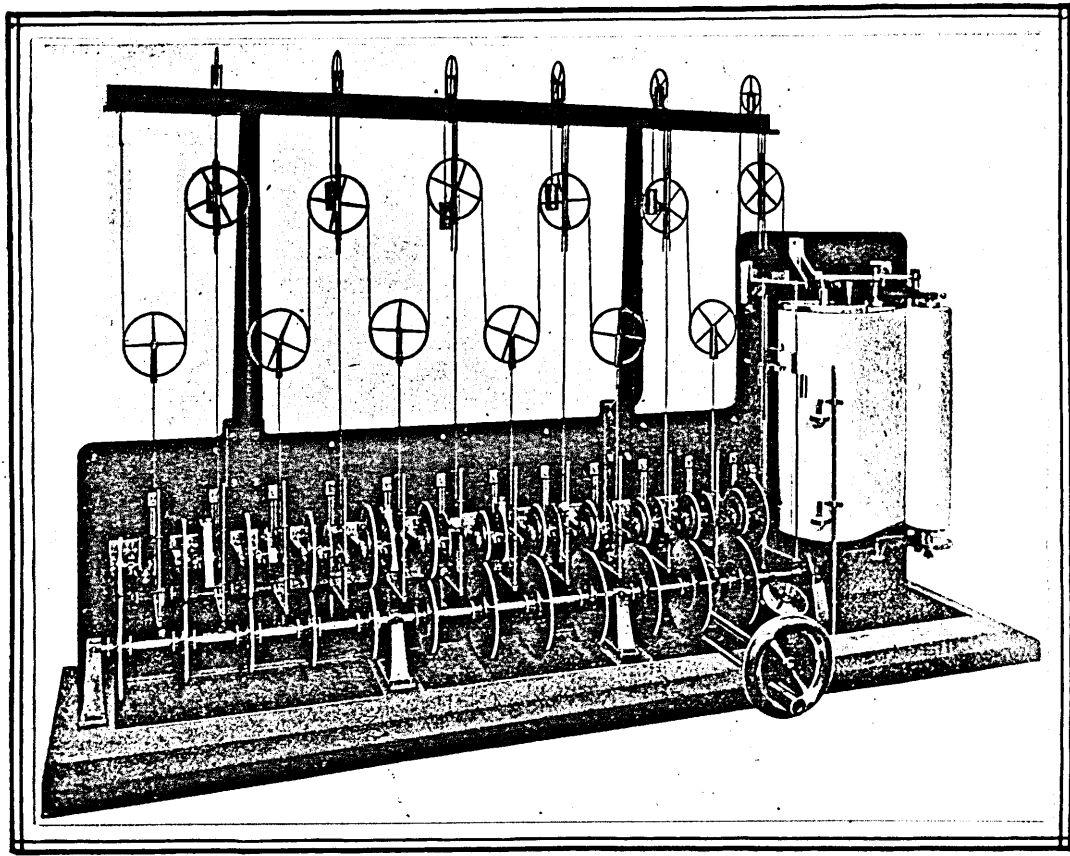


FIG. 101. Kelvin's tide predictor.



FIGURES No.96 and 97. Kelvin Tidal Synthesizer

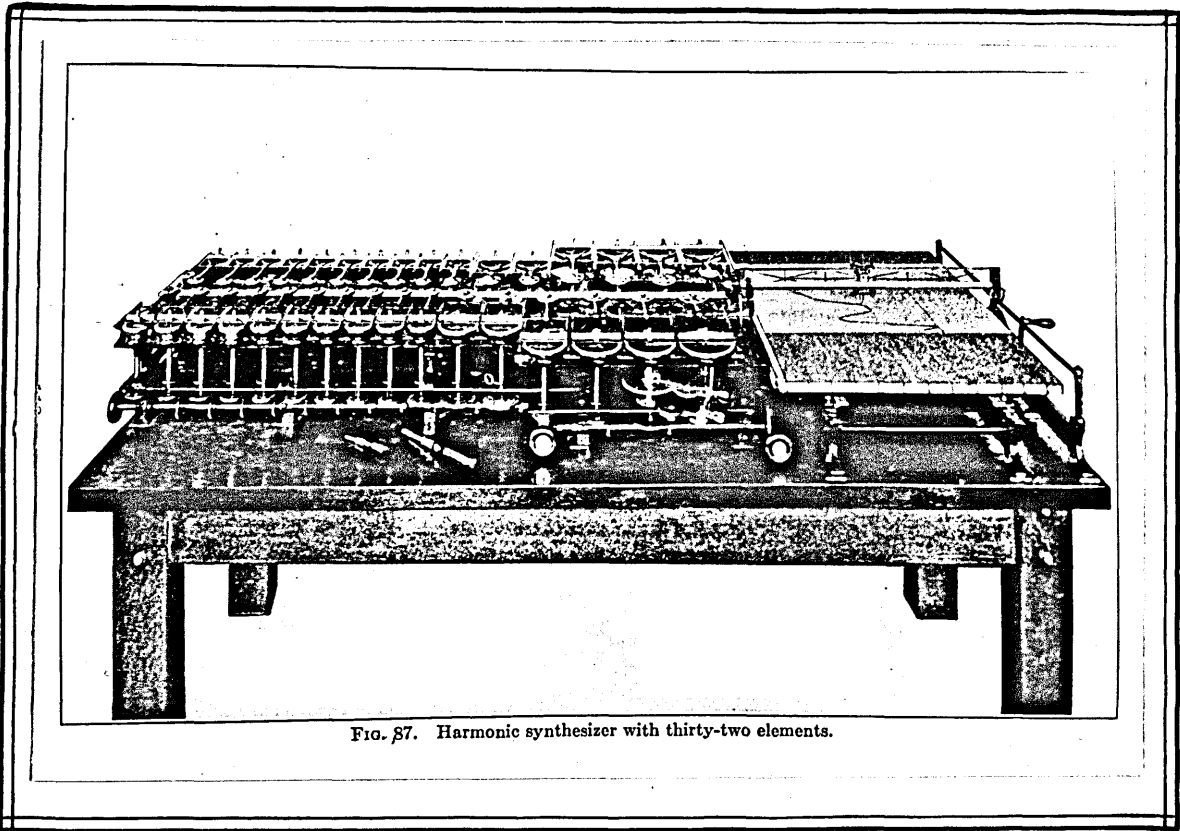
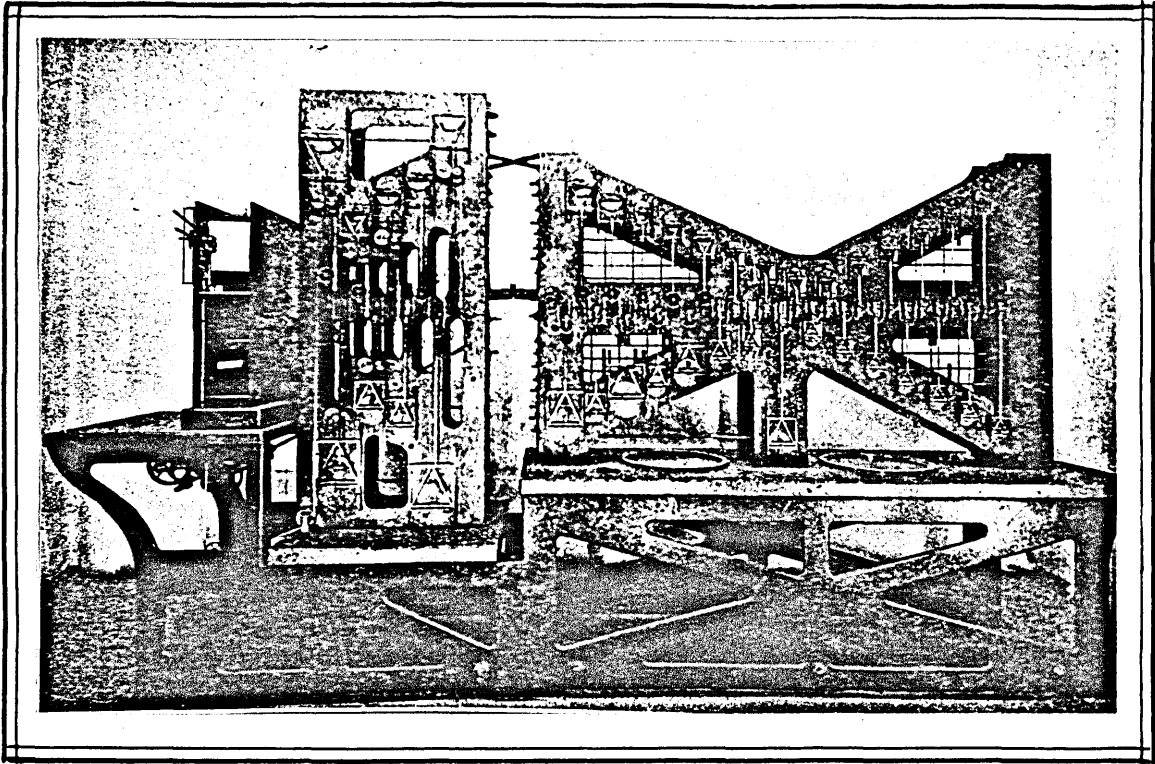


FIG. 87. Harmonic synthesizer with thirty-two elements.

FIGURES. No.98 and 99. Two Types of Modern Synthetizers.

OTHER DEVICES AND DETAILS CON-
NECTED WITH HARMONIC ANALYSIS.

OTHER DEVICES WHICH MIGHT BE USED FOR HARMONIC ANALYSIS.

There have been proposed a large number of devices for solving equations which could be applied to harmonic analysis. These involve electric methods, hydrostatic methods, and mechanical systems.

One of the most ingenious electrical methods is described by Arthur Wright, (Philosophical Magazine, 6th-Ser. Vol. 18, p. 291, 1909). He uses elements consisting of resistance wire wound on a logarithmically shaped form as shown in Fig. 100. The equation of the curve YPC is :

$$y = k \cdot 10^{-x/h}$$

Where: $OY = k$, $OS = h$, and $SC = k/10$.

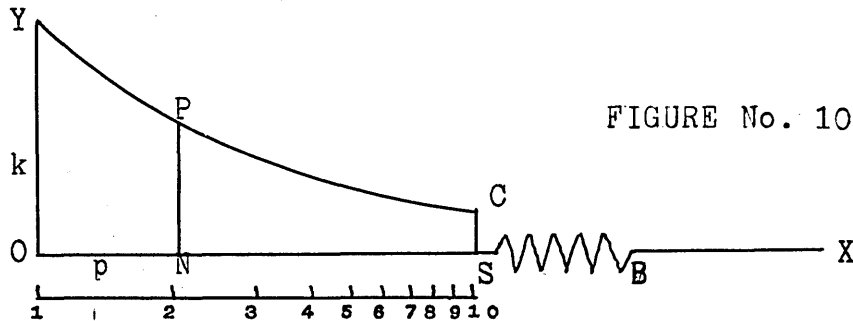


FIGURE No. 100.

The plate is wound uniformly with #36 resistance wire and an external resistance connected at SB.

Then: $ON = h \cdot \log p$, $y = PN = k/p$

The resistance of the wire between O and N will be approximately proportional to area ONPY, which is:

$$\int y \cdot dx = k \int 10^{-x/h} dx = [hk / \log_e 10] [1 - (y/k)]$$

If the wire on the frame have a resistance $9R/10$ and the coil SB in series have a resistance of $R/10$, then the resistance from N to B will equal $(y/k)R = R/p$.

If n of these resistances are connected in parallel their resistance will be such that the sum of the current flowing will be : $E/R(p_1 + p_2 + p_3 + p_4 \dots)$

The scales may be reversed in direction and the resistances connected in series, in which case the current will again read the sum of the scale readings. Thus addition may be performed in either of two ways.

Multiplication is performed much the same as with a slide rule. The slider is made with a movable contact finger OP. (Fig. 101). The scale lies at right angles to the resistance. The resistance is then provided with a scale also. The resistance is moved to the multiplier and the scale index set to the multiplicand, the resistance in circuit then being proportional to the product if the contact finger OP makes an angle of 45° .

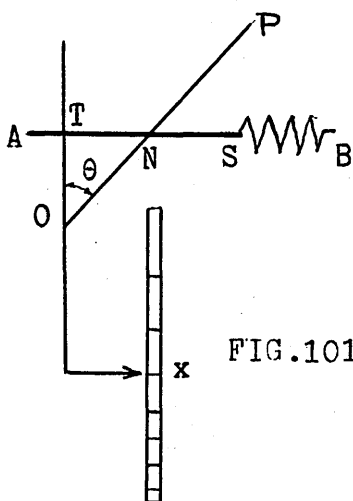


FIG. 101.

The angle of the contact finger θ determines the exponent of x .

$$\begin{aligned} TN &= OT \cdot \tan \theta = h \cdot \tan \theta \cdot \log x = \\ &= h \cdot \log x^{\tan \theta} \end{aligned}$$

Thus by the proper choice of θ , all all positive integral or fractional powers of x may be calculated.

By bending the contact fingers to the shape of empirical curves, these may also be introduced into equations.

The resistances are connected on the two sides of a resistance bridge circuit as shown in Fig. 102. All negative values are considered as being on one side and positive on the other. the proper settings are made and an extra resistance adjusted until the bridge balances, when the reading upon the scale of the balancing resistance will give the root of the equation.

Mr. Wright closes by stating that it seems particularly suitable to harmonic analysis, since the integrals representing the coefficients of $\sin.kx$ and $\cos.kx$ can be readily determined. There are many combinations given in the original article referred to which are not here mentioned owing to this being merely an abstract to indicate the underlying idea.

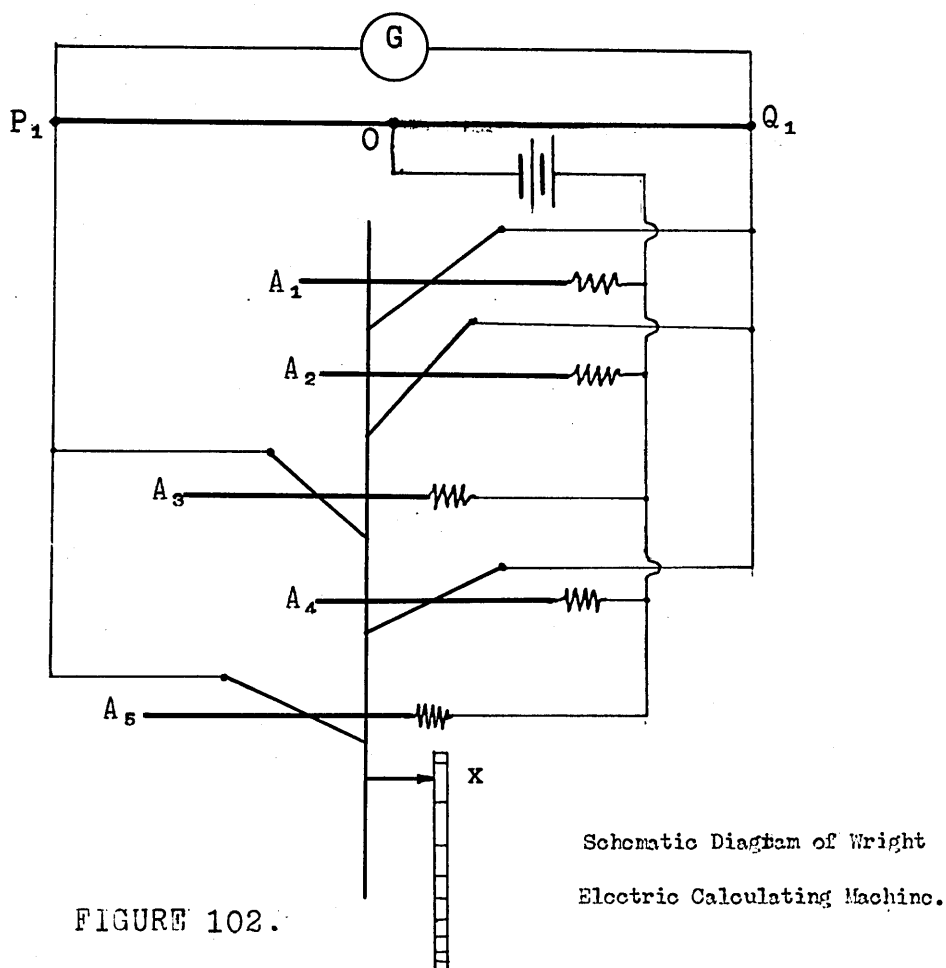


FIGURE 102.

Alexander Russell who worked on the above described machine with Wright, has devised several electrical devices of similar nature for the purpose of calculating and harmonic analysis, but no descriptions of his work appear to have been published.

M. F. Lucas, Bethenod and others have proposed various electric schemes for similar purposes. Among these is a method actually put into use in which vertical wires carrying Direct Current are brought up through a table. The resultant field due to the wires and the Earth's Field is plotted by means of following a small compass with a pencil. By properly adjusting the currents in the wires both real and imaginary roots of equations may be obtained.

from the locii of intersecting points.

Bethenod proposes the use of a number of air core Direct Current Generators similar to the motors used in Thomson Wattmeters. For the solution of a polynomial the machines are excited from a direct current source and connected in series to a galvanometer or voltmeter. The voltage of each machine is proportional to the field current and speed. Therefore if the speed is made proportional to the variable and the field current proportional to the coefficient of each term, the galvanometer deflection will be proportional to:

$$f(x) = k_1x + k_2x^2 + k_3x^3 + \text{etc.}$$

If it is desired to introduce a constant term this may be done by connecting a battery of the proper voltage in series with the generators. A null method may be used by adjusting an extra generator until the galvanometer reads zero. Then if there are more than one root there will be several adjustments giving zero deflection, each corresponding to a root. The rotation of the generator would be reversed for negative roots. The chief difficulty is to devise a method for conveniently adjusting the constants of the circuit to give electric results identical with the equations.

MECHANICAL SYSTEMS OF CORDS WEIGHTS AND PULLEYS have been used or proposed by various people. The Harmonic Synthesizer of Lord Kelvin, already described, is the best adapted to Fourier's Series, although others could be used for this purpose but do not seem to have actually been so used.

Peddie, Exner, Boys, Berard and Lalanne have developed various special cases of such a device. In some of them lever arms provided with pulleys, over which runs a cord, are used. The settings of the pulleys correspond to the coefficients and the amount of thread pulled through the machine, one end being fixed, corresponds to the independent variable. The positions of the levers give the different roots. Another method is to have a compound balance, consisting of a large number of balance arms, interlinked by some mechanism. The size and position of the various weights is adjusted according to the constants in the equation, and a final balancing arm used to equilibrate the whole system. The values used to balance the final arm for each position of equilibrium give the roots of the equation. A great many variations can be built upon this principle, and it is not necessary to more than mention them at this point, since they do not deal directly with harmonic analysis.

HYDROSTATIC BALANCES for the solution of equations have been developed, notably by Demanet and Meslin. The method used by Demanet consists in providing a number of vessels whose shapes are solids of revolution corresponding to the various powers of x entering into the equation. The equation is converted into such a form that all the variables are on one side and the constant is on the other. An amount of water is then poured into the system proportional to the constant term. All the vessels are connected by a common pipe at the bottom and placed level. If the volume of each vessel for a given height is then proportional to the power of x for each term, the height to which the water rises in the vessels will give the solution of the equation. This assumes that all signs are positive. If negative signs are present a solid is introduced corresponding to the term, and placed inside the vessels, so that it decreases the volume by the amount of the negative term. The device is difficult to use for equations of higher order than the cubic.

Meslin has improved upon this by using a hydrostatic balance as shown in Fig. 103. Solids of revolution are provided which may be hung upon the balance arm.

These solids displace an amount of water proportional to the powers of x in the equation. Their depth is proportional to x for a given immersion, and the distance at which they are hung upon the balance arm is proportional to the coefficient of the corresponding term. Negative terms are hung on one side of the balance arm and positive terms upon the other side. The balance is then adjusted for equilibrium by auxiliary weights with no water in the tanks. Next a weight is attached at one end corresponding to the constant term of the equation. This disturbs the equilibrium. The equilibrium is restored by allowing water to flow into the tanks, which tends to buoy up the various weights. When equilibrium is reached the height of the water will be the solution of the equation. If there is more than one root then there will be several heights of water for which balance is obtained.

An ELECTRO-CHEMICAL METHOD has been proposed by Lucas in which the equation is converted into partial fractions with real and definite numerators. Charges of electricity proportional to the numerators are properly spaced and the nodal points of the electrostatic field determined by chemical reaction, then the nodal points will be the roots of the equation. He also proposes to use magnetic charges and determine the nodal points by means of iron filings. This method will determine real as well as imaginary roots.

Since the solution of Fourier's Series according to the equations (6) and (7) is a polynomial of the first degree, it can be easily solved by any of these methods, the chief criticism being that the amount of complication involved is not required for a solution that can be obtained by simpler means, and these devices were proposed chiefly to solve equations which would not be soluble by ordinary mathematical processes.

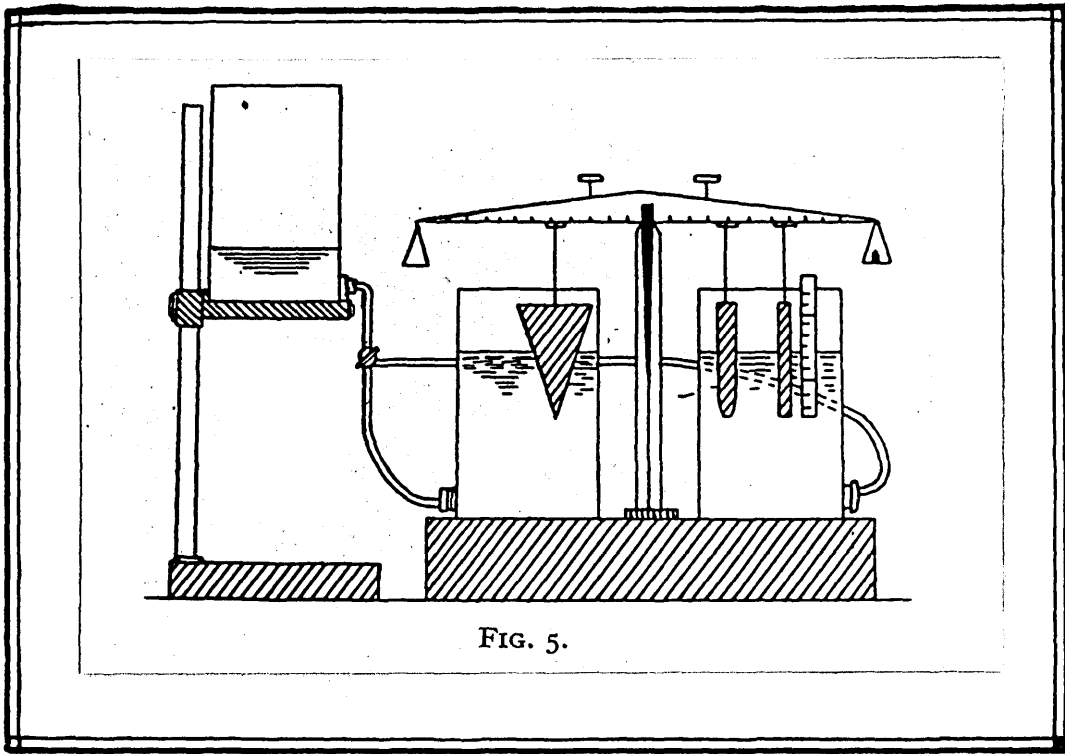


FIGURE No. 103. Hydrostatic Balance proposed by
Meslin for the Solution of Equations.

POSSIBLE METHODS OF ANALYSIS NOT PREVIOUSLY
PROPOSED.

ALINEMENT CHARTS. It would be possible to construct alinement charts in several ways that would perform the operations necessary for harmonic analysis according to formulas (6) and (7). It would be similar in form to the Beattie Charts already discussed. One way would be as indicated in Fig. 104. The supports are graduated as sine or cosine of the angle and tabulated as angles. The values of y_k are read off the supports, which automatically multiplies them by the sine or cosine of the corresponding angle. A large number of supports would be necessary, one for each value of y . There does not seem to be any advantage over the Beattie method, and the resulting charts would be limited in scope and complicated in use.

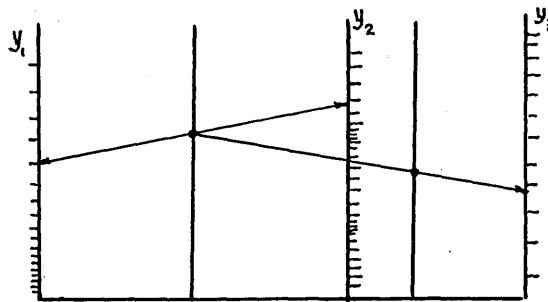


Fig. No. 104.

SLIDE RULES. It would be possible to make a multi-slide rule which would determine the coefficients according to schedule analysis. Theoretically it could be carried to any number of harmonics, but practically it would only be useful for two or three of lower order. The arrangement is shown in Fig. 105.

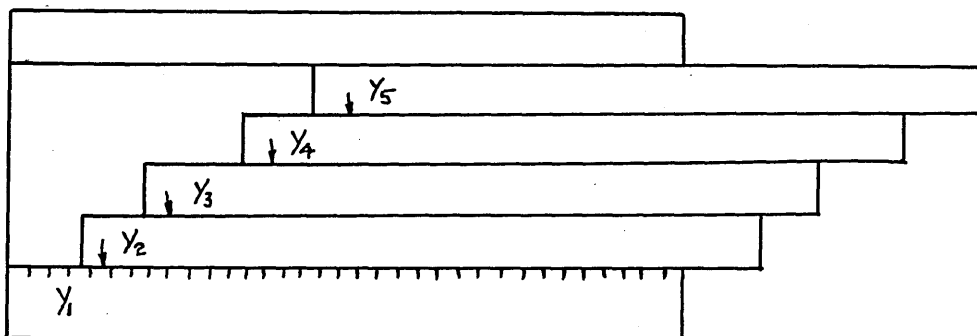


FIGURE No. 105.

The slides would each carry two scales on opposite edges. One would be $\log.\cos \theta$ or $\log.\sin \theta$ and the other would be $\log y$. Then the values of the ordinates read from the curve would be set over the corresponding angles. The final scale at the top would be fixed to the same base as the scale at the bottom and would be graduated to read the sum of the various products of $y.\sin$ or $\cos \theta$ obtained on the different slides.

This could also be accomplished by a two slide rule with a runner. The slides would carry one scale each and the bottom a series of log scales for various values of sine and cosine functions. The top scales would be arithmetic and the bottom slide would be a log scale of y . That is scales A and B would be arithmetic. Scales C and D logarithmic. Then multiplication can be done on lower scales and addition on upper scales. In operation the values of y would be successively read from the curve and multiplied by the proper trigonometric function by means of scales C and D. The index of scale B would be set to the first product on scale A, the slider then run to the second product on scale B. This would give the sum of the two. Index of B would then be moved to the runner position and the third product read on B and the runner again moved up. The final summation would be given by the final runner position and would be proportional to the coefficient desired. The range would only be limited by the number of different multiple angular functions that could be given on the scales at D. The manipulation would be laborious for more than a few harmonics. The general arrangement is shown in Fig. 106.

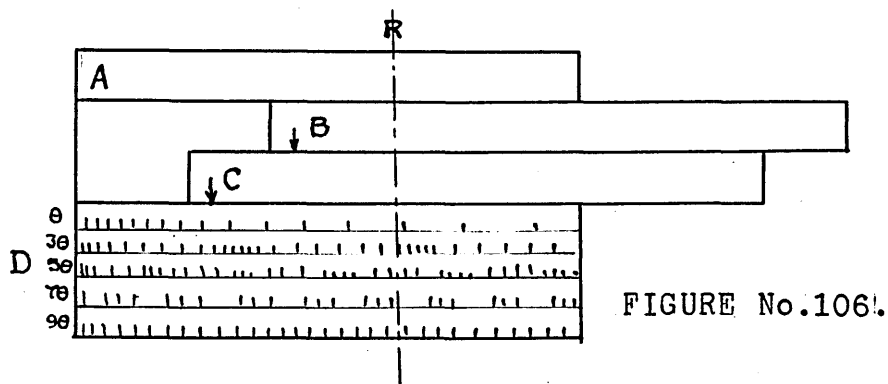


FIGURE No.106.

CALCULATING MACHINES. There are a number of calculating machines on the market which perform any arithmetical calculations by manipulating keys and cranks. It is possible to perform a schedule analysis upon many of these without putting down any figures but the results. For accurate work the schedule analysis must be done by a calculating machine. It would be possible, but very expensive with most types, to arrange the machine so that it would perform multiplication by any trigonometric function without manipulation of more than one key. The procedure with this improvement would be to set the keys to the value of the first ordinate and push the proper key for the angular sine or

cosine multiplier. This would eliminate the time necessary to crank out a multiplication of a function multiplication with several decimals, which is the largest time item involved in machine calculations. Most of the machines do not lend themselves readily to this modification, but there is one known as the "Millionaire" which should be capable of easy modification.

The general arrangement is shown in Fig. 108. There are nine parallel toothed racks ZZ and transverse axes upon which slide the pinions T, displaced by the knobs upon the face-plate. These parts make up the carrying mechanism. The recording mechanism consists of the wheels RR which transmit to the record wheels G the motion of the pinions T. The multiplying mechanism consists of nine tongue plates shown in Fig. 107. The lever H carries the proper tongue piece into line with the racks Z, and the tongue piece carriage is moved by the crank on end of arm K. The recording wheels are disengaged during the return travel of the racks. In operation suppose that it is desired to multiply 516 by 8. The knobs on top are pulled down so that the pinions T engage the 5th, 1st and 6th racks respectively. Lever H is then set to 8, which brings the 8x tongue plate in line with the racks. Rotation of the crank K thrusts the tongue plate twice against the racks Z and gives the pinions T a rotation corresponding to the displacement of the rack, which is adjusted by the tongue plate so that the rotation is 8 times the setting of the knobs. The carrying over is produced by displacement of the recording dials. It will be seen that the principle depends upon the amount the racks Z are shifted by the tongue plates, and the corresponding number of rotations of the pinions T.

It would not require very much change in construction to arrange a bank of pinions and racks representing the number of decimal places to which it was desired to carry the sine or cosine multipliers. These would all be connected to the same recording pinions so that the total displacements would add up. Tongue plates would be provided for the proper value of each bank. That is if four decimal places were used and the multiplier were .9659, then there would be four banks of racks and pinions and the successive tongue plates would be set to multiply by 9, 6, 5 and 9 respectively. The sum of the products would be given upon the recording wheels. The machine would then be cleared, the next value of y set upon the knobs and the tongue plates shifted to the proper position for the next multiplier, the product being again added to the recording dials. The final reading would be $n/2$ times the value of the coefficient desired. Great accuracy and speed could be obtained by a machine of this sort and its cost would not be greater than that of a Chubb or Henrici-Coradi analyser. From times given for straight multiplication it is estimated that coefficients up to the eleventh could be determined in about 6 seconds per coefficient after the ordinates were read from the curve.

OTHER PLANIMETER COMBINATIONS. The uses of a polar planimeter for harmonic analysis are pretty well covered by machines already described. Some other combinations are possible, but are such slight modifications of those already existing that they offer no material difference. It would be possible to increase the number of components obtained in one trace of the curve by making a multiple planimeter with a machine such as that of Le Conte. If the planimeter in this case carried a ring of wheels instead of just one wheel, and each one of the ring ran upon a platen operated from a crank at successively higher harmonic speeds, the crank being driven from the trace in the X-axis direction as with the Le Conte, then the successive wheels would read the values of the successive harmonics. Constructional difficulties would make this type either inaccurate, expensive or complicated.

No attempt seems to have been made to make use of the fact that changing the length of the arm of the planimeter harmonically will introduce the sine or cosine term desired. This could be easily done, but does not offer any apparent advantage.

The Prytz or Hatchet planimeter has not been proposed for use as an adjunct of an analyser. There are some features of this type which might make it simpler to use than the polar or rolling type with recording wheel. The Sharpe analyser might be considered as an adaptation of this type of planimeter. The knife edge may be replaced by sharp edged wheels, and if the wheels be made to run upon a track given simple harmonic motion of the proper period, the final deflection of the planimeter will be proportional to the coefficient sought.

An ELECTRIC ANALYSER based upon the Fischer-Hinnen method might be constructed, resistance wires being supported upon the joints of a "Lazytongs" frame. The wires would be strung between two members, operating together so that they would always be parallel and proportionally spaced, but the actual size of the spaces could be varied at will. Each wire is provided with a sliding contact, and alternate wires are all connected in series. All the odd wires are connected in series to one side of a bridge and all even wires to the other side of the bridge. If the frame be laid over the curve and the wires adjusted so that they divide it into the proper number of parts according to the Fischer-Hinnen rules, then the resistance necessary to balance the bridge when the sliders are made to coincide with the curve, the bases of the wires being at the X-axis, will give the coefficient for which it is set.

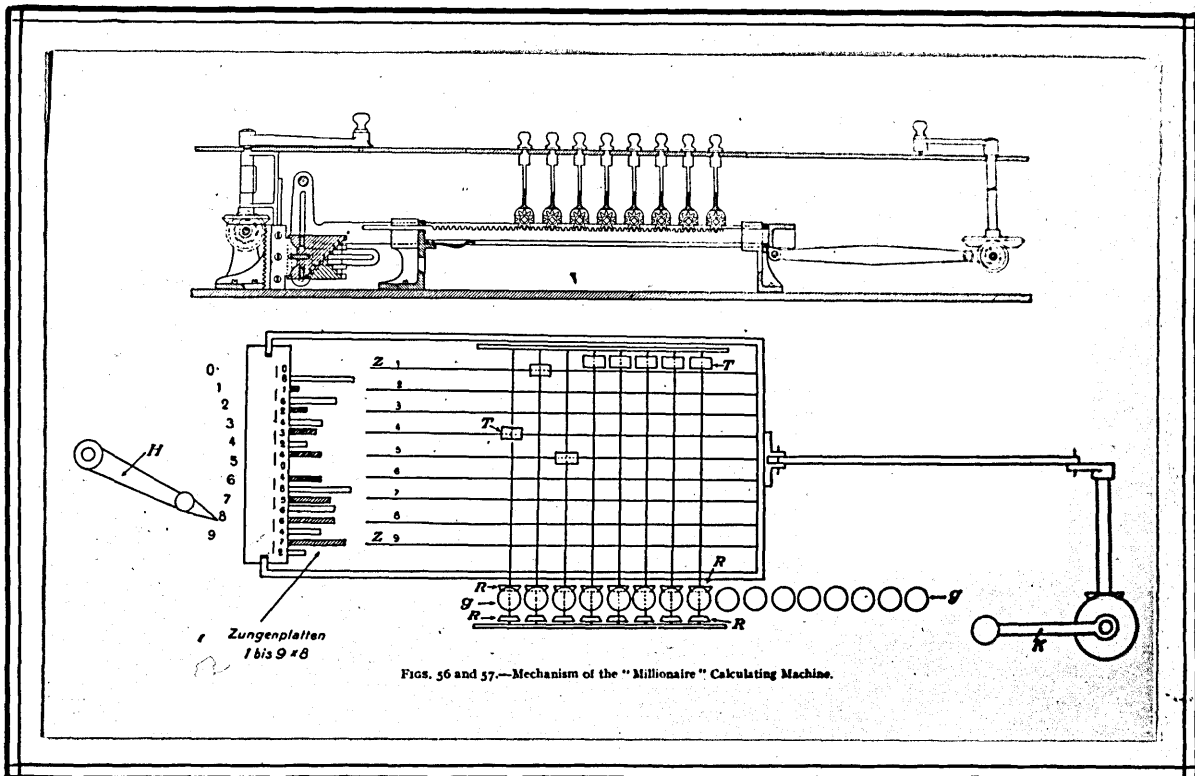
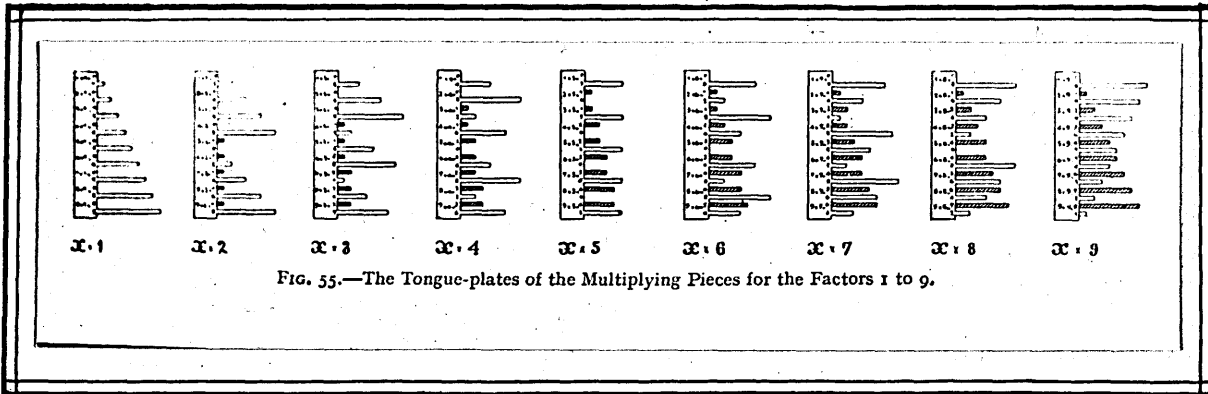


FIGURE No.107 and 108. Millionaire Calculating Machine.

DETERMINATION OF AMPLITUDE FROM COMPONENTS.

The harmonic analysis usually results in the sine and cosine components of the different harmonic coefficients being determined separately. The results are usually desired in the form of the percent of equivalent sine wave with phase angles of harmonics. Considerable labor is required to make this conversion. D.C. Miller seems to be the only one who has attempted to simplify this part of the procedure. His device for combining the two parts of the component so as to give the amplitude and phase angle is shown in figure 109.

A board is provided with scales at right angles upon two of its edges. Upon these scales slide runners carrying an hypotenuse bar, pivoted on the lower scale runner, and sliding through the upper scale runner. The lower end of the hypotenuse bar is provided with a protractor and index to indicate its angle with respect to the lower scale. The sine and cosine components are then set off on the two scales and the runners adjusted to the scalar values. The scale reading upon the hypotenuse bar will then give the true amplitude, or square root of the sum of the squares of the two components. The angle of the harmonic is indicated upon the protractor. The angle must be corrected for the signs of the components so that it gives the harmonic in the proper quadrant.

The value of the amplitude may be found even more simply by the use of an alinement chart. The two outside supports are graduated in squared scale and marked with the numbers squared. The middle support is the same to double scale. The line joining the values of the components upon the outer supports will then give the true amplitude of the harmonic upon the middle support. This does not, however, give any value of angle. A similar alinement chart may be designed to give angles from similar settings, but this requires two operations for the complete determination of the harmonic.

A simpler arrangement would be an alinement chart with the outside supports divided in such a way that the reading upon the middle support would be the ratio of the two components. This is the tangent of the phase angle, and so could have a scale graduated directly in degrees. Beside this scale there could be a scale of sines or cosines of the angles found. Then the proper component divided by this sine or cosine value would give the amplitude, which could be done by slide rule or two further supports upon the chart.

After obtaining the total amplitudes and phase angles of the individual harmonics it is generally desirable to express their magnitude as a percent of the equivalent sine wave. In order to do this the square root of the sums of the squares of all the harmonics must be determined, which gives the amplitude of the equivalent sine wave, and the various harmonic coefficients or amplitudes are then divided by this quantity to obtain their value in percent. In this way it is not usually necessary to have the absolute value of any of the quantities, but merely their relative values, the total ratios being comparable to meter readings when reduced to equivalent sine wave percentages as above.

The procedure of taking the square root of the sum of the squares of a large number of harmonics is rather tedious, and does not seem to have been considered by any of those working with the processes of harmonic analysis. There is no very evident method for doing it easily, but several ways suggest themselves. Thus an alignment chart could be made with a number of supports representing the various harmonic amplitudes. This usually becomes rather complex if more than three or five supports are used however. A slide rule could be devised which would be divided as an addition rule and marked with the square roots of the numbers used for divisions. The final reading could be transferred to a square root scale, and the series of sums worked out upon it in the same way as a series of multiplications is worked out upon the ordinary form of slide rule.

It thus appears that in addition to the actual determination of the components of the harmonic coefficients in any complex wave it is also necessary to perform two further operations which require an amount of time equal, or perhaps greater than the analysis itself, and that therefore attention should be directed towards the simplification of these processes as well as the actual analysis.

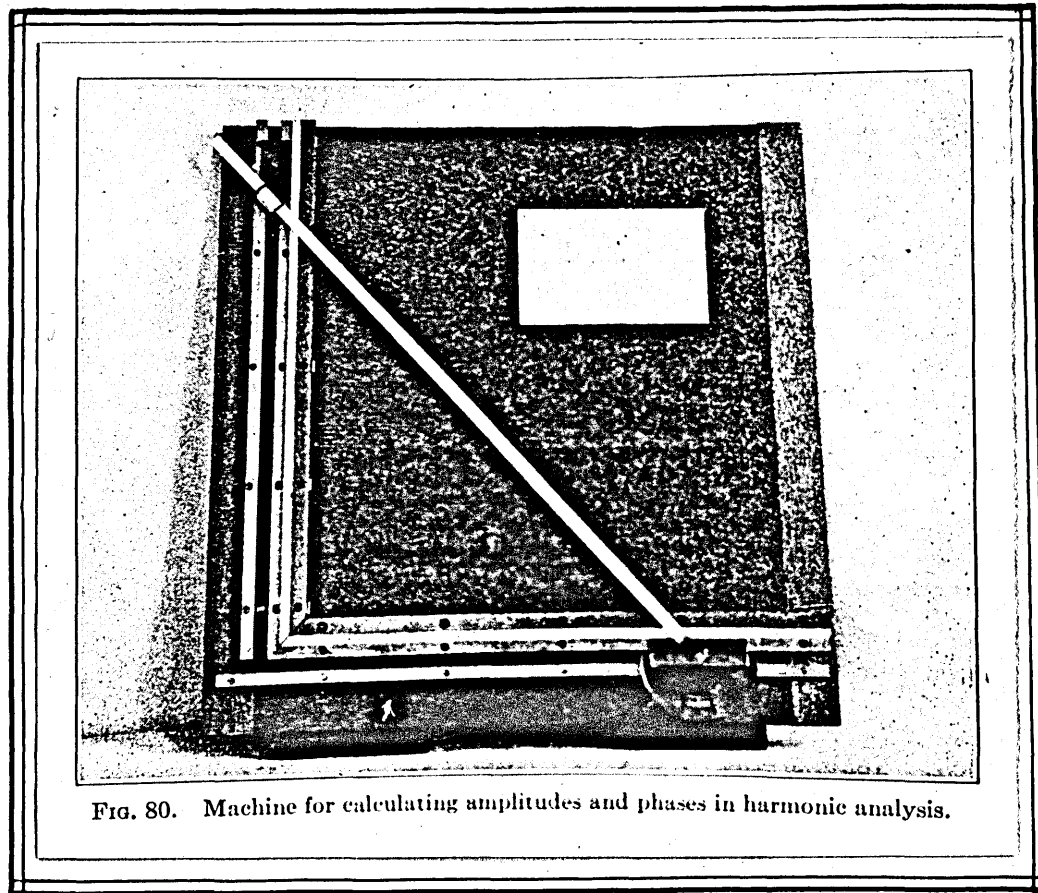


FIG. 80. Machine for calculating amplitudes and phases in harmonic analysis.

FIGURE No. 109. Device for Calculating the
Amplitudes and Phases of Harmonic Coefficients
from the Sine and Cosine Components.

DATA ON THE TIME REQUIRED FOR HARMONIC ANALYSIS.

METHOD.	TIME.	NO. HARM- ONICS DE- TERMINED.	MINUTES PER COEFF.	AUTHORITY.
Steinmetz	10hrs.	10	60	D.C. Miller.
Schedule	3hrs.	8	22.5	D.C. Miller.
Schedule	1hr.	8	7.5	F.W. Grover.
Schedule	2.5hrs.	17	10.6	Author.
Schedule	15mins.	3	5	D.C. Miller.
Coradi Mch.	13mins.	10	1.3	D.C. Miller.
Coradi Mch.	7mins.	5	1.4	D.C. Miller.
Schedule	30mins.	6	5.0	Author.
Electric Mch.	3.5mins.	6	0.6	Author.
Woodbury Mch.	5mins.	3	1.7	E.J. Arnold.
Rowe Mch.	2hrs.	8	15.0	G.H. Rowe.

The times are not all strictly comparable. The Steinmetz results are probably the most accurate possible for a mathematical method. The schedule methods of D.C. Miller and F.W. Grover do not check well at all, although the same schedule was used in both cases. This is probably because Grover was very familiar with the schedule method and Miller did not use it to any very great extent. Although it is not definitely stated, the Coradi machine times do not appear to include the time required to prepare the curve of the proper base, 16" in this case. The time for the electric machine is taken from the moment of beginning to set the ordinates upon the machine. The time upon the Woodbury machine is approximate, but includes all preparation since the readings are made directly from the oscillograph film, which is placed upon the machine. The time for the Rowe machine, while given by its originator, looks too long, and it is probable that the curve was followed very slowly, since the time should compare favorably with the Coradi machine. Since the latter obtains five harmonics with one setting, the Rowe machine should take five times as long, whereas the time given is 10 to 12 times as long.

The schedule analyses by the author were done upon a Marchant Calculating Machine. While greater speed is possible with a slide rule, the errors are very greatly increased. In general the time required increases rapidly with the number of harmonics determined. In the schedule types the time increases about as the square of the number of harmonics, while with the mechanical analysers it increases about directly with the number of harmonics.

D I S C U S S I O N A N D C O N C L U S I O N S.

DISCUSSION AND COMPARISON OF METHODS OF
HARMONIC ANALYSIS.

Since the Fourier Integral evaluating the coefficients of Fourier's Series represents the area of the curve obtained by the product of $f(x)$ with a sine wave, the most evident procedure is to measure this area with a planimeter and provide a machine for tracing the area with one of its members which carry the planimeter, while the stylus traces the actual curve. This is the basis of most of the analysers proposed or built by the various men mentioned. The machines due to Chubb, Yule, Bush, Le Conte, Boucherot, Mader, Rowe and some others all operate upon this general principle and involve the use of a standard form of planimeter.

This system involves several principle defects. First the curve must be traced at least once for each coefficient determined, and usually twice to determine the sine and cosine coefficients separately. This means a good deal of labor if more than one coefficient required. It may also introduce different errors in each coefficient determined due to not following the curve closely. The machine cannot be combined to give more than one coefficient at a time without extreme complication.

Second, in nearly all designs the curve must be of a definite base length per period, which in most cases will require enlargement or reduction, either photographically or by pantograph, from the original curve. If the base length is short large errors are introduced in the higher harmonics and if it is made long, bulky curve tracings are required, and if obtained photographically will require rather cumbersome and expensive equipment. The pantograph might be included in the machine so that it could be set to take any curve. This does not appear to have been tried, but would not seem to complicate the device unduly, and involves no great difficulty. The Chubb analyser obtains long base length of curve in small space by using a polar form which is compact and easy to apply to the analyser. It also offers other advantages, but the Cartesian form of curve gives so much better physical idea of the function that it is generally to be preferred.

On account of these difficulties attempts have been made to either do away with the planimeter or modify it for better adaptation to the task in hand. The most successful example of this is the Henrici-Coradi machine, where a special form of planimeter is used. On account of its special design five sine and five cosine coefficients can be determined with one trace, and adjustments easily made for other values. There can be little criticism of the ability and speed of the Henrici machine, but it is complicated, requires workmanship of the very highest grade, is very expensive, and must be very care-

fully taken care of. It is therefore out of reach of the ordinary laboratory or test room.

The Wiechert and Sommerfeld machine is a departure from the planimeter practice, but involves such a complicated motion that its mechanism prevents it from competing with the types using a standard planimeter, since it does not involve any advantages.

The Terada analyser falls in much the same class as the last mentioned, but from complications in operation rather than mechanism. Several traces must be made from the curve, and it is difficult to obtain the correct adjustment from the curve without knowing something about the harmonics to begin with. The results obtained are in the form of a curve which must be measured to determine its amplitude etc, and not in the form of definite figures or values for the coefficient.

The Sharpe analyser is possessed of one definite advantage. The sine, cosine and combined coefficients for any harmonic are determined by one trace, together with the phase relation of the harmonic. It also does not involve the use of a planimeter and does not require a preliminary setting. All of these points are very much in its favor, and the only objection over the other analysers mentioned is the probability of some mechanical difficulties in making the linkages operate without too much friction.

This covers all of the direct analyser types, and the Henrici Coradi is preeminently the best. For use with the polar oscillograph the Chubb analyser is very satisfactory and accurate, but requires a large amount of time, which may be lessened for the operator by using a motor to drive the machine, but does not decrease the time required to actually get the results, so that it is not a saving if there is no further work to do until the analyses have been obtained. The Bush analyser is undoubtedly the simplest, and is excellent for determining one or two harmonics, but becomes laborious if many harmonics are necessary. The Sharpe analyser appears to be worth further investigation to see if the mechanism cannot be improved so that it will determine several components for one trace, and operate with less difficulty.

All of the analysers thus far mentioned require a curve of definite period length, and only determine coefficients of one frequency, with the exception of the Henrici-Coradi. In order to obviate the time required for converting the curve to some definite size, and of making repeated traces, the Dellenbaugh Electric analyser was developed. The values are read from the curve and then set on the machine so that any curve may be used. It can be arranged to give any number of harmonic coefficients from one setting, by manipulation of gang switches.

It may also be manufactured out of materials found in any spot connected with electrical experimental work. Thus it obviates a number of the objections to harmonic analysers as a whole. However, it transfers the mechanical complexity of the Henrici-Coradi machine in electrical complexity, which increases rapidly with the number of harmonics required. It also can only be used within the range for which it is built, and does not give as great accuracy as other methods. While the cost is small for one to cover only a few harmonics, the cost rapidly increases with larger numbers of harmonics included owing to the resistances required and the complicated switching arrangements. Its accuracy could probably be improved by more careful construction, but it would always retain the tantalizing habit of electrical networks for indulgence in open and short circuits at critical moments. Therefore, while it appears on paper to have solved the problem for a simple and rapid analyser, it does not actually accomplish these results unless the harmonics desired would never be greater than the 5th or 7th, a condition which is not met in practice.

The further investigation by D.O. Woodbury developed an analyser which seem to come nearer to the ideal than any yet proposed. The machine is simple to make and requires no expensive parts regardless of the number of harmonics required. While it requires a fixed base for the curve, a new notching bar can be easily and quickly made for the curve base desired, so that it is not limited by internal structure to one size of curve as with many of the analysers, but only by the spacing of the notches upon one bar. These can be combined upon a bar of small size so that many harmonics can be determined for one bar, up to the 21st for odd harmonics being included in the bar shown in connection with its description. It still retains the disadvantage of having to make separate settings from the curve for each harmonic, and for each sine and cosine coefficient, but owing to the Fischer-Hinnen method used as a basis this labor is small for the lower order harmonics, and also only requires the location of definite points on the curve and not the accurate trace of the whole cycle.

The Michelson and Stratton Analyser has purposely been left until the last for discussion. This machine is fundamentally a synthesizer, but its originators have shown that it may also be used for analysis. In this case the ordinates of the curve are set upon the machine and it is then caused to draw a curve whose ordinates at definite points give the coefficients. One curve must be drawn for sine and one for cosine coefficients. Thus it complies with two desirable features: The base of the curve is immaterial and the values of all coefficients are given with one trace for one of the component parts. Two traces are necessary for complete analysis. But this is an automatic trace and does not require any care on the part of the operator, and is connected with the original curve only through the first setting of the ordinates upon the machine.

The machine as made by its originators consists of 80 elements, which will cover almost any analysis intended to be made, but requires a good deal of labor in setting the machine from the curve ordinates. For this purpose a template of the curve may be cut out and the links of the machine pressed against the template, but this means reducing the curve to a definite period length. For most analyses a smaller number of elements would be satisfactory.

It is interesting that in various discussions of this machine it is pointed out as one of its remarkable characteristics that it may be used as either a synthesizer or analyser, while as a matter of fact any synthesizer may be used as an analyser. This point appears to have been generally overlooked. It is mentioned by Michelson and Stratton of course, in connection with their machine, and the basis is given in the discussion of their machine. It might be worth while to give a little fuller discussion of this inverse property of synthesizers in order to investigate the limits of its use.

If the curve to be analysed is divided up as for schedule analysis, values of ordinates y_1 y_2 y_3 etc. are obtained at positions θ_1 θ_2 θ_3 etc. The value of the fundamental sine component will be given by :

$$a_1 = y_1 \sin \theta_1 + y_2 \sin \theta_2 + y_3 \sin \theta_3 \text{ etc.}$$

But if the division of the curve has been in even spaces,

$$\theta_2 = 2\theta_1 \quad \theta_3 = 3\theta_1 \text{ etc.}$$

$$\text{Thus: } a_1 = y_1 \sin \theta_1 + y_2 \sin 2\theta_1 + y_3 \sin 3\theta_1 \text{ etc.}$$

If the values y_1 y_2 y_3 etc. are set upon the successive harmonic wheels of a synthesizer, then the curve drawn by the synthesizer will be, for wheels starting at $\theta=0^\circ$:

$$y = y_1 \sin \theta + y_2 \sin 2\theta + y_3 \sin 3\theta \text{ etc.}$$

Thus if $\theta = \theta_1$ the ordinate of the curve will be the same as the evaluation of the sine fundamental coefficient, and the analysis has been made.

For the value of the second harmonic it is necessary to determine the value of the expression:

$$a_2 = y_1 \sin 2\theta_1 + y_2 \sin 2\theta_2 + y_3 \sin 2\theta_3 \text{ etc.}$$

Which can be transformed as before into:

$$a_2 = y_1 \sin 2\theta_1 + y_2 \sin 4\theta_1 + y_3 \sin 6\theta_1 \text{ etc.}$$

Therefore if the point of the synthesized curve $\theta = 2\theta_1$ be taken, the value of the ordinate will be the value of a_2 . In this way all the sine term coefficients may be evaluated from the synthetic curve, by reading the ordinates at points along the X-axis equivalent to θ 2θ 3θ etc. In all cases θ is taken as the angle of the fundamental synthesizer wheel or gear. For the sine components all gears are started at 0° . For the cosine components it is only necessary to repeat the synthesis of the curve with all harmonic wheels or gears starting at 90° instead of 0° . The ordinates of the curve at the proper values of θ will then give the cosine component coefficients.

The required number of elements in a synthesizer for a given analysis will depend upon similar rules to those used in schedule analysis. The value of θ_1 must be divisible into 360° the same number of times as the order of the highest harmonic plus one, since otherwise there would not be enough points in the cycle synthesized to determine the harmonics. For odd harmonics only a half wave of the curve to be analysed may be taken, but then only a half wave of the synthesized curve can be considered. Thus if $\theta_1 = 30^\circ$ up to the 11th harmonic may be determined for a whole wave, but only to the 5th for a half wave. The synthesizer must contain the even as well as the odd harmonic wheels in order to be used as an analyser, and thus will have twice the number of elements for analysis as for synthesis where only odd harmonics are considered, and the same number for both purposes where both even and odd harmonics exist. Obviously the synthesizer may have as many more elements above the necessary minimum as desired.

CONCLUSIONS.

It is amazing the number of methods of harmonic analysis available all subject to rather serious objections.

Mathematical methods are extremely laborious. Direct reading methods, such as resonance type, are not very accurate and require considerable power from the source of supply to operate, which may distort the wave from its natural form. Machines are available in many forms, but practically all of them require a definite base of curve for one period, are limited to one component per trace, and so require many traces of curve, and consist of rather complicated machinery.

The present best known types are the Henrici-Coradi, the Chubb-Westinghouse and the Michelson Stratton. The first is undoubtedly the best for analysis, is speedy, but expensive, difficult to obtain and requires great care to maintain. The second is rather expensive, requires polar curves for analysis, and is not very speedy, but the mechanism is extremely rugged. The last is expensive, but can be made in a greater number of elements than the other types. It will act as both synthesizer and analyser, and will give all harmonic components coefficients with two synthesized curves from one set of readings of ordinates from the curve.

Three new types of harmonic analysers have been recently introduced, those of Bush, Dellenbaugh and Woodbury. The first is undoubtedly the simplest yet constructed, and is ideal for the determination of one or two harmonics whether of high or low period. It has the advantage over the other two that any harmonic determined will be correct, within the limitations of accuracy of the machine, regardless of the other harmonics present. The Dellenbaugh Analyser is possible of extension and gives great speed in obtaining the coefficients, but suffers from the usual difficulties of hastily made electric circuits. It is also limited in the same manner as the schedule method upon which it is based. The Woodbury machine appears to be a very good all around machine, and approaches nearer the ideal than most of the other types. It is cheap to build, easy and quick to operate, reasonably accurate and not limited in application. It operates upon the basis of the Fischer-Hinnen method of analysis, and so a complete analysis must be made, the individual determinations of harmonics not being possible.

There is still much room for improvement, and it would appear that the planimeter type of analyser is fundamentally unsatisfactory. Thus the probable best direction to look for improvements is along the lines of machines such as the Michelson and Stratton, the Sharpe and the Woodbury.

A P P E N D I X I.

E X A M P L E S O F W A V E S H A P E A N D C O N V E R -
G E N C E.

EXAMPLES OF WAVE SHAPE AND CONVERGENCE OF SERIES.

Figs. 110 and 111 (C.E. Magnusson, Alternating Currents, McGraw-Hill, 1916). These two sets of curves show the change in wave shape due to the addition of a third or a fifth harmonic at various phase relations with the fundamental. These are usually the most prominent harmonics found in alternating current waves. The data for each curve is given in the figure.

Fig. 112 (D.C. Miller, The Science of Musical Sounds, The Macmillan Co, 1916). This curve shows a typical curve containing both even and odd harmonics, with the harmonic components separately drawn. There is also a constant term represented by the distance between the lines a'b' and ab. a'b' is the true or geometrical axis. The equation of this curve is:-

$$\begin{aligned}
 y = & b_0 + 96.5\sin(\theta + 76^\circ) + 66.0\sin(2\theta + 319^\circ) \\
 & + 36.5\sin(3\theta + 337^\circ) + 19.2\sin(4\theta + 354^\circ) \\
 & + 10.3\sin(5\theta + 330^\circ) + 8.4\sin(6\theta + 347^\circ) \\
 & + 6.4\sin(7\theta + 354^\circ) + 8.9\sin(8\theta + 290^\circ) \\
 & + 4.3\sin(9\theta + 252^\circ) + 2.3\sin(10\theta + 252^\circ) \\
 & + 2.2\sin(11\theta + 230^\circ) + 1.5\sin(12\theta + 211^\circ).
 \end{aligned}$$

Later synthesis of this equation (not given here) shows practically exact coincidence with the originally recorded curve.

Fig. 113 (D.C. Miller, loc. cit.) This page gives the analysis of an entirely arbitrary curve, i.e. the profile of a photograph of a head. The curve headed O is the original and the curve headed S is a synthesised copy from the equation, given on the bottom of the page. Of course if the synthesis is continued the curve repeats indefinitely, the part shown representing one period,

Figs. 114 and 115. (W.E. Byerly, Fourier's Series and Spher-

ical Harmonics, Ginn and Co, 1893>) These two pages show the convergence of Fourier's Series with increasing numbers of terms. It will be noted that the convergence is much quicker with some types of functions than with others. The four here illustrated are all geometrical figures. Each curve gives the first four approximations. The equations of the different curves are:-

$$I. y = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots$$

$$y=0, \text{ when } x=0, y=\pi/4, \text{ from } x=0 \text{ to } x=\pi; y=0, \text{ when } x=\pi$$

$$II. y = 2[\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots]$$

$$y=x \text{ from } x=0 \text{ to } x=\pi, y=0 \text{ when } x=\pi$$

$$III. y = 4/\pi[(1/1^2)\sin x - (1/3^2)\sin 3x + (1/5^2)\sin 5x - (1/7^2)\sin 7x + \dots]$$

$$y=x \text{ from } x=0 \text{ to } x=\pi/2; \text{ and } y=(\pi-x) \text{ from } x=\pi/2 \text{ to } x=\pi$$

$$IV. y = \frac{1}{2} \sin x + \frac{2}{3} \sin 2x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{2}{6} \sin 6x + \frac{1}{7} \sin 7x + \dots$$

$$y=0 \text{ when } x=0, y=\pi/2 \text{ from } x=0 \text{ to } x=\pi/2, y=0 \text{ from } x=\pi/2 \text{ to } \pi$$

Fig. 116. (D.C. Miller, loc. cit.) This shows similar approximations to case II. immediately above; except that the curves were actually drawn upon a synthesiser, and are carried to 10 terms. It is interesting to note that the form of each term is the same as Case II, but the sign of the even terms has changed from - to +, which has the effect of reversing the curve from left to right. It is easy to see, in case I, for instance, why the derivative will not be the same as the actual derivative, for the slopes of the approximating curve are entirely different from the actual curve, but the areas and thus integrals will be approximately the same.

Fig. 117 (A.A. Michelson and S.W. Stratton, A New Harmonic Analyser, Philosophical Magazine, Vol. 45, p. 85, 1898>) This series of curves shows a still closer approximation, being carried to 79 terms. The exact equation is not given, but it is undoubtedly the same as that for Case I, Fig. 114. The slight kink at

points $\pi, 2\pi,$ and 3π appears to be due to a mechanical defect in the synthesiser, since it also appears in the fundamental.

Figs. 118 and 119. (D.C. Miller. loc. cit.) These pages show four characteristic curves obtained from a synthesiser two of them again being similar to those already shown but having a different number of terms. Comparison of Figs. 114, 117 and 119 will give a very good idea of the degree of approximation that can be obtained with a given number of terms for a rectangular wave. It is probable that the synthesiser of D.C. Miller works with greater accuracy than that of Michelson and Stratton, although the latter has a greater number of elements, and is also adaptable as an analyser as well.

Figs. 120 to ~~130~~ 131 show the equations for a number of wave shapes that are commonly met with in alternating current practice, the majority being taken from ~~Bragstad and La Cour~~ P. Janet.

Figs. 132 to 136 inclusive, show a great variety of waves that may be obtained from different series, and were drawn by a Michelson and Stratton Synthesiser. (loc. cit.) It is interesting to note the way in which one type of wave may be superimposed upon another by the additions of their series. Thus curve 15 is the sum of curve 3 and curve similar to curve 7.

THE EQUATION OF A SEMICIRCULAR WAVE FORM.

The coefficients of the series expressing a semi-circular wave form maybe written in the following form:

$$\begin{aligned} & \pi^2/4 \left\{ \sin \alpha \sum_{n=0}^{\infty} (-1)^{n+1} / n! n-1! (\pi/4)^2 (n-1) \right. \\ & - \sin 3\alpha \sum_{n=0}^{\infty} [(-1)^{n+1} / n! n-1!] [(\pi/4)3]^2 (n-1) \\ & \left. + \sin 5\alpha \sum_{n=0}^{\infty} [(-1)^{n+1} / n! n-1!] [(\pi/4)5]^2 (n-1) \dots \dots \dots \right. \end{aligned}$$

The values of the first three terms calculate:-

$$y = 1.781 \sin \alpha - 0.2948 \sin 3\alpha + 0.1332 \sin 5\alpha - \dots \dots \dots$$

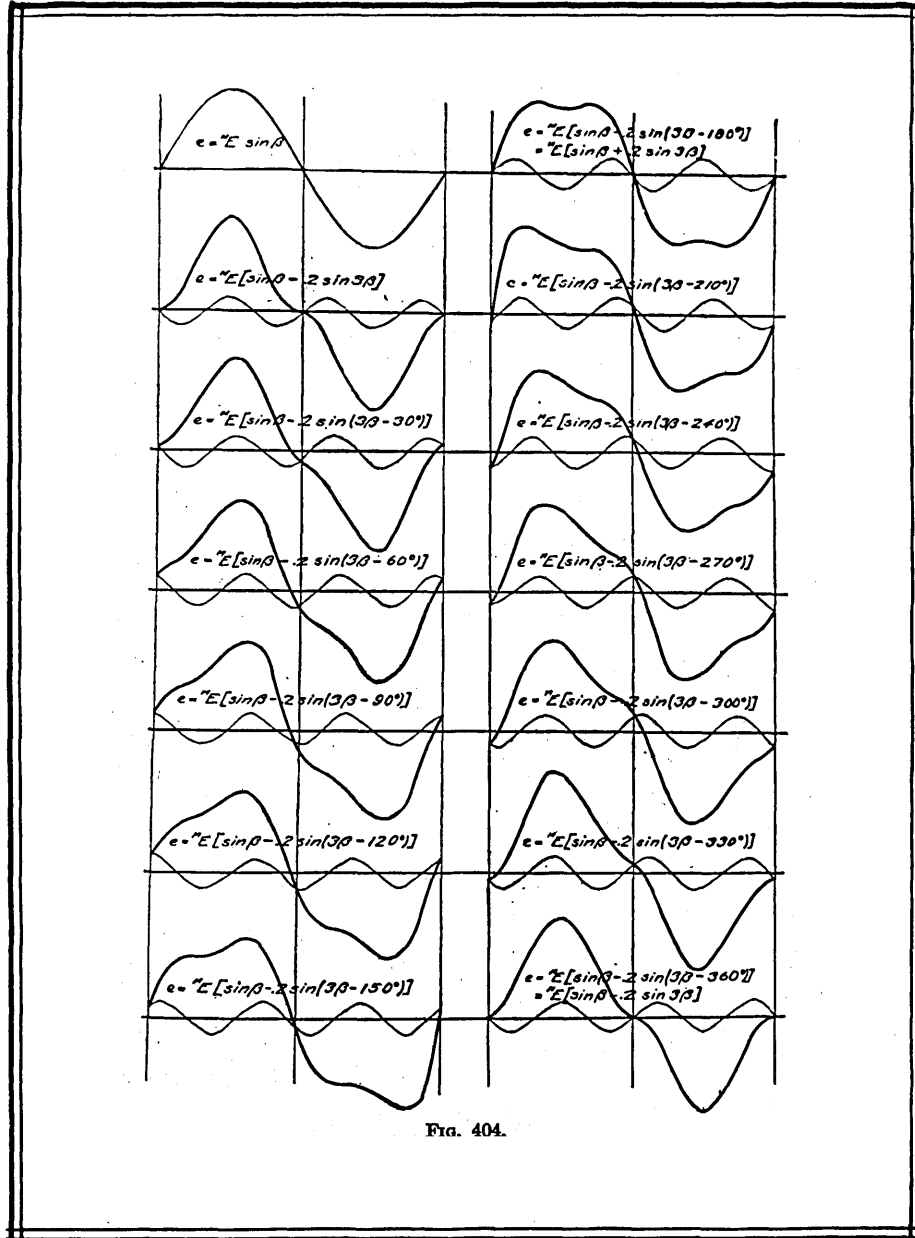


FIG. 404.

FIGURE No. 110.

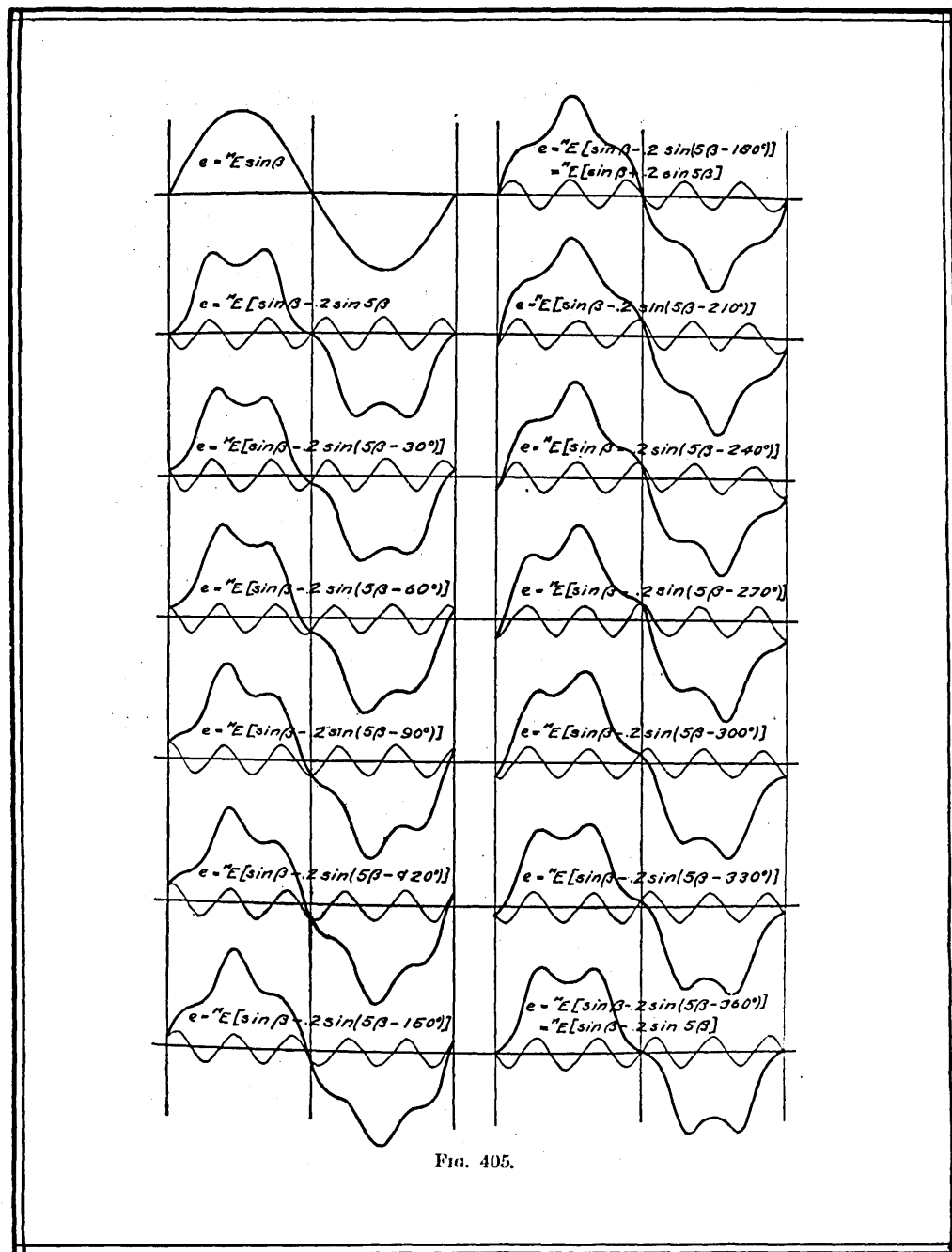


FIG. 405.

FIGURE No. 111.

ANALYSIS AND SYNTHESIS OF HARMONIC CURVES

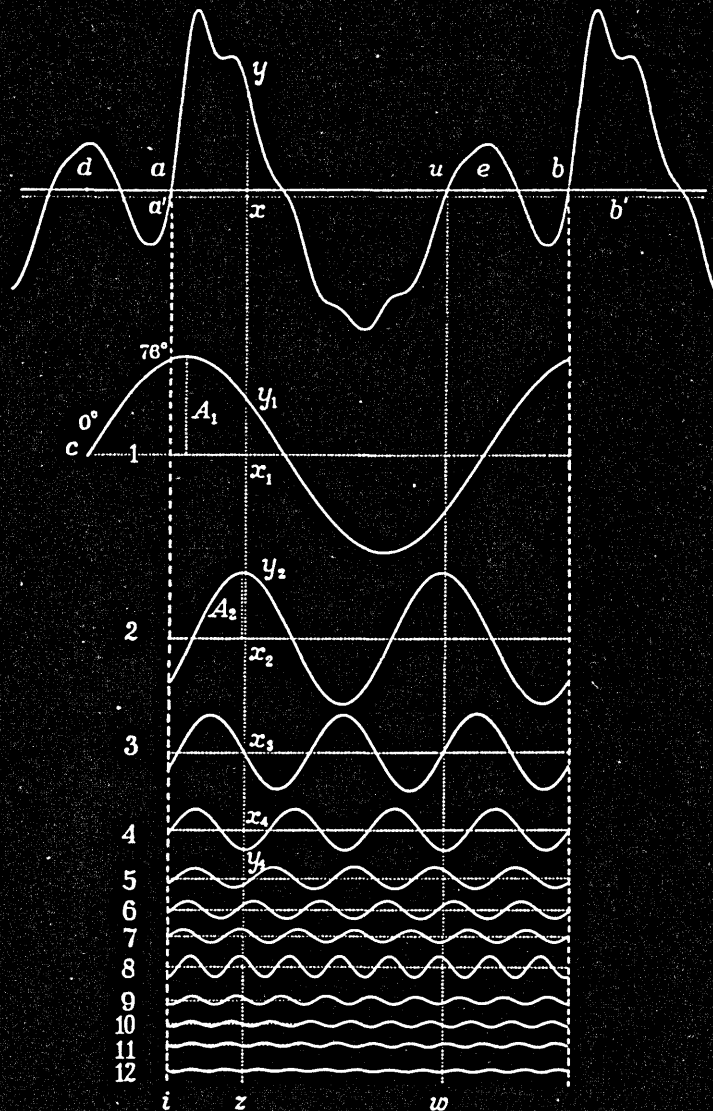


FIG. 98. An organ-pipe curve and its harmonic components.

FIGURE No. 112.

ANALYSIS AND SYNTHESIS OF HARMONIC CURVES

profile is given at the left, *O*. The curve was analyzed to thirty terms, but the coefficients of the terms above the eighteenth were negligibly small. The equation of the

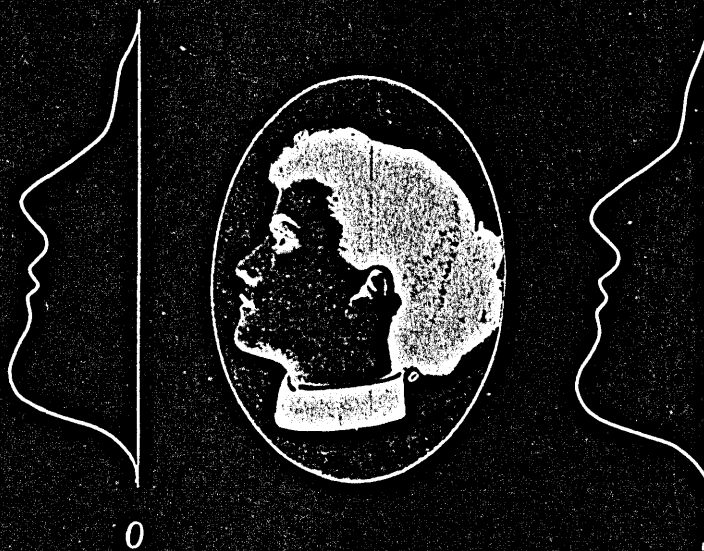


FIG. 94. Reproduction of a portrait profile by harmonic analysis and synthesis.

curve is as follows, the numerical values corresponding to a wave length of 400 :

$$\begin{aligned}
 y = & 49.6 \sin (\theta + 302^\circ) + 17.4 \sin (2\theta + 298^\circ) \\
 & + 13.8 \sin (3\theta + 195^\circ) + 7.1 \sin (4\theta + 215^\circ) \\
 & + 4.5 \sin (5\theta + 80^\circ) + 0.6 \sin (6\theta + 171^\circ) \\
 & + 2.7 \sin (7\theta + 34^\circ) + 0.6 \sin (8\theta + 242^\circ) \\
 & + 1.6 \sin (9\theta + 331^\circ) + 1.3 \sin (10\theta + 208^\circ) \\
 & + 0.3 \sin (11\theta + 89^\circ) + 0.5 \sin (12\theta + 229^\circ) \\
 & + 0.7 \sin (13\theta + 103^\circ) + 0.3 \sin (14\theta + 305^\circ) \\
 & + 0.4 \sin (15\theta + 169^\circ) + 0.5 \sin (16\theta + 230^\circ) \\
 & + 0.5 \sin (17\theta + 207^\circ) + 0.4 \sin (18\theta + 64^\circ).
 \end{aligned}$$

This equation was set up on the synthesizer, and the portrait, as drawn by the machine, is shown at the right, *S*, Fig. 94.

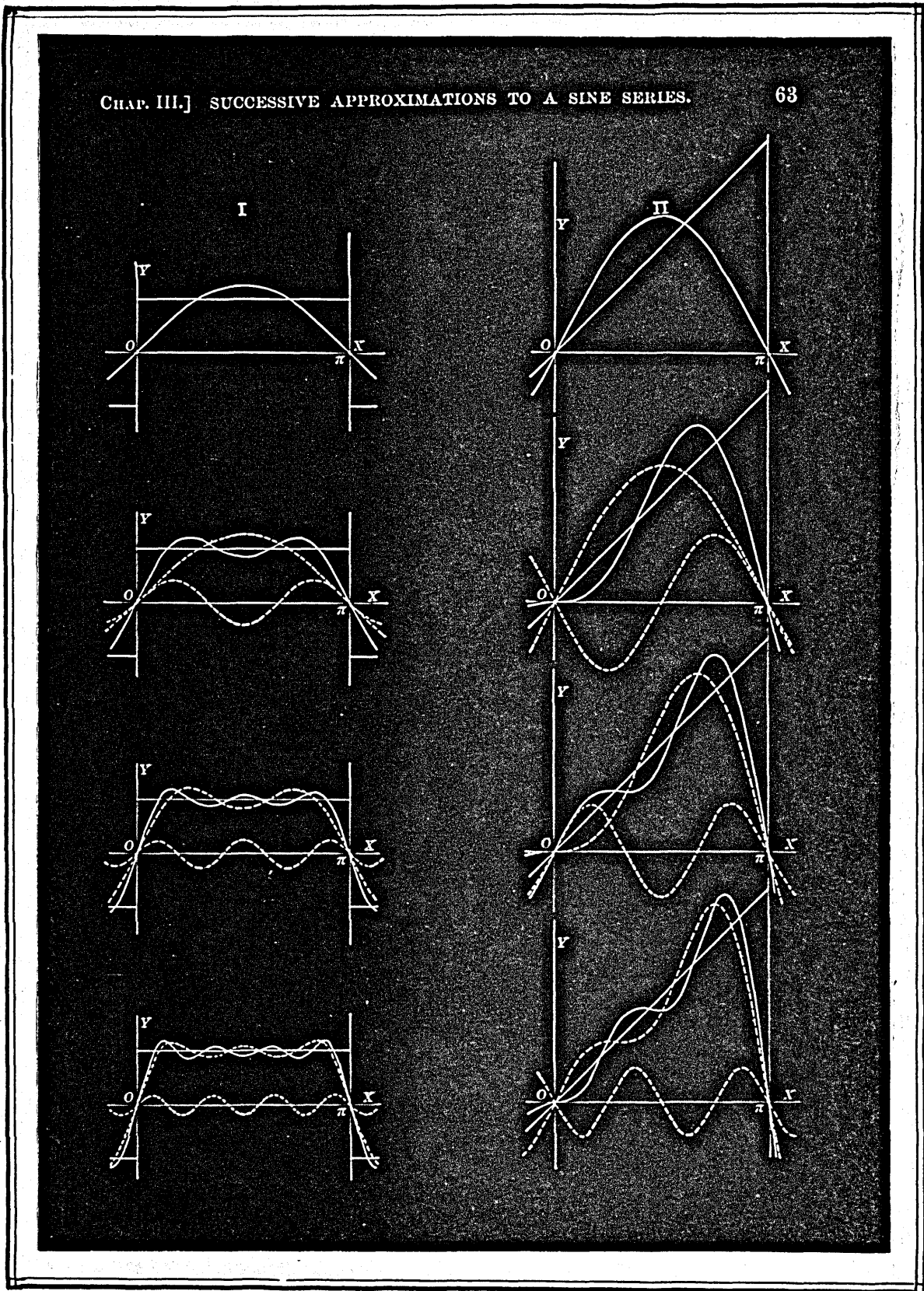


FIGURE No.114.

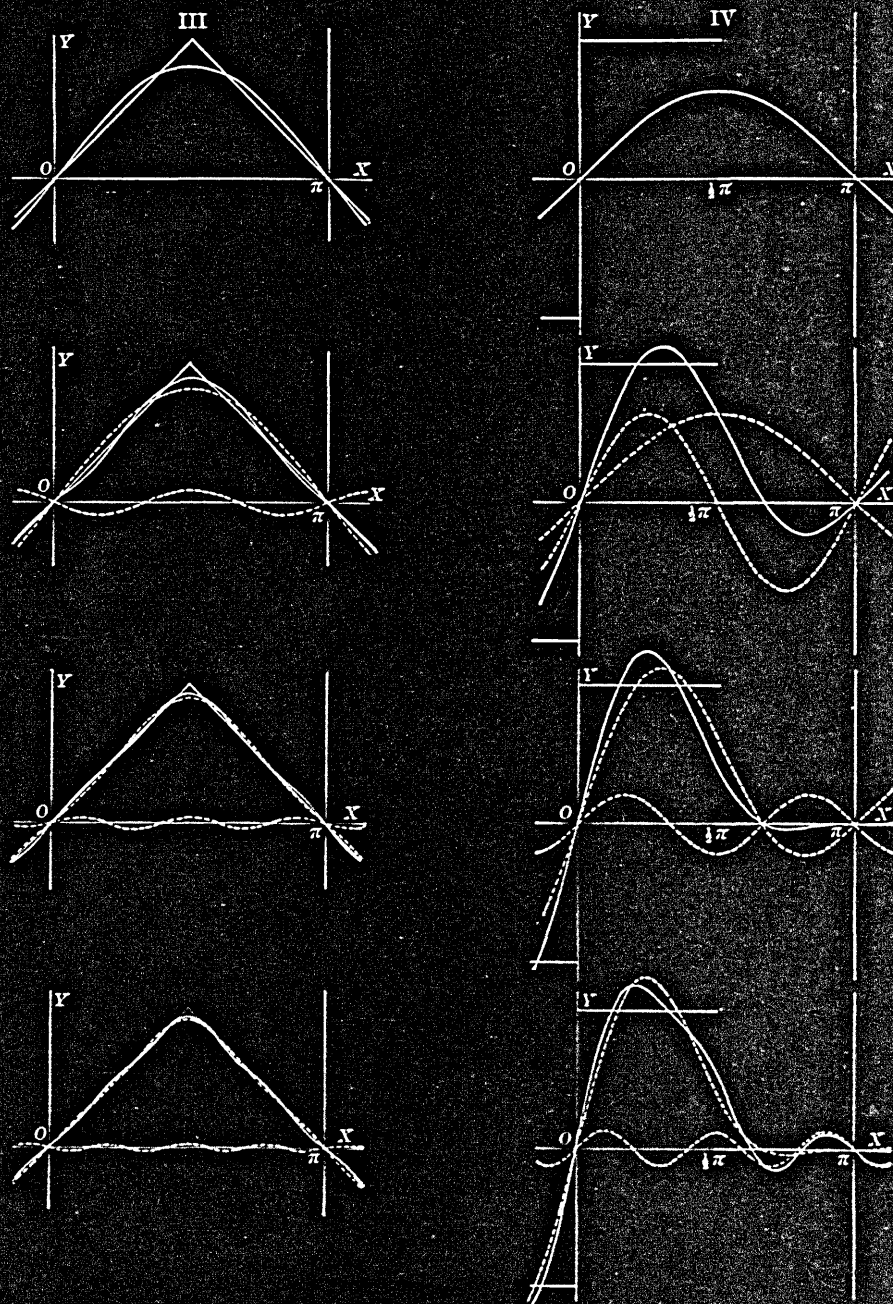


FIGURE No. 115.

THE SCIENCE OF MUSICAL SOUNDS

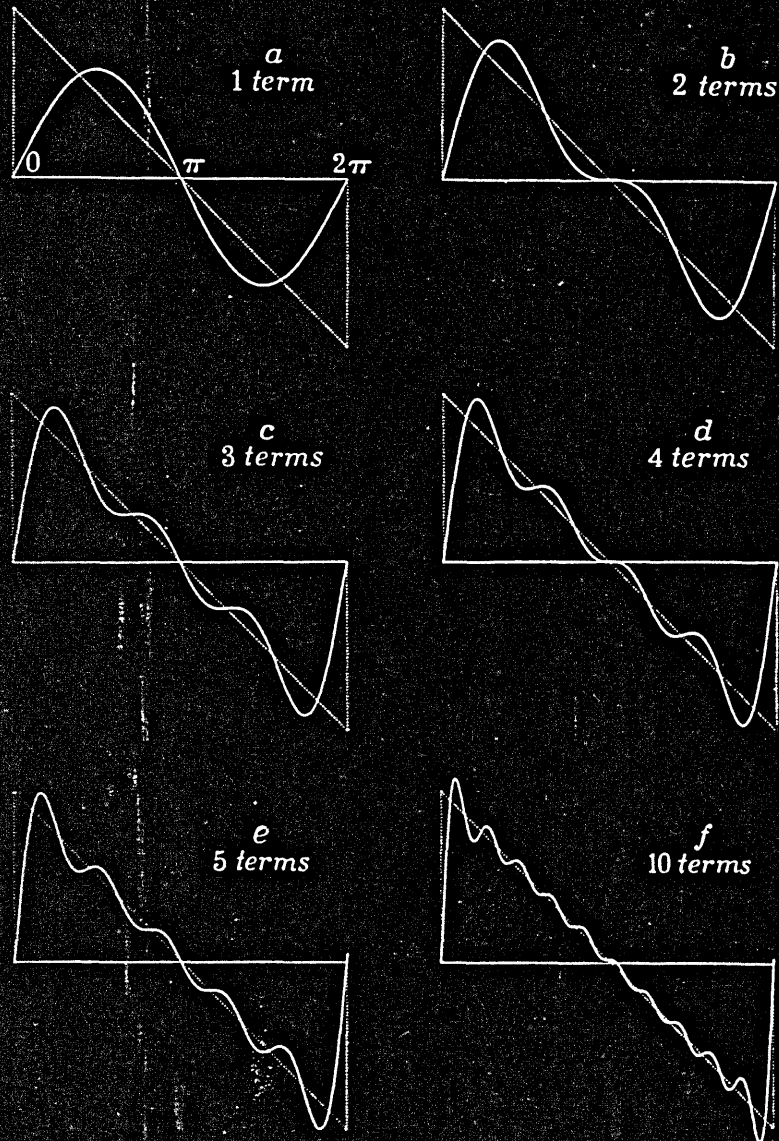


FIG. 89. Forms obtained by compounding 1, 2, 3, 4, 5, and 10 terms of the series
 $y = 2[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots]$.

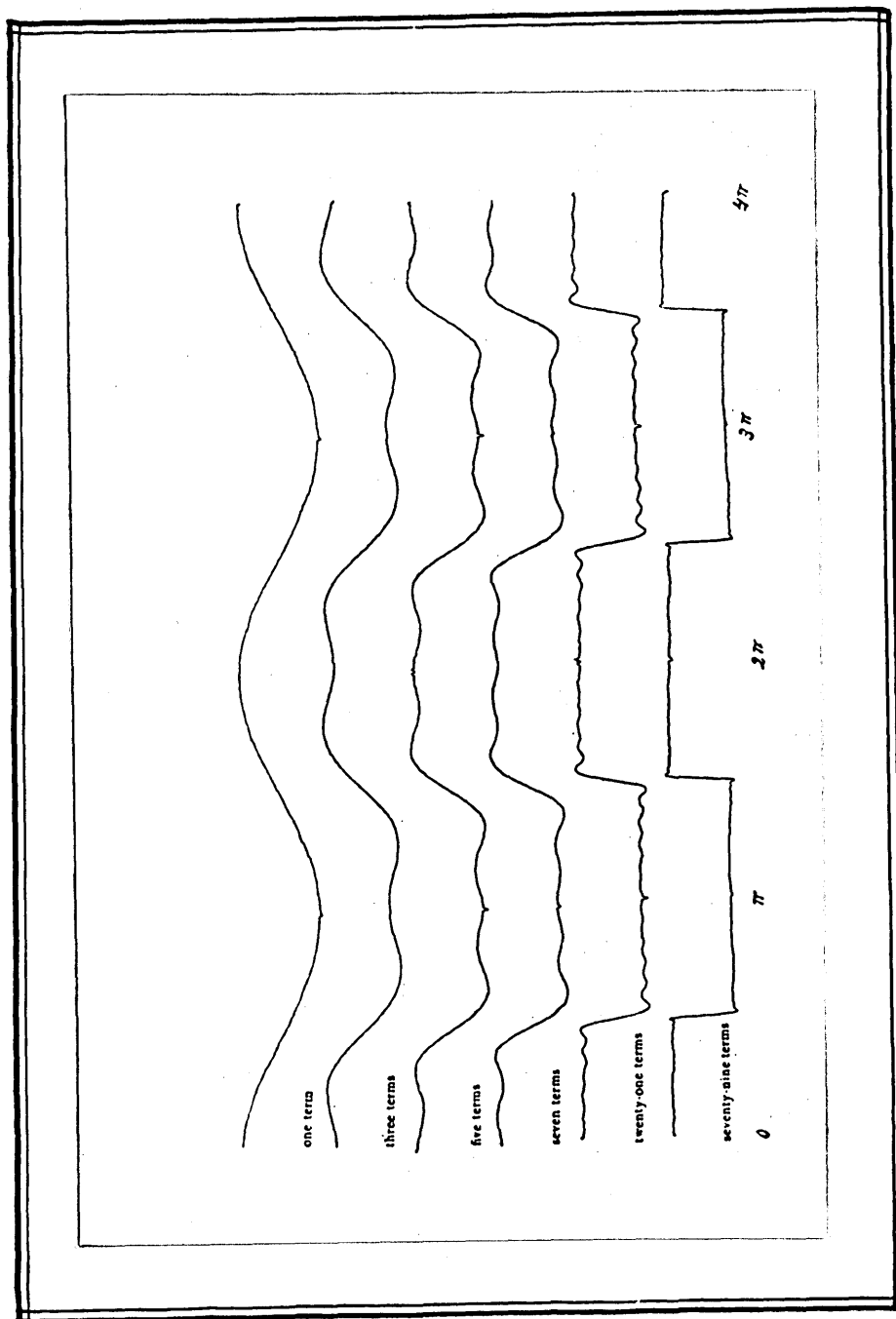


FIGURE No. 117.

ANALYSIS AND SYNTHESIS OF HARMONIC CURVES

Fig. 91 is a curve made up of the same components as enter into the curve shown in Fig. 90; the only difference is

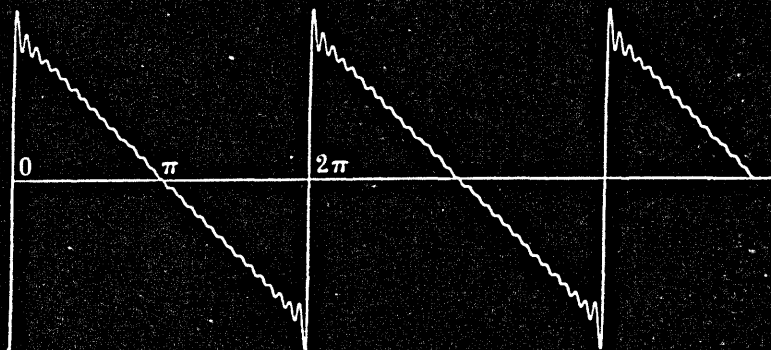


FIG. 90. Curve obtained by compounding 30 terms of the series $y = 2[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots]$.

that the phase of each component has been changed by 90° ; that is, the sines become cosines.

A further interesting variation is obtained by using the

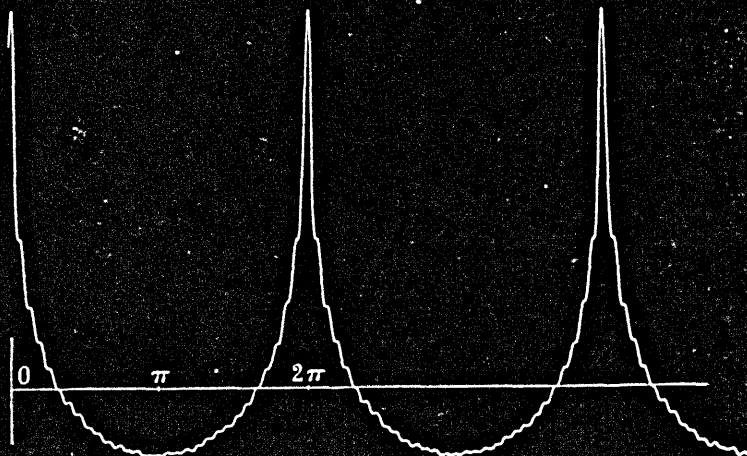


FIG. 91. Curve obtained by compounding 30 terms of the series $y = 2[\sin(x + 90^\circ) + \frac{1}{2} \sin(2x + 90^\circ) + \frac{1}{3} \sin(3x + 90^\circ) + \dots]$, which is equivalent to $y = 2[\cos x + \frac{1}{2} \cos 2x + \frac{1}{3} \cos 3x + \dots]$.

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odd-numbered terms only of the first series, producing the form shown in Fig. 92.

If the phases of the alternate terms of the odd-term series

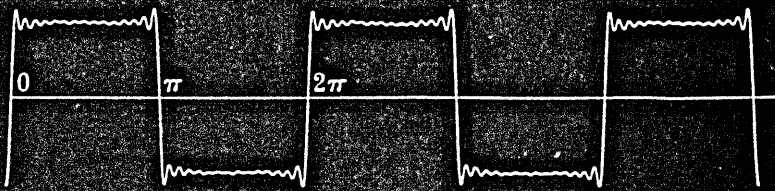


FIG. 92. Curve obtained by compounding 15 terms of the series $y = 2 [\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots]$.

are changed by 180°, the curved form shown in Fig. 93 is obtained.

The arbitrary nature of the curves that may be studied

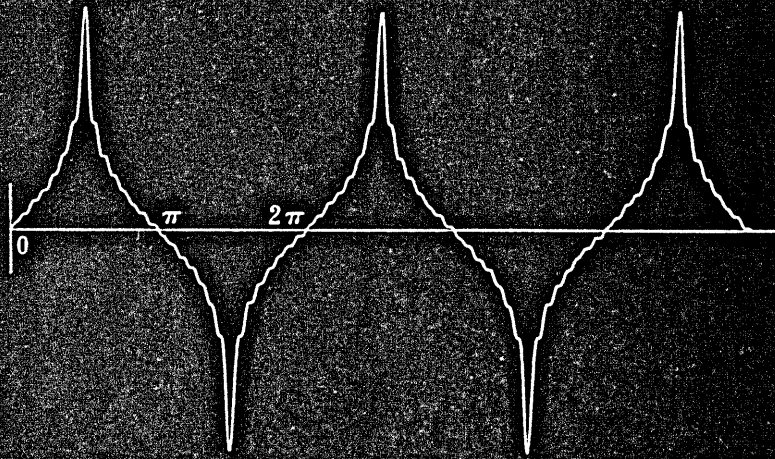


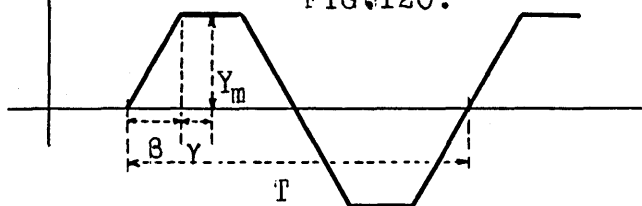
FIG. 93. Curve obtained by compounding 15 terms of the series $y = 2 [\sin x + \frac{1}{3} \sin (3x + 180^\circ) + \frac{1}{5} \sin 5x + \frac{1}{7} \sin (7x + 180^\circ) + \dots]$, or $y = 2 [\sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x - \frac{1}{7} \sin 7x + \dots]$.

by the Fourier method is further illustrated by the analysis and synthesis of a portrait profile. The original portrait is shown in the center of Fig. 94, while a tracing of the

FIGURE No. 119.

TRAPEZOIDAL WAVE.

FIG.120.



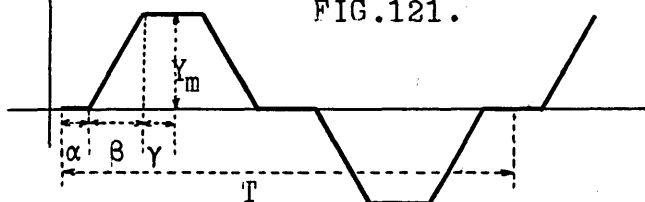
$$\text{Effective Value} = Y_m \sqrt{1 - \frac{8\beta_1}{3}}$$

$$\alpha_1 = \alpha/T \quad \beta_1 = \beta/T \quad \gamma_1 = \gamma/T$$

$$y = 1.053 Y_m [\sin \omega t - (1/25) \sin 5\omega t + (1/49) \sin 7\omega t - (1/121) \sin 11\omega t + \dots]$$

TRAPEZOID WITH ZERO SECTIONS.

FIG.121.



$$\text{Effective Value:- } Y_m [4\beta_1/3 + \gamma_1]$$

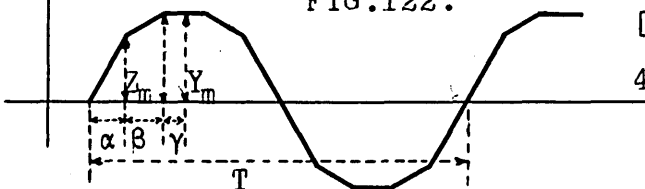
$$\alpha_1 = \alpha/T \quad \beta_1 = \beta/T \quad \gamma_1 = \gamma/T$$

y = Terms made up from:-

$$A_{2k+1} = [4Y_m/\pi^2\beta_1] [1/(2k+1)^2] \sin(2k+1)\pi\beta_1 \cos(2k+1)\pi(2\alpha_1 + \beta_1).$$

BROKEN TRAPEZOIDAL LINE.

FIG.122.



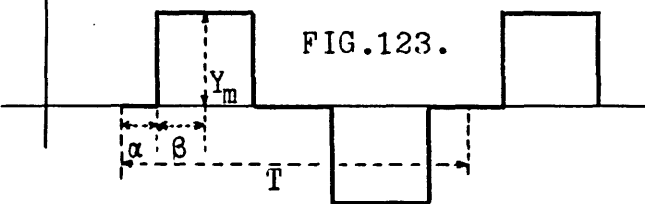
$$[\text{Effective Value}]^2 =$$

$$4[(1/3)(\alpha_1 + \beta_1)Z_m^2 + (\beta_1/3 + \gamma_1)Y_m^2 + (\beta_1/3)Y_m Z_m].$$

$$A_{2k+1} = [4(Y_m - Z_m)/\pi^2\beta_1] [1/(2k+1)^2] \sin(2k+1)\pi(2\alpha_1 + \beta_1) \cos(2k+1)\pi\beta_1 - [4/\pi^2] [1/(2k+1)^2] [(Y_m - Z_m/\beta_1) - (Z_m/2\alpha_1)] \sin(2k+1)\pi\alpha_1$$

RECTANGLES WITH ZERO SECTIONS.

FIG.123.

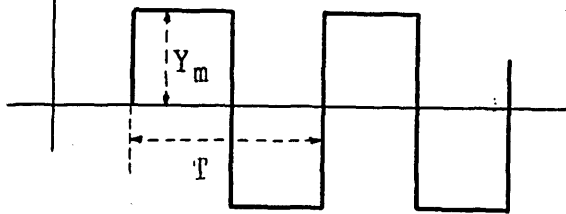


$$\text{Effective Value} = 2Y_m \sqrt{\beta_1}$$

$$A_{2k+1} = [4(Y_m/\pi)] [1/(2k+1)] \cos 2(2k+1)\alpha_1$$

FIG.124.

RECTANGULAR WAVE.

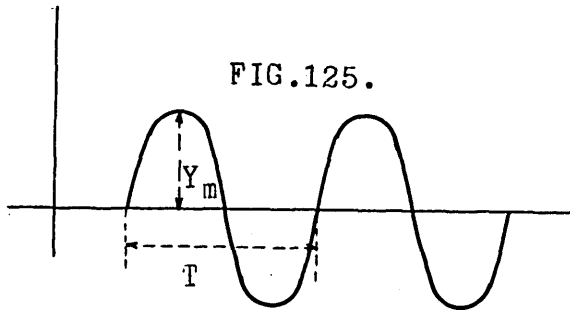


Effective Value = Y_m

$$y = 4Y_m/\pi[\sin \omega t + \frac{1}{3}\sin 3\omega t + \dots \dots \dots \frac{1}{2k+1} \sin(2k+1)\omega t + \dots]$$

FIG.125.

PARABOLIC WAVES.



Equation for half period:-

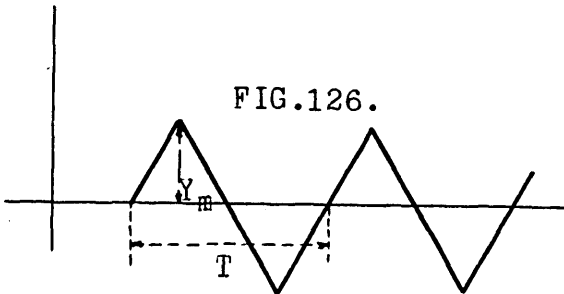
$$y = 8Y_m/T^2(T - 2t)t$$

Effective Value = $Y_m\sqrt{8/15} = 0.733Y_m$

$$y = 32Y_m/\pi^3[\sin \omega t + \frac{1}{3^3}\sin 3\omega t + \dots \dots \dots \frac{1}{(2k+1)^3} \sin(2k+1)\omega t]$$

FIG.126.

ISOSCELES TRIANGULAR WAVE.

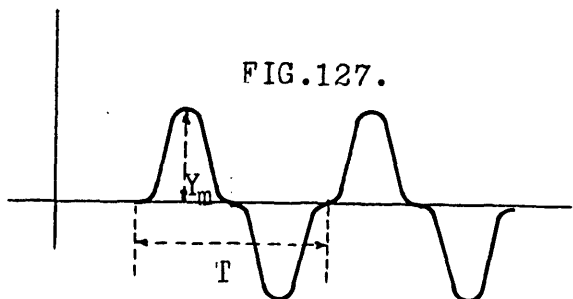


Effective Value = $Y_m\sqrt{1/3} = 0.577Y_m$

$$y = 8Y_m/\pi^2[\sin \omega t + \frac{1}{9}\sin 3\omega t + \dots \dots \dots \frac{1}{(2k+1)^2} \sin(2k+1)\omega t]$$

FIG.127.

PORTIONS OF SINUSOIDS.



Equation for half period:-

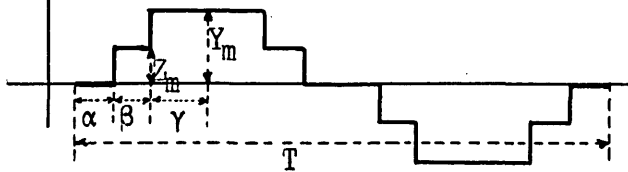
$$y = Y_m/2(1 - \cos 2\omega t).$$

Effective Value = $Y_m\sqrt{3/8} = 0.612Y_m$

$$y = 8Y_m/\pi[\frac{1}{1.3}\sin \omega t - \frac{1}{1.3.5}\sin 3\omega t - \frac{1}{3.5.7}\sin 5\omega t - \dots]$$

FIG.128.

STEPPED WAVE WITH TWO STEPS AND ZERO REGION!



Effective Value:-

$$(\text{Eff. Val})^2 = 4(Z_m^2 \beta_1 + Y_m^2 \gamma_1)$$

$$\alpha_1 = \alpha/T \quad \beta_1 = \beta/T \quad \gamma_1 = \gamma/T$$

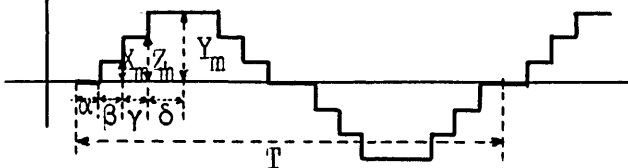
$$A_{2k+1} = (2Y_m/\pi)(1/2k+1) [\cos 2(2k+1)\pi(\alpha_1 + \beta_1) + \cos 2(2k+1)\pi\alpha_1] \quad ; \quad \text{if } Z_m = Y_m/2$$

$$\text{or } = (4(Y_m - Z_m)/\pi)(1/2k+1) \cos 2(2k+1)\pi(\alpha_1 + \beta_1) + 4Z_m/\pi(1/2k+1) \cos 2(2k+1)\pi\alpha_1$$

(For the general case).

FIG.129.

STEPPED WAVE WITH THREE STEPS AND ZERO REGION!



[Effective Value]² = 4(X_m²β₁ + Z_m²γ₁ + Y_m²δ₁)

$$\alpha_1 = \alpha/T \quad \beta_1 = \beta/T \quad \gamma_1 = \gamma/T \quad \delta_1 = \delta/T$$

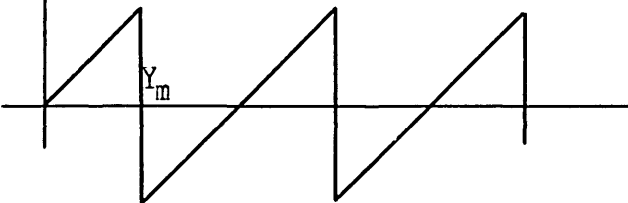
$$A_{2k+1} = (4X_m/\pi)(1/2k+1) \cos 2(2k+1)\pi\alpha_1 +$$

$$+ [4(Z_m - X_m)/\pi][1/2k+1] \cos 2(2k+1)\pi(\alpha_1 + \beta_1)$$

$$+ [4(Y_m - Z_m)/\pi][1/2k+1] \cos 2(2k+1)\pi(\alpha_1 + \beta_1 + \gamma_1)$$

FIG.130.

RIGHT TRIANGULAR WAVE.



Compare with isosceles triangular wave of previous page.

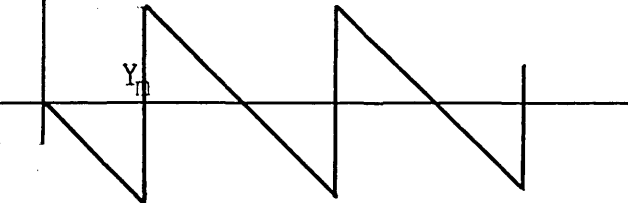
$$Y_m = 2/\pi$$

$$y = 2[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x \dots \dots \dots]$$

FIG.131

RIGHT TRIANGULAR WAVE.

Reverse of that above!



$$Y_m = 2/\pi$$

$$y = 2[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \dots \dots \dots]$$

FIGURE No. 132.

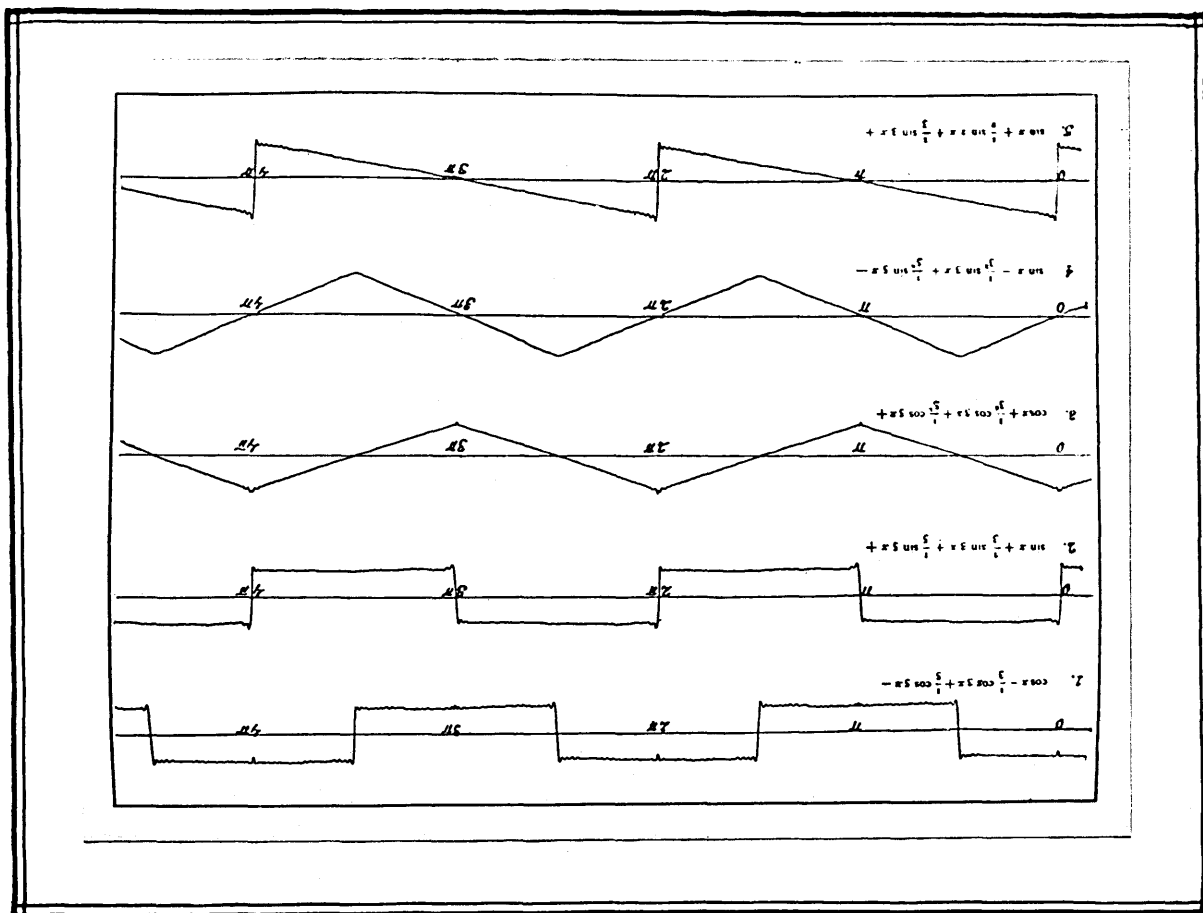
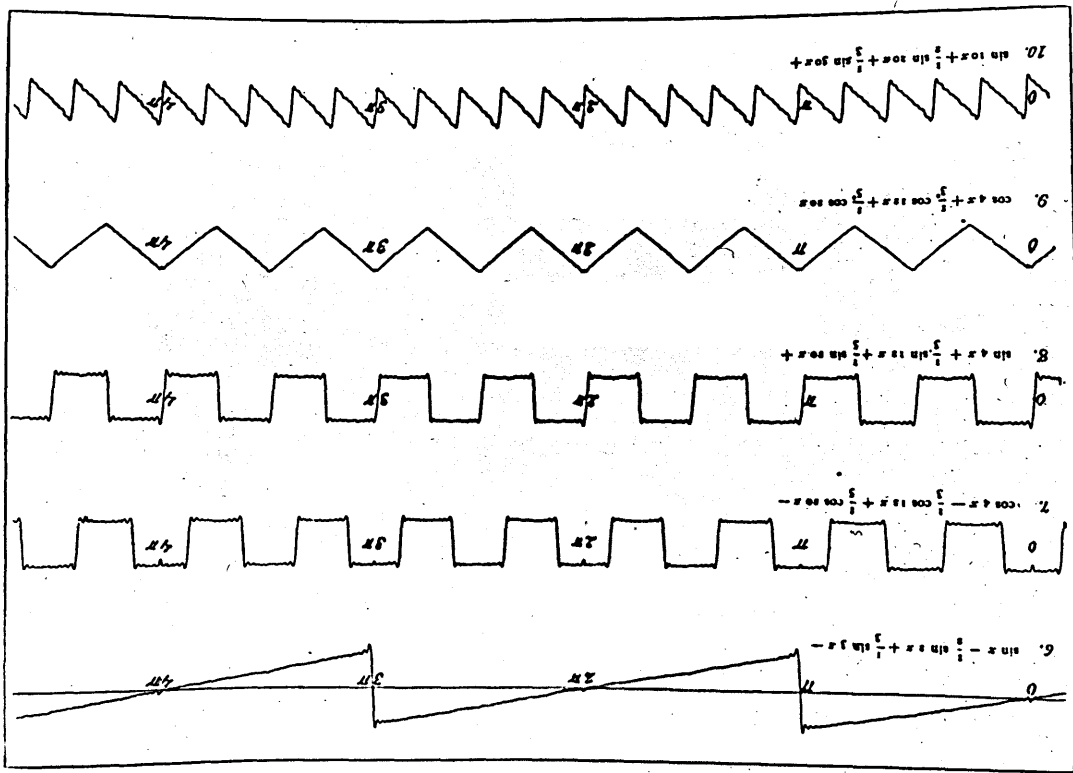


FIGURE NO. 133.



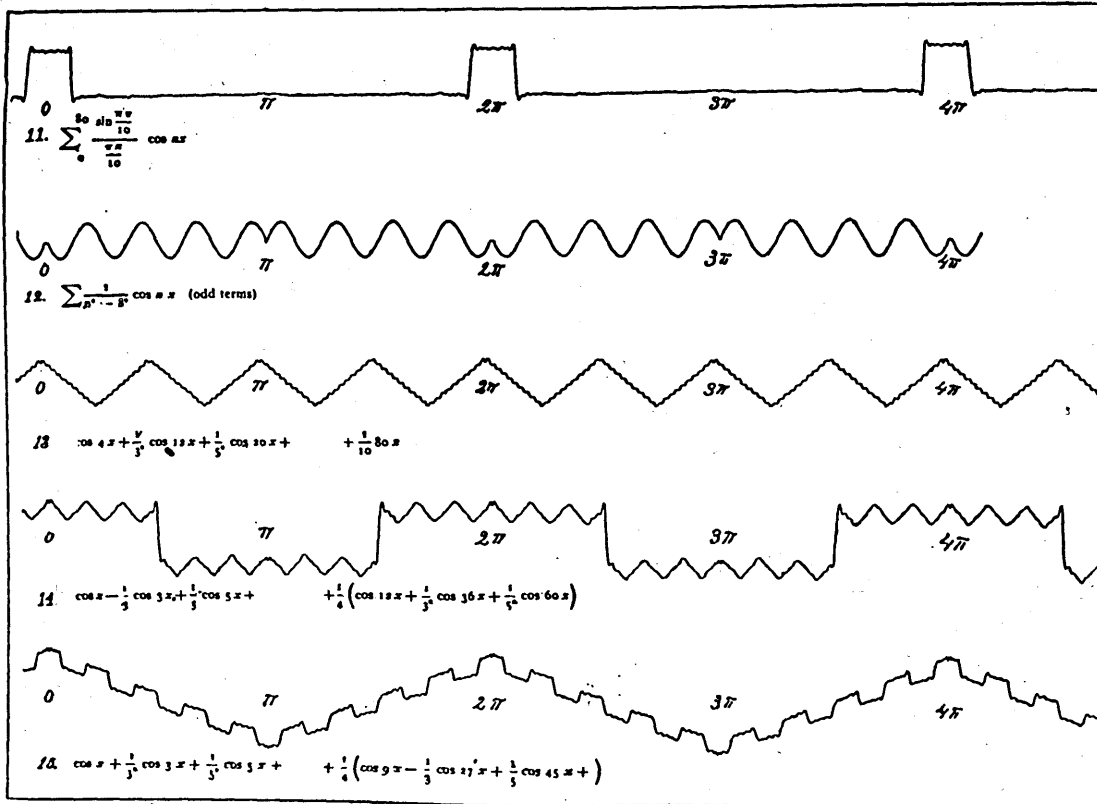


FIGURE No.134.

FIGURE NO. 135.

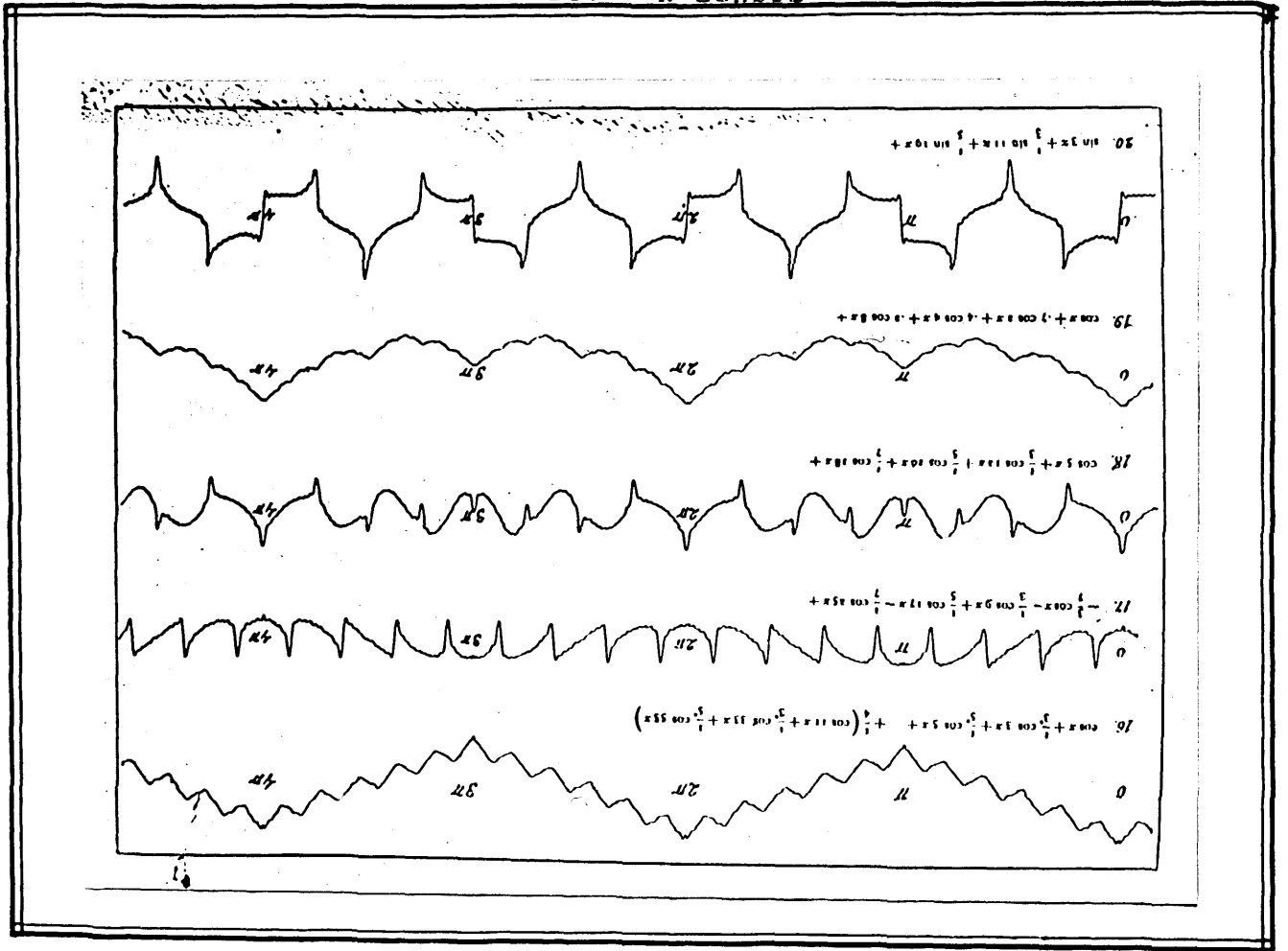
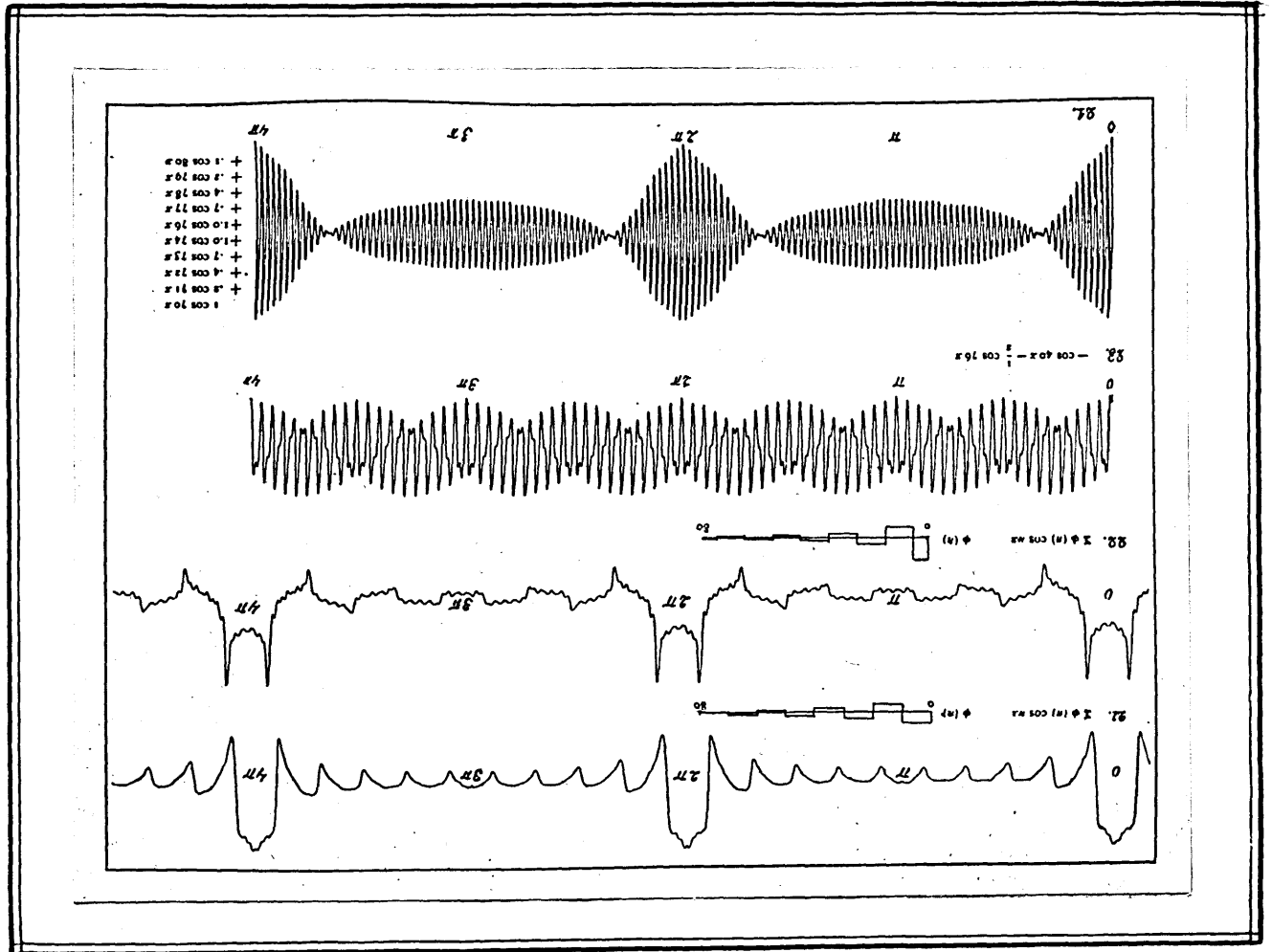


FIGURE NO. 136.



A P P E N D I X II.

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B I B L I O G R A P H Y.

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