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ASYMPTOTIC STABILITY, IDENTIFICATION, AND THE HORIZON PROBLEM

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ABSTRACT: Recent literature in discrete adaptive control has emphasized the importance of asymptotic stability of the adaptive controller in obtaining convergence of system parameter estimates to their true values. This paper studies the relationship between these results and the problem of the convergence of first period decisions in planning models as the planning horizon time increases. The primary results to date have been based on stationary and purely quadratic cost functions. This paper extends these results to cost functions containing linear terms and to discounted cost functions. The main results are a set of sufficient conditions on the nature of cost and system parameters under which first period decisions converge to a fixed value as the optimization time horizon increases. A characterization of the optimal asymptotic controls is given for the discounted and undiscounted cases.

I. INTRODUCTION

Consider the following linear system.

(1.1) $x_{k+1} = Ax_k + Du_k + v_k$

(1.2) $y_k = Mx_k + w_k$

In (1.1), x_k is a p-dimensional column vector which represents the state of the system at time k; A is a pxp transition matrix; D is a pxr control matrix; u_k is an r-dimensional control vector; and v_k represents a random disturbance. In (1.2), y_k is a q-dimensional vector representing an observation made on the system at time k; M is a qxp observation matrix; and w_k represents noise. We will assume that $v_o, v_1, \ldots$ and $w_1, w_2, \ldots$ are independent sequences of zero mean, independent and identically distributed random vectors with covariance matrices V and W respectively and that x_o is independent of the v_i s and w_j s and has finite covariance matrix.

Linear least squares prediction and filtering may be done for the system (1.1)-(1.2) using the Kalman Filter [4], which yields the projections $x_{t|k}$ and $y_{t|k}$ of x_t and y_t on the Hilbert subspace spanned by $y_1, y_2, \ldots, y_k$. These projections are given by

(1.3) $$x_{k|k} = (I - \Lambda_k)Ax_{k-1|k-1} + \Lambda_k y_k, \qquad k \geq 1$$
$$x_{t|k} = A^{t-k} x_{k|k}$$
$$y_{t|k} = Mx_{t|k}, \quad t>k,$$

where I denotes the pxp identity matrix and $x_{o|o}=E[x_o]$. The weighting matrix Λ_k in (1.3) is determined by

(1.4a) $$\Lambda_k = S_k M'[MS_k M' + W]^{\dagger}, \qquad k \geq 1$$

(1.4b) $$S_k = AP_{k-1}A' + V, \qquad k \geq 1$$

(1.4c) $$P_k = [I - \Lambda_k M]S_k, \qquad k \geq 1$$

where † denotes pseudo-inverse, ' denotes transpose, and P_o is the covariance matrix of x_o.

Now consider the following optimization problem.

(1.5) $$\text{Min } E\{ \Sigma_{k=1}^{N} [x_k' Q_{1,k} x_k + u_{k-1}' Q_{2,k-1} u_{k-1} + G_{1,k} x_k + G_{2,k-1} u_{k-1}] \}$$

subject to (1.1),(1.2), k=0,1,...,N-1, where for k=1,2,...,N, $Q_{1,k}$ and $Q_{2,k-1}$ are symmetric positive semi-definite (psd) matrices and $G_{1,k}$, $G_{2,k-1}$ are 1xp and 1xr row vectors respectively, and one of the following conditions is satisfied: (a) $Q_{2,k-1}$ is positive definite (pd); (b) $Q_{1,k}$ is pd and rank(D)= min(p,r). Either (a) or (b) is required to insure the existence of a finite minimum of the performance criterion and to insure the existence of matrix inverses for the dynamic programming solution to the problem, (1.5).

In an extension of Gunckel and Franklin's result [3] for the pure quadratic loss function, it can be shown (see [7]) that the optimal controls, u_k, in (1.5) are given by

(1.6) $$u_k = -C_k x_{k|k} - H_k^{-1} Z_k'(1/2) \qquad 0 \leq k \leq N-1;$$

where C_k and H_k are determined by

(1.7a) $$H_k C_k = D'(F_{k+1} + Q_{1,k+1})A$$

(1.7b) $$H_k = Q_{2,k} + D'(F_{k+1} + Q_{1,k+1})D$$

(1.7c) $$F_k = A'(F_{k+1} + Q_{1,k+1})A - C_k' H_k C_k, \qquad F_N = 0$$

(1.8a) $$Z_k = G_{1,k+1}D + G_{2,k} + B_{k+1}D$$

(1.8b) $$B_k = (G_{1,k+1} + B_{k+1})A - Z_k C_k, \qquad B_N = 0,$$

where it is shown in the dynamic programming solution to the problem that F_{k+1} is psd and therefore, by the assumptions on $Q_{1,k+1}, Q_{2,k}$, and D, H_k is nonsingular and C_k is determined uniquely from (1.7a).

As Kalman has shown, there is a close relationship between equations (1.4) and (1.7) which allows results obtained for the filter equations, (1.4), to be applied to the optimization equations (1.7) and (1.8). In section II of this paper we use this relationship and the uniform asymptotic stability of the Kalman filter to show that the matrices $C_o = C_o(N)$ and $H_o^{-1}Z_o' = H_o^{-1}(N)Z_o'(N)$ in (1.6) converge to a fixed point as the time horizon N increases, when the costs in (1.5) are stationary. This result is extended in section III to the case of discounted costs. In section IV we explore the implications of these results for the aggregate planning problem in production scheduling. Briefly, these results imply that when sales are generated by a linear autoregressive system, then the first period decision rules approach a fixed point as the time horizon is increased. Finally, generalizations to the above results are discussed in the concluding remarks.

The above results for the undiscounted case with pure quadratic loss function were originally proven by Kalman [5], although his results on the "separation theorem" were known earlier in another form as the certainty equivalence theorem, see Simon [8]. Kalman's results were applied and extended to the problem of aggregate planning and information system design

by G. B. Kleindorfer [6].[1] Anderson et. al. extended Kleindorfer's work in their study of the identification problem and they laid most of the groundwork for the analytical methods to be used here. The present work extends past results on the convergence of the first period decision rules to the case where the objective function contains linear terms in the state and control variables as well as extending known results to the discounted case.

II. THE UNDISCOUNTED CASE

We begin by studying the asymptotic behavior of (1.4). We may combine equations (1.4) to obtain

$$(2.1) \qquad S_{k+1} = A(S_k - S_k M'[MS_k M' + W]^{\dagger} MS_k)A' + V, \qquad k \geq 1.$$

Moreover, it is clear from (1.4) that the matrices $\{S_1, S_2, \ldots\}$ uniquely determine the corresponding sequences $\{\Lambda_1, \Lambda_2, \ldots\}$ and $\{P_1, P_2, \ldots\}$. We therefore restrict our attention to a study of (2.1) and note the following result proven in Anderson et. al. [1].

Theorem 1: Let M=I and let V be pd and W psd; define Φ on the set T of pd matrices by

$$(2.2) \qquad \Phi(S) = AS(S + W)^{-1} WA' + V, \qquad S \in T$$

so that $S_k = \Phi(S_{k-1})$, $k \geq 2$. Then Φ has a unique pd fixed point S_o, and $\Phi^n(S) \to S_o$ uniformly on T as $n \to \infty$, where Φ^n denotes the nth iterate of Φ. Moreover, $\Lambda_k \to \Lambda_o = S_o(S_o + W)^{-1}$, and $P_k \to P_o = S_o - S_o(S_o + W)^{-1} S_o$ as $k \to \infty$.

This result was proven originally by Kalman [5] under the assumption that the system (1.1) and (1.2) is completely observable and completely controllable. His proof also allows for M, the observation matrix, to be non-square.

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[1] Results of a similar nature are also contained in some unpublished research of Professor Lance Taylor of Harvard University.

Theorem 1 may also be easily generalized to include a non-square observation matrix.

Corollary 1: Let V and W be pd and let rank(M) = min(q,p); define Φ on the set T of pd matrices by

(2.3) $$\Phi(S) = A(S^{-1} + M'W^{-1}M)^{-1}A' + V, \qquad S \in T.$$

Then for any S_1 pd, $S_k = \Phi(S_{k-1})$, $k \geq 2$. Moreover, Φ has a unique pd fixed point, S_o, and $\Phi^n(S) \to S_o$ uniformly on T as $n \to \infty$, where Φ^n denotes the nth iterate of Φ. Furthermore, $\Lambda_k \to \Lambda_o = S_oM'[MS_oM' + W]^{-1}$, and $P_k \to P_o = S_o - S_oM'[MS_oM' + W]^{-1}MS_o$ as $k \to \infty$.

Proof: We first verify that for any S_1 pd, $S_k = \Phi(S_{k-1})$, $k \geq 2$. By (2.1) and the fact that $[MS_kM' + W]^{\dagger} = [MS_kM' + W]^{-1}$ since W is pd, we only need to show that

(2.4) $$(S^{-1} + M'W^{-1}M)^{-1} = S - SM'[MSM' + W]^{-1}MS, \qquad S \in T.$$

To demonstrate the validity of (2.4) consider the following calculations.

(2.5) $$\begin{aligned}&(S^{-1} + M'W^{-1}M)(S - SM'[MSM' + W]^{-1}MS)\\ &= I - M'[MSM' + W]^{-1}MS + M'W^{-1}MS - M'W^{-1}MSM'[MSM' + W]^{-1}MS\\ &= I + M'(W^{-1} - [MSM' + W]^{-1} - W^{-1}MSM'[MSM' + W]^{-1})MS\end{aligned}$$

However,

(2.6) $$\begin{aligned}W^{-1}MSM'[MSM' + W]^{-1} &= W^{-1}(MSM' + W - W)[MSM' + W]^{-1}\\ &= W^{-1}(I - W[MSM' + W]^{-1})\\ &= W^{-1} - [MSM' + W]^{-1},\end{aligned}$$

so that substitution of (2.6) into (2.5) yields the desired result.

Corollary 1 now follows from theorem 1 since when M is of full rank, $M'W^{-1}M$ is invertible and

(2.7) $$(S^{-1} + M'W^{-1}M)^{-1} = S[S + (M'W^{-1}M)^{-1}]^{-1}(M'W^{-1}M)^{-1}$$

so that identifying $(M'W^{-1}M)^{-1}$ with W in (2.2) yields the desired result.

We now consider the relationship between the control equations (1.7) and

the Kalman filter equations (1.4). In fact, equations (1.7) may be combined to yield

(2.8) $$F_k = A'(F_{k+1} + Q_{1,k+1})A - A'(F_{k+1} + Q_{1,k+1})D[Q_{2,k} + D'(F_{k+1} + Q_{1,k+1})D]^{-1}D'(F_{k+1}+Q_{1,k+1})A$$
$$F_N = 0$$

or letting $R_k = F_k + Q_{1,k}$, $k = 1,2,\dots,N$, we obtain

(2.9) $$R_k = A'(R_{k+1} - R_{k+1}D[Q_{2,k} + D'R_{k+1}D]^{-1}D'R_{k+1})A + Q_{1,k}, \quad k=1,\dots,N-1;$$
$$R_N = Q_{1,N}$$

Comparing (2.9) with (2.1) we see that these difference equations are of precisely the same form if we identify $R \leftrightarrow S$, $A \leftrightarrow A'$, $D \leftrightarrow M'$, $Q_{2,k} \leftrightarrow W$, $Q_{1,k} \leftrightarrow V$, $k \leftrightarrow N-k$. Thus, when $Q_{1,k} = Q_1$, $Q_{2,k} = Q_2$ for all k, it is clear that a study of the asymptotic behavior of $F_1 + Q_1 = R_1 = R_1(N)$ as $N \to \infty$ may be obtained from corollary 1. Indeed, corollary 1 and the above remarks imply

Theorem 2: Let rank(D) = min(p,r) and let Q_1 and Q_2 be pd. Define Ψ on the set of pd matrices T by

(2.10) $$\Psi(R) = A'(R - RD[Q_2 + D'RD]^{-1}D'R)A + Q_1, \qquad R \in T,$$

so that $R_k = \Psi(R_{k+1})$. Then Ψ has a unique pd fixed point R_* and $\Psi^N(R) \to R_*$ uniformly on T as $N \to \infty$, where Ψ^N denotes the Nth iterate of Ψ. Moreover, $H_o(N) = Q_2 + D'\Psi^{N-1}(Q_1)D \to Q_2 + D'R_*D$ and $C_o(N) = [Q_2 + D'\Psi^{N-1}(Q_1)D]^{-1}D'\Psi^{N-1}(Q_1)A \to C_* = [Q_2 + D'R_*D]^{-1}D'R_*A$ as $N \to \infty$.

Now let us consider the asymptotic behavior of $B_1 = B_1(N)$ in (1.8) which is required for the computation of $Z_o(N)$, used with $C_o(N)$ and $H_o(N)$ in (1.6) for the computation of the first period controls, u_o. From (1.8) we obtain

(2.11) $$B_k = B_{k+1}(A - DC_k) + G_{1,k+1}(A - DC_k) + G_{2,k}C_k, \quad k=1,2,\dots,N-1;$$
$$B_N = 0$$

In order to show that $B_1 = B_1(N)$ converges to a fixed point as $N \to \infty$, we will need the following lemmas, the first of which is due to Stein and is proven

in [1].

Lemma 1: Let Y be a square matrix and let $\rho(Y)$ be the spectral radius of Y. If there exists a pd matrix L for which $L - Y'LY$ is pd then $\rho(Y) < 1$.

Lemma 2: Let $C_* = [Q_2 + D'R_*D]^{-1}D'R_*A$, where R_* is the unique pd fixed point of Ψ. Then, $\rho(A - DC_*) < 1$.

Proof: By definition of C_*, we have

$$(2.12) \qquad \begin{aligned} A - DC_* &= A - D[Q_2 + D'R_*D]^{-1}D'R_*A \\ &= (I - D[Q_2 + D'R_*D]^{-1}D'R_*)A \\ &= R_*^{-1}(R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_*)A \end{aligned}$$

Now a calculation similar to the proof of (2.4) in corollary 1 shows that

$$(2.13) \qquad R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_* = [R_*^{-1} + DQ_2^{-1}D']^{-1}$$

Therefore, we have from (2.12)

$$(2.14) \qquad A - DC_* = R_*^{-1}[R_*^{-1} + DQ_2^{-1}D']^{-1}A$$

To prove the assertion it will suffice by lemma 1 to exhibit a pd matrix L for which $L - (A - DC_*)'L(A - DC_*)$ is pd. But $L = R_*$ is such a matrix, for by definition, $R_* = \Psi(R_*)$, and therefore

$$(2.15) \qquad \begin{aligned} R_* - (A - DC_*)'R_*(A - DC_*) &= A'(R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_*)A + \\ &\quad Q_1 - (A - DC_*)'R_*(A - DC_*) \end{aligned}$$

Now using (2.13) and (2.14) in (2.15) we obtain

$$(2.16) \qquad \begin{aligned} R_* - (A - DC_*)'R_*(A - DC_*) &= A'(R_* - R_*D[Q_2 + D'R_*D]^{-1}D'R_*)A - \\ &\quad A'[R_*^{-1} + DQ_2^{-1}D']^{-1}R_*^{-1}R_*R_*^{-1}[R_*^{-1} + DQ_2^{-1}D']^{-1}A + Q_1 \\ &= A'[R_*^{-1} + DQ_2^{-1}D']^{-1}(I - R_*^{-1}[R_*^{-1} + DQ_2^{-1}D']^{-1})A + Q_1 \\ &= A'[R_*^{-1} + DQ_2^{-1}D']^{-1}\{I - (R_*^{-1} + DQ_2^{-1}D' - DQ_2^{-1}D')[R_*^{-1} + DQ_2^{-1}D']^{-1}\}A + Q_1 \\ &= A'[R_*^{-1} + DQ_2^{-1}D']^{-1}DQ_2^{-1}D'[R_*^{-1} + DQ_2^{-1}D']^{-1}A + Q_1 \end{aligned}$$

QED.

We now show that $B_1(N)$ converges to a given finite vector, B_*. For convenience, we reverse the time index in (2.11) so that $k \leftrightarrow N-k$ and

$$(2.17) \qquad B_{t+1} = B_t(A - DC_{t+1}) + G_1(A - DC_{t+1}) + G_2C_{t+1}, \quad B_o = 0 .$$

<u>Theorem 3</u>: Let B_t be defined by (2.17). Then B_t converges to B_* given by

(2.18) $$B_* = [G_1(A - DC_*) + G_2C_*][I - A + DC_*]^{-1}$$

where C_* is given in lemma 2. Consequently, $B_1(N)$ converges to B_* as $N \to \infty$.

<u>Proof</u>: Define the matrices Ξ_t and Ω_t by

(2.19) $$\Xi_t = A - DC_{t+1}$$

(2.20) $$\Omega_t = G_1(A - DC_{t+1}) + G_2C_{t+1}$$

so that (2.17) becomes

(2.21) $$B_{t+1} = B_t\Xi_t + \Omega_t , \qquad B_o = 0 .$$

Now since $C_t \to C_*$, $\Xi_t \to \Xi_* = A - DC_*$, and by lemma 2, $\rho(\Xi_*) < 1$, so that (2.21) is asymptotically stable. From (2.21) we obtain

(2.22) $$B_{t+1} = B_o\Pi_{j=o}^{t} \Xi_j + \Sigma_{k=o}^{t}\Omega_k(\Pi_{j=k+1}^{t}\Xi_j)$$

We now remark that since $\rho(\Xi_*) < 1$, $I - \Xi_*$ is invertible, and

(2.23) $$(I - \Xi_*)^{-1} = \Sigma_{k=o}^{\infty}\Xi_*^k ,$$

so that letting $B_t^* = \Sigma_{k=o}^{t} \Omega_*(\Pi_{j=k+1}^{t}\Xi_*)$, where $\Omega_* = G_1(A - DC_*) + G_2C_*$, we have $B_t^* \to B_*$. Thus it suffices to show that $|B_t - B_t^*| \to 0$, where $|\cdot|$ is the Euclidian norm. To show this, we note that $\rho(\Xi_*) < 1$ implies (see Varga [10], p. 67) the existence of an integer $r \geq 1$ for which $|\Xi_*^r| < 1$, and since $\Xi_t \to \Xi_*$, there exist $\rho_o < 1$ and an integer $k_o \geq 1$ for which

(2.24) $$\text{Max } [|\Pi_{j=k}^{k+r} \Xi_j|, |\Xi_*^r|] < \rho_o , \qquad k \geq k_o.$$

Since Ω_k is a convergent sequence, there exists a uniform upper bound, U_1, such that $|\Omega_k| \leq U_1$ for all k. Moreover, (2.24) implies for $s \geq 1$, and $k \geq k_o$ that

(2.25) $$\text{Max } [|\Pi_{j=k+1}^{t} \Xi_j|, |\Xi_*^{t-k-1}|] \leq \rho_o^s \text{Max } [|\Pi_{j=k+1+rs}^{t} \Xi_j|, |\Xi_*^{t-k-1-rs}|]$$

In particular, $\ell im_{t\to\infty}|\Pi_{j=j_o}^{t} \Xi_j| = 0$ for every $j_o \geq 0$, so that

(2.26) $$\begin{aligned} \ell im_{t\to\infty} \sup |B_{t+1} - B_{t+1}^*| &\leq \ell im_{t\to\infty} \sup U_1\{\Sigma_{k=o}^{t}|\Pi_{j=k+1}^{t}\Xi_j| + |\Pi_{j=k+1}^{t}\Xi_*|\} \\ &= \ell im_{t\to\infty} \sup U_1\{\Sigma_{k=j_o}^{t}|\Pi_{j=k+1}^{t}\Xi_j| + |\Xi_*^{t-k-1}|\} \\ &\leq \ell im_{t\to\infty} \sup \rho_o^s U_1\{\Sigma_{k=j_o}^{t}|\Pi_{j=k+1+rs}^{t} \Xi_j| + |\Xi_*^{t-k-1-rs}|\} , \quad j_o \geq k_o \end{aligned}$$

But (2.25) implies for $j_o \geq k_o$ and for $i \geq 0$

(2.27) $$\sum_{k=j_o}^{t} \left| \prod_{j=k+i}^{t} \Xi_j \right| \leq \sum_{p=o}^{p_o} \{ \sum_{k=j_o+i}^{j_o+i+r} \left| \prod_{j=j_o+i}^{j_o+i+k} \Xi_j \right| \} \rho_o^p$$

where p_o is the largest integer less than or equal to $(t-j_o-i)/r$. Therefore,

(2.28) $$\ell im_{t\to\infty} \sup \Sigma_{k=j_o}^{t} \left| \Pi_{j=k+i}^{t} \Xi_j \right| \leq U_2/(1-\rho_o)$$

Then (2.26) and (2.28) imply

(2.29) $$\ell im_{t\to\infty} \sup \left| B_{t+1} - B_{t+1}^{*} \right| \leq 2\rho_o^s U_1 U_2/(1-\rho_o)$$

which may be made arbitrarily small by proper choice of s. This completes the proof of theorem 2.

We may summarize the results of this section as follows. Let $u_o(N)$ be the optimal first period controls given by (1.6) as

(2.30) $$u_o(N) = -C_o(N)x_{o|o} - 1/2\ H_o^{-1}(N)Z_o'(N)\ .$$

Let D be of full rank and let Q_1 and Q_2 be pd. Then $u_o(N)$ converges to

(2.31) $$u_* = -C_* x_{o|o} - 1/2\ H_*^{-1}\ Z_*'$$

as $N \to \infty$. The values of the parameters in (2.31) are given below and $x_{o|o}$ is the linear least squares estimate of the system at the present time.

(2.32) $$R_* = A'(R_* - R_* D[Q_2 + D'R_* D]^{-1} D'R_*)A + Q_1$$

(2.33) $$C_* = [Q_2 + D'R_* D]^{-1} D'R_* A$$

(2.34) $$H_* = Q_2 = D'R_* D$$

(2.35) $$Z_* = G_1 D + G_2 + B_* D$$

(2.36) $$B_* = [G_1(A - DC_*) + G_2 C_*][I - A + DC_*]^{-1}$$

It should be remarked that in practice one would determine R_* by iteration of (2.9) with $k \leftrightarrow N-k$ until successive values of R_k and R_{k+1} were within a desired tolerance. In this process it should be recalled that convergence of R_k to R_* is uniform. In fact, it can easily be shown on the basis of the proof of the above theorem 1 in Anderson <u>et</u>. <u>al</u>. [1], that (with $k \leftrightarrow N-k$)

(2.37) $$R_N = \Psi^N(Q_1) \leq \Psi^N(A'[DQ_2^{-1}D']^{-1}A + Q_1) \leq A'[DQ_2^{-1}D']^{-1}A + Q_1$$

where $X \leq Y$ in (2.37) if $Y - X$ is psd. Thus, since both $\Psi^N(Q_1)$ and $\Psi^N(A'[DQ_2^{-1}D']^{-1}A + Q_1)$ are converging to the fixed point R_*, one can use the (some convenient) norm of the difference of these two quantities to obtain a precise bound on $|\Psi^N(Q_1) - R_*|$.

III. THE DISCOUNTED CASE

Let α be a given discount factor, $0 < \alpha < 1$, and consider the minimization (1.5) with

(3.1) $$Q_{i,k} = \alpha^k Q_i, \quad G_{i,k} = \alpha^k G_i, \quad i=1,2;\ k \geq 0.$$

In this case equations (1.7) and (1.8) yield

(3.2) $$R_k = A'(R_{k+1} - R_{k+1}D[D'R_{k+1}D + \alpha^k Q_2]^{-1}D'R_{k+1})A + \alpha^k Q_1, \quad 1 \leq k \leq N-1;$$

(3.3) $$B_k = B_{k+1}(A - DC_k) + \alpha^{k+1}G_1(A - DC_k) + \alpha^k G_2 C_k, \quad B_N = 0, \ 0 \leq k \leq N-1;$$

where C_k is determined by (1.7) and where

(3.4) $$R_k = F_k + \alpha^k Q_1, \quad R_N = \alpha^N Q_1.$$

We now show that, by redefining the system parameters, (3.2) can be put in the form of (2.9) with stationary parameters, so that the desired asymptotic properties follow from theorem 2. We begin by defining the parameters

(3.5) $$\underline{A} = \beta A, \quad \underline{Q}_1 = Q_1, \quad \underline{D} = D, \quad \underline{Q}_2 = (1/\alpha)Q_2, \quad \beta^2 = \alpha.$$

Then letting $\underline{R}_k = \alpha^{-k}R_k$, we have from (3.2)

(3.6) $$\underline{R}_k = A'(\alpha^{-k}R_{k+1} - \alpha^{-k}R_{k+1}D[D'R_{k+1}D + \alpha^k Q_2]^{-1}D'R_{k+1})A + Q_1$$

$$= (\beta A)'(\alpha^{-k-1}R_{k+1} - \alpha^{-k-1}R_{k+1}D[D'(\alpha^{-k-1}R_{k+1})D+\alpha^{-1}Q_2]^{-1}D'\alpha^{-k-1}R_{k+1})(\beta A) + Q_1$$

or, using (3.5),

(3.7) $$\underline{R}_k = \underline{A}'(\underline{R}_{k+1} - \underline{R}_{k+1}\underline{D}[\underline{D}'\underline{R}_{k+1}\underline{D} + \underline{Q}_2]^{-1}\underline{D}'\underline{R}_{k+1})\underline{A} + \underline{Q}_1$$

Thus, (3.7) is precisely of the same form as (2.9) and theorem 2 therefore implies that $R_1(N) = \alpha\underline{R}_1(N)$ converges uniformly to a fixed point, $\alpha\underline{R}_*$, as $N \to \infty$. This also implies the convergence of $C_o(N)$ and $H_o(N)$ to $C_* = H_*^{-1}D'\alpha\underline{R}_*A$ and

$H_* = Q_2 + D' \underline{R}_* D$, respectively.

Similarly, the equation (2.11) for B_k becomes for the discounted case

(3.8) $$B_k = B_{k+1}(A - DC_k) + \alpha^{k+1} G_1 (A - DC_k) + \alpha^k G_2 C_k, \quad B_N = 0.$$

Let $\underline{B}_k = \alpha^{-k} B_k$. Then (3.8) implies

(3.9) $$\underline{B}_k = \underline{B}_{k+1}\alpha(A - DC_k) + G_1 \alpha (A - DC_k) + G_2 C_k \ .$$

If we identify $\Xi_k = \alpha(A - DC_k)$ and $\Omega_k = G_1\alpha(A - DC_k) + G_2 C_k$, then we may proceed as in theorem 3 to prove the convergence of $\underline{B}_1(N)$ to a finite vector $\underline{B}_*$ provided that $\rho(\alpha[A - DC_*]) < 1$, where C_* is the fixed point of $C_o(N)$ defined above. To show this we note that

(3.10) $$\begin{aligned} C_* &= [Q_2 + D'\alpha\underline{R}_* D]^{-1} D'\alpha\underline{R}_* A \\ &= [\alpha^{-1} Q_2 + D'\underline{R}_* D]^{-1} D' R_* A \\ &= \beta[\underline{Q}_2 + \underline{D}'\underline{R}_*\underline{D}]^{-1} \underline{D}'\underline{R}_*\underline{A} \end{aligned}$$

and therefore, using (3.5), we obtain

(3.11) $$\alpha(A - DC_*) = \beta(\underline{A} - \underline{DC}_*)$$

where $\underline{C}_*$ is the fixed point of the stationary system with parameters, $\underline{A}$, $\underline{D}$, $\underline{Q}_1$, and $\underline{Q}_2$, corresponding to $\underline{C}_o(N)$. But $\rho(\beta[\underline{A} - \underline{DC}_*]) = \beta\rho(\underline{A} - \underline{DC}_*)$ and $\rho(\underline{A} - \underline{DC}_*) < 1$ by lemma 2, so that the desired result follows as in the proof of theorem 3. We may summarize the results of this section in the following manner.

<u>Theorem 4</u>: Let $u_o(N)$ be the optimal first period controls given by

(3.12) $$u_o(N) = -C_o(N) x_{o|o} - 1/2\ H_o^{-1}(N) Z_o'(N) \ .$$

Let D be of full rank and let Q_1 and Q_2 be pd. Then $u_o(N)$ converges to u_* given by

(3.13) $$u_* = -C_* x_{o|o} - 1/2\ H_*^{-1} Z_*'$$

as $N \to \infty$. In (3.13) $x_{o|o}$ is the linear least squares estimate of the system state at the present time and the parameters C_*, H_*, and Z_* are determined by

(3.14) $$\underline{R}_* = \underline{A}'(\underline{R}_* - \underline{R}_*\underline{D}[\underline{Q}_2 + \underline{D}'\underline{R}_*\underline{D}]^{-1}\underline{D}'\underline{R}_*)\underline{A} + \underline{Q}_1$$

$B_j = Q_j + D^{-1}_{j} \varepsilon_j D_j$ respectively.

Similarly, the equation [illegible]

(3.8) [illegible]

Let [illegible]

(3.9) [illegible]

If we [illegible]

[illegible]

[illegible]

above. [illegible]

(3.10) [illegible]

[illegible]

(3.15) $C_* = [Q_2 + D'\alpha R_* D]^{-1} D'\alpha \underline{R}_* A$

(3.16) $H_* = Q_2 + D'\alpha \underline{R}_* D$

(3.17) $Z_* = \alpha G_1 D + G_2 + \alpha \underline{B}_* D$

(3.18) $\underline{B}_* = [G_1 \alpha(A - DC_*) + G_2 C_*][I - \alpha(A - DC_*)]^{-1}$

where $\underline{A}$,$\underline{D}$, $\underline{Q}_1$, and $\underline{Q}_2$ are given in (3.5).

IV. APPLICATIONS AND EXTENSIONS

Although the results of the preceding sections have wider applicability, we restrict ourselves to a brief exploration of their implications for the aggregate planning of production and work force (see Holt *et*. *al*. [2]). This discussion will serve to highlight as well the limitations of the above analysis.

Following Holt *et*. *al*. [2], we first assume the following model for the aggregate planning problem.

(4.1) $\text{Min } E\{\Sigma_{k=1}^{N} f_k(I_{k+1}, P_k, W_k, U_k)\}$

subject to

(4.2) $I_{k+1} = I_k + P_k - S_k$

(4.3) $W_{k+1} = W_k + U_k$

(4.4) I_o, W_o given,

where f_k is a quadratic-linear cost function in its arguments and represents period k costs. I_k is the inventory at the beginning of period k, P_k is the aggregate production in period k, S_k is the sales in period k, W_k is the work force at the beginning of period k, and U_k is the change in work force during period k.

In order to reduce the above problem to the form of (1.1) and (1.2), we must assume that the sales are generated by a first order autoregressive scheme of the form

(4.5) $\xi_{k+1} = \Gamma \xi_k + v_k$

(4.6) $\lambda_k = M\lambda_k + w_k$

where $\xi_k' = (S_k, S_{k-1}, \ldots, S_{k-n})$. Then (4.5) and (4.6) could be incorporated into (4.2) to yield a system of the form of (1.1) and (1.2). Besides only being able to consider sales generated by an autoregressive scheme , the present results are also limited to costs which are separable in the state and control variables. This would rule out, for example, costs of the form (see [2])

$$\text{Cost of Overtime}_k = c_1(P_k - c_2W_k)^2 + c_3P_k + c_4W_k, \quad c_1, c_2 > 0,$$

since such costs lead to terms of the form $-2c_1c_2P_kW_k$. For the above reasons it seems appropriate to generalize the fundamental model (1.1)-(1.2) to the form

(4.7) $x_{k+1} = Ax_k + Du_k + s_k + v_k$

(4.8) $y_k = Mx_k + w_k$

where all quantities above are defined as in (1.1)-(1.2) except s_k which is a deterministic p-vector.

Recent work (see [9]) in Kalman filter techniques has generalized the underlying model to which the Kalman filter is applicable to the form of (4.7)-(4.8). The fundamental filter equations (1.4) remain unchanged in this case and therefore the results on their asymptotic behavior are still applicable. Moreover, (4.7) and (4.8) are clearly directly related to the form of the aggregate planning problem (4.1)-(4.4). It remains to be determined whether the essential properties of equations (1.6)-(1.8) will hold for the system (4.7)-(4.8). It is my conjecture that the results of theorems 2,3,and 4 hold for the system (4.7)-(4.8) whenever the cost function in (1.5) is a quadratic-linear function (including state-control cross-product terms) provided that the cost function is convex and strictly convex in the controls, u_k, and when, in addition, the terms, s_k, are bounded by a stationary linear system. Verification

of this conjecture involves first resolving the dynamic program leading to (1.6)-(1.8) with added cross product terms in u_k and x_k and subject to (4.7) and (4.8) instead of (1.1) and (1.2). The present work and that reported in [1] and [7] provides a foundation for further studies in this direction.

REFERENCES

1. W. N. Anderson, Jr., G. B. Kleindorfer, P. R. Kleindorfer, and M. B. Woodroofe, "Consistent Estimates of the Parameters of a Linear System," To appear in Annals of Math Stat, Dec 1969.

2. C. C. Holt, F. Modigliani, J. F. Muth, and H. A. Simon, Planning Production, Inventories, and Work Force. Prentice-Hall, Englewood Cliffs, N. J., 1960.

3. T. L. Gunckel, III, and G. F. Franklin, "A General Solution for Linear Sampled-data Control," J. Basic Eng Trans, ASME, Vol 85D (June, 1963), 197-201.

4. R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," J. Basic Eng Trans, ASME, Vol 82D (1960), 35-45.

5. R. E. Kalman, "New Methods in Wiener Filtering Theory," In Proc First Symposium on Engineering Applications of Random Function Theory and Probability, Wiley, New York, 1963, 270-388.

6. G. B. Kleindorfer, "An Adaptive Control Model for Information System Design," Unpublished doctoral dissertation, Carnegie-Mellon University, Pittsburgh, Pa, May, 1968.

7. G. B. Kleindorfer and P. R. Kleindorfer, "Quadratic Performance Criteria with Linear Terms in Discrete-Time Control," IEEE Trans on Automatic Control, Vol. AC-12, (June, 1967), 320-321.

8. H. A. Simon, "Dynamic Programming Under Uncertainty with a Quadratic Criterion Function," Econometrica, Vol. 24, (1956), 74-81.

9. H. W. Sorenson, "Kalman Filtering Techniques," In C. T. Leondes (ed.), Advances in Control Systems, Vol. 3, (1966), Academic Press, New York, 219-292.

10. R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1962.

REFERENCES

1. H. N. Anderson, Jr., [illegible] To appear in [illegible]

2. [illegible] duction [illegible] 1960.

3. [illegible] 201.

4. [illegible]

5. [illegible]

6. [illegible]

7. [illegible]

8. [illegible]

9. [illegible]

10. [illegible]