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Abstract

We study a problem that is the "inverse" of Merton (1971). For a given consumption-portfolio policy, we provide necessary and sufficient conditions for it to be optimal for "some" agent with an increasing, strictly concave, time-additive, and state independent utility function when the risky asset price follows a general diffusion process. These conditions involve a set of consistency and state independency conditions and a partial differential equation satisfied by the consumption-portfolio policy. We also provide an integral formula which recovers the utility function that supports a given optimal policy. The inverse optimal problem studied here should be viewed as a dynamic recoverability problem in financial markets with continuous trading.

1 Introduction

The study of an individual's optimal consumption-portfolio policy in continuous time under uncertainty has been a central topic in financial economics; see, for example. Merton (1971), Cox and Huang (1989, 1991), and He and Pearson (1991). The main question addressed in this literature is: Does there exist an optimal consumption-portfolio policy for an economic agent represented by a time-additive and state-independent utility function and what are the properties of his/her optimal policy if it indeed exists?

In this paper we address the "inverse" of the above consumption-portfolio problem. For a general specification of the asset price process, we investigate the necessary and sufficient conditions for a given consumption-portfolio policy to be optimal for "some" increasing, strictly concave, time-additive, and state independent utility function. A consumption-portfolio policy that can be "supported" by such a utility function is called an *efficient* policy. We also provide an integral formula which recovers the utility function that supports a given efficient policy.

The inverse problem studied here can be viewed as a dynamic recoverability problem in financial markets with continuous trading; see Kurz (1969) and Chang (1988) for related problems. Our objective here is to recover an economic agent's preferences from the *observed* consumption-portfolio policy that has been specified for a given asset price process. Since our emphasis is in analyzing an individual's consumption-portfolio policy in a continuous time securities market environment, the inverse problem studied here and the solution method employed in this paper are very different from those of Kurz (1969) and Chang (1988), who study an inverse problem in the theory of optimal growth.

Cox and Leland (1982) are the first to characterize efficient consumption-portfolio policies when the asset price follows a geometric Brownian motion; also see Black (1988). Our contribution in this paper lies in giving a characterization of the efficient consumption-portfolio policies when the asset price follows a general diffusion process. Since our characterization of efficient consumption-portfolio policies is derived for a general specification of the price process, we can also use the same approach to answer a related question: Can a given consumption-portfolio policy be optimal for a given utility function and "some" diffusion price process or for "some" utility function and "some" diffusion price process?

The motivation for studying the inverse optimal problem when the asset price follows a general diffusion process is twofold. First, accumulating empirical evidence suggests that the stock price processes, and especially the price processes for portfolios, are not best described by a geometric

Brownian motion; see, for example, Black (1976) and Lo and MacKinlay (1988). As a result, for the study of optimal consumption-portfolio policies one needs to consider price processes more general than the geometric Brownian motion. Second, when the price process is not a geometric Brownian motion, the calculation of an optimal consumption-portfolio policy is extremely complicated; see, for example, Cox and Huang (1989) and He and Pearson (1991). As a result, in practice, certain rules of thumb policies are usually followed. It is thus important to have the necessary and sufficient conditions for a given policy to be consistent with utility maximization.

Our strategy to solve the inverse optimal problem consists of two steps. First, we take as given two real-valued functions, the first of which gives the consumption and the second of which gives the dollar amount invested in the single risky asset, for any given levels of the individual's wealth, the risky asset price, and the time. These two functions completely specify the consumption-portfolio policy of an individual. We then characterize the efficiency of this consumption-portfolio policy through a set of necessary and sufficient conditions that are imposed solely on the given consumption-portfolio policy. Second, we present an integration procedure that recovers the utility function supporting an efficient consumption-portfolio policy.

Besides deriving the necessary and sufficient conditions for efficient consumption-portfolio policies, we also obtain some characterizations of an efficient policy that are of independent interest. For example, it is shown that, when there is no intermediate consumption, an efficient policy must make the risk tolerance of the indirect utility function in units of the riskless asset a martingale (subject to some regularity conditions) under the so-called risk neutral probability (to be defined formally later). As will be made clear later, this result turns out to be the most significant restriction for a given consumption-portfolio policy to be efficient. It implies in particular that, when the price process is a geometric Brownian motion and there is no intermediate consumption, the present value of the dollar amount invested in the stock at any future date is equal to the dollar amount currently in the stock. This holds true for all efficient consumption-portfolio policies.

Our paper is related to the characterization of efficient portfolios in a one-period setting due to Peleg (1975), Peleg and Yaari (1975), Dybvig and Ross (1982), and Green and Srivastava (1985, 1986). Because of the single period setting, dynamic trading rules are not considered in this literature. In contrast, the emphasis of this paper is to analyze efficiency of and to recover utility functions from dynamic trading rules. Finally, our paper is related to the classical "integrability" problem of revealed preference, which asks whether preferences are determined by the entire demand correspondence; see, for example, Mas-Colell (1977) and Geanakoplos and Polemarchakis (1990).

The rest of this paper is organized as follows. Section 2 formulates a dynamic consumption-

portfolio problem and derives necessary conditions satisfied by an indirect utility function. Section 3 shows how to express these necessary conditions solely in terms of the given consumption-portfolio policy and presents the necessary conditions for efficiency. This section also gives some examples to demonstrate how to use the necessary conditions for efficiency and provides some characterizations of an optimal consumption-portfolio policy that are of independent interest. Section 4 shows that the necessary conditions for efficiency are also sufficient under some regularity conditions. It is then demonstrated how to recover the utility function that supports an efficient policy. Section 5 contains more examples illustrating our results and Section 6 presents some concluding remarks.

2   The Setup

Consider a securities market economy with a finite horizon $[0, T]$ in which there is one stock and one bond available for trading.[1] The bond price grows exponentially at a constant rate r, the riskless interest rate. The stock does not pay dividends[2], and its price follows a diffusion process whose dynamics is described by the stochastic differential equation[3]

$$dS(t) = \mu(S(t), t)S(t)dt + \sigma(S(t), t)S(t)dw(t), \quad t \in [0, T],$$

where w is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. For notational simplicity, we assume that starting from any $x > 0$ and any time $t \in [0, T)$, the price process can access any $y > 0$ over any time interval $[t, t + \epsilon]$ for however small $\epsilon > 0$, and zero is always inaccessible.[4] This assumption implies that the stock price is strictly positive with probability one.[5] Investors are assumed to have access only to the information contained in historical prices, that is, the information the investors have at time t is the sigma-field generated by $\{S(s); 0 \le s \le t\}$. For brevity, we will sometimes simply use $\mu(t)$ and $\sigma(t)$ to denote $\mu(S(t), t)$ and $\sigma(S(t), t)$, respectively.

We assume that there exists an equivalent martingale measure, or a risk neutral probability[6] Q for the price process. Given our current setup, this equivalent martingale measure must be uniquely represented by

$$Q(A) = \int_A \xi(\omega, T)P(d\omega) \quad \forall A \in \mathcal{F},$$

---

[1] In applications, one can take the risky asset to be an index portfolio.

[2] Nothing will be affected if the stock pays dividends at rates that depend on the stock price only.

[3] Implicit in this is the hypothesis that a solution to the stochastic differential equations exists.

[4] Otherwise, we have to qualify many statements to be made so that they are only valid over the range of the stock price. This does not change the essence of the results but will complicate the notation, which is already heavy.

[5] We will use weak relations throughout. For example, positive means nonnegative, concave means weakly concave, and so forth. For strict relations, we use, for example, strictly increasing, strictly concave, and so forth.

[6] See Harrison and Kreps (1979) for the former and Cox and Ross (1976) for the latter.

where

$$\xi(t) \equiv \exp\left\{ \int_0^t \hat{\kappa}(S(s),s)dw(s) - \frac{1}{2}\int_0^t |\hat{\kappa}(S(s),s)|^2 ds \right\} \tag{1}$$

and

$$\hat{\kappa}(S(t),t) \equiv -\frac{\mu(S(t),t) - r}{\sigma(S(t),t)}.$$

Under Q, the stock price dynamics becomes

$$dS(t) = rS(t)dt + \sigma(t)S(t)dw^*(t), \quad t \in [0,T],$$

where w^* is a standard Brownian motion under Q.

A consumption and portfolio policy (C,A) is a pair of functions, $(C(W,S,t), A(W,S,t))$, denoting the consumption rate and the dollar amount invested in the risky asset at time $t \in [0,T]$, respectively, when the wealth is W and the price of the risky asset is S. For simplicity, we will often use $C(t)$ and $A(t)$ to denote $C(W,S,t)$ and $A(W,S,t)$, respectively. Let $W(t)$ denote the wealth at time t. From Merton (1971), the dynamic budget constraint a consumption-portfolio policy must satisfy is, starting from any $t \in [0,T]$,

$$dW(s) = [rW(s) + A(s)(\mu(s) - r) - C(s)]ds + A(s)\sigma(s)dw(s), \quad s \in [t,T]. \tag{2}$$

A policy (C,A) is admissible if

A1. the drift and the diffusion term of the wealth process satisfy a linear growth condition and a local Lipschitz condition,[7] and starting from any $x > 0$ and any time $t \in [0,T)$, the wealth process can access any $y > 0$ over any time interval $[t, t+\epsilon]$ for however small $\epsilon > 0$.

This condition ensures that there exists a unique solution to the stochastic differential equation (2).

The policy (C,A) is said to be *efficient*, if it is admissible and there exist a utility function for intermediate consumption and a utility function for final wealth, $u(x,t) : \Re_+ \times [0,T] \to \Re \cup \{-\infty\}$ and $V(x) : \Re_+ \to \Re \cup \{-\infty\}$, respectively, which are twice continuously differentiable, increasing and concave in x, and either $u(x,t)$ for almost all t or $V(x)$ is strictly concave in x such that (C,A)

---

[7]A function $f : \Re^N \times [0,T] \to \Re$ is said to satisfy a linear growth condition if $|f(x,t)| \le K(1+|x|)$ for all x and t, where $|x|$ denotes the Euclidean norm of x and K is a strictly positive scalar. A function $f : \Re^n \times [0,T] \to \Re$ satisfies a local Lipschitz condition if for any $M > 0$ there is a constant K_M such that for all $y,z \in \Re^n$ with $|y| \le M$ and $|z| \le M$ and $t \in [0,T]$, we have $|f(y,t) - f(z,t)| \le K_M|y - z|$, where $|y|$ denotes the Euclidean norm.

is the solution to the following dynamic consumption and portfolio problem:[8]

$$\sup_{C \geq 0, A} \quad \mathbf{E}_t^{xy} \left[\int_t^T u(C(s), s)dt + V(W(T)) \right]$$

$$\text{s.t.} \quad (2) \text{ holds,}$$

$$(A, C) \text{ is admissible,}$$

$$W(s) \geq 0 \quad s \in [t, T], \tag{3}$$

$$W(t) = x,$$

$$S(t) = y,$$

for any $x > 0$, $y > 0$, and $t \in [0, T]$, where $\mathbf{E}_t^{xy}[\cdot]$ is the expectation at time t conditional on $W(t) = x$ and $S(t) = y$. The third condition is a positive wealth constraint that rules out the possibility of creating something out of nothing; see Dybvig and Huang (1988).

Note also that there is a vast literature on the existence and the characterization of an optimal consumption-portfolio policy for a *given* pair of utility functions (u, V); see Merton (1971), Cox and Huang (1989, 1991), and He and Pearson (1991), for example. Our purpose here is different from that of this literature. We take a consumption-portfolio policy (C, A) as given and ask: What are the necessary and sufficient conditions for it to be an optimal policy for *some* pair of utility functions (u, V)? As we don't know the utility functions to begin with, these necessary and sufficient conditions can only involve the given policy (C, A).

Now suppose that (C, A) is efficient. Then there exists (u, V) so that (C, A) solves (3) for any $x > 0$, $y > 0$, and $t \in [0, T]$. Let $J(W, S, t)$ be the value of the objective function of (3), or the *indirect utility function*, given that the wealth and the risky asset price at time t are W and S, respectively. By the monotonicity and the strict concavity of either $u(x, t)$ for almost all t or $V(x)$ in x, J must be increasing and strictly concave in W. We will restrict our attention to efficient policies (C, A) such that the following conditions hold:

A2. $A(W, S, t)$ is continuous and $C(W, S, t)$ is continuously differentiable in W and S for $W > 0$, $S > 0$, and for all $t \in [0, T]$, and $C(W, S, t)$ is such that

$$\mathbf{E} \left[\int_0^T |C(W(t), S(t), t)|^2 dt \right] < \infty$$

for the wealth process defined by (C, A);

A3. for $(W, S, t) \in [0, \infty) \times (0, \infty) \times [0, T]$, $R(W, S, t) \equiv -\frac{J_W(W, S, t)}{J_{WW}(W, S, t)}$ and $H(W, S, t) \equiv -\frac{J_{WS}(W, S, t)}{J_{WW}(W, S, t)}$, and for $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T]$, $N(W, S, t) \equiv \frac{J_{Wt}(W, S, t)}{J_W(W, S, t)}$ are well-defined; and for

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[8]Imposing strict concavity on u ensures that the consumption function is continuous in W, which we will assume later.

$(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T]$, R and H are twice continuously differentiable in (W, S) and continuously differentiable in t, and N is continuously differentiable in (W, S), where the subscripts denote partial derivatives;

A4. J satisfies a polynomial growth condition[9] and for any feasible policy (C, A) and its associated wealth process defined by (2)

$$\mathbf{E}\left[\int_{\tau}^{T} |\mathcal{L}J(s) + J_t(s)| ds\right] < \infty, \quad \tau \in [0, T)$$

where

$$\mathcal{L}J \equiv \frac{1}{2}J_{WW}A^2\sigma^2 + J_{WS}A\sigma^2 S + \frac{1}{2}J_{SS}\sigma^2 S^2 + J_W[rW + A(\mu - r) - C] + J_S\mu S$$

and

A5. the wealth never reaches zero before time T.

The interpretation of the terms defined in A3 will be given later. Henceforth, subscripts denote partial derivatives unless mentioned otherwise.

Condition A2 requires the policy (C, A) to be sufficiently smooth and C to be square-integrable. Condition A3 implies that J is continuous on its domain and allows us to work with many derivatives of J. Condition A4 enables us to work with the Bellman's equation. Condition A5 needs some explanation. Cox and Huang (1989, proposition 3.1) have shown that the optimally invested wealth for any (u, V) must not reach zero before T when the stock price follows a geometric Brownian motion. Merton (1990, theorem 16.2) generalizes this result to any diffusion price process obeying two regularity conditions satisfied by most of the processes with which financial economists have worked. Thus Condition A5 can be viewed as a necessary condition for (C, A) to be efficient.

Later in this section, we will add another regularity condition A6 that (C, A) must satisfy. Until then, an efficient (C, A) will be understood to satisfy Conditions A1–A5.

Now let (C, A) be efficient with corresponding utility functions (u, V). By Conditions A3 and A4, J satisfies the Bellman's equation (see, for example, Fleming and Soner (1992, chapter 3)):[10]

$$0 = \max_{C \geq 0, A} \left\{ u(C, t) + J_t + [rW + A(\mu - r) - C]J_W + \mu S J_S + \frac{1}{2}\sigma^2 A^2 J_{WW} + \sigma^2 S A J_{WS} + \frac{1}{2}\sigma^2 S^2 J_{SS} \right\}$$

---

[9]A function $f : \Re^N \times [0, T] \to \Re$ is said to satisfy a polynomial growth condition if $|f(x, t)| \leq K(1 + |x|^\gamma)$ for all x and t, where $|x|$ denotes the Euclidean norm of x and K and γ are two strictly positive scalars.

[10]It is certainly not necessary for an indirect utility function to satisfy the Bellman's equation and a polynomial growth condition. For the former see Cox and Huang (1989) and for the latter see Footnote 20.

for all $(W, S, t) \in (0, \infty) \times (0, \infty) \times (0, T)$, with the boundary conditions

$$
\begin{aligned}
J(W, S, T) &= V(W), \\
J(0, S, t) &= \int_t^T u(0, s) ds + V(0),
\end{aligned}
\tag{4}
$$

where we have suppressed the arguments of J, C, and A.

Note that the second boundary condition is a consequence of the positive wealth constraint and it necessitates immediately that

$$
\begin{aligned}
C(0, S, t) &= 0, \quad \text{and} \tag{5} \\
A(0, S, t) &= 0. \tag{6}
\end{aligned}
$$

The first order necessary conditions for the dynamic consumption and portfolio problem (3) are, for all $(W, S, t) \in (0, \infty) \times (0, \infty) \times (0, T)$,

$$
\begin{aligned}
u_c(C(W, S, t), t) &\leq J_W(W, S, t), \quad \text{if } C(W, S, t) = 0, \\
u_c(C(W, S, t), t) &= J_W(W, S, t), \quad \text{if } C(W, S, t) > 0;
\end{aligned}
\tag{7}
$$

$$
A(W, S, t) = \left(\frac{\mu(S, t) - r}{\sigma^2(S, t)} \right) R(W, S, t) + S H(W, S, t),
\tag{8}
$$

where we have used the notation defined in Condition A3. Note that $R(W, S, t)$ on the right-hand side of (8) is the inverse of the Arrow-Pratt measure of the absolute risk aversion of the indirect utility function, henceforth the *risk tolerance*, and the second term is the "hedging demand against adverse changes in the consumption/investment opportunity set".[11] By Condition A3, A must be twice continuously differentiable in $(W, S) \in (0, \infty) \times (0, \infty)$ and is continuously differentiable in $t \in (0, T)$.

Furthermore, (7) and the chain rule of differentiation imply that $u_{cc} C_W = J_{WW}$ when $C > 0$. Since $J_{WW} < 0$, we must have $C_W > 0$ when $C > 0$. That is, strictly positive consumption can only occur when the marginal propensity to consume is strictly positive.

The first order necessary conditions and the Bellman equation place strong restrictions on (C, A). But these restrictions are expressed in terms of the partial derivatives of J, which is not known (as u and V are unknown). However, it will be shown in the next section that these necessary conditions can be transformed so that they can be expressed solely in terms of (C, A). Therefore, one can directly check whether a given pair (C, A) is a candidate for an efficient policy without any other information. Furthermore, with some regularity conditions, we will show that the necessary conditions for (C, A) to be efficient are also sufficient.

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[11] See Breeden (1984) and Merton (1973).

Clearly, for all $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T)$, the assumed differentiability of J implies that

$$\frac{\partial}{\partial S}\left(\frac{1}{R}\right) = -\frac{\partial}{\partial W}\left(\frac{H}{R}\right), \tag{9}$$

$$\frac{\partial N}{\partial S} = \frac{\partial}{\partial t}\left(\frac{H}{R}\right), \tag{10}$$

$$\frac{\partial N}{\partial W} = \frac{1}{R^2}\frac{\partial R}{\partial t}. \tag{11}$$

Now, for all $W > 0$, $S > 0$, and $t \in [0, T]$, define three functions

$$O(W, S, t) \equiv \int_{\underline{W}}^{W} \frac{1}{R(z, S, t)}dz, \tag{12}$$

$$X(S, t) \equiv \int_{\underline{S}}^{S} \frac{H(\underline{W}, \eta, t)}{R(\underline{W}, \eta, t)}d\eta, \tag{13}$$

$$Y(t) \equiv \int_{0}^{t} N(\underline{W}, \underline{S}, \tau)d\tau, \tag{14}$$

where $\underline{W}$ and $\underline{S}$ are two arbitrary strictly positive constants, and where we have used the convention that $\int_a^b = -\int_b^a$ when $a > b$. In addition, define a function $\mathcal{U}$,

$$\ln \mathcal{U}(W, S, t) \equiv -O(W, S, t) + X(S, t) + Y(t). \tag{15}$$

Note that, for all $W > 0$, $S > 0$, $t \in [0, T)$, by the continuity of R, H, and N;

$$\ln \mathcal{U}(W, S, t) = \ln J_W(W, S, t) - \ln J_W(\underline{W}, \underline{S}, 0)$$
$$\begin{cases} = \ln u_c(C(W, S, t), t) - \ln J_W(\underline{W}, \underline{S}, 0), & \text{if } C(W, S, t) > 0; \\ \geq \ln u_c(0, t) - \ln J_W(\underline{W}, \underline{S}, 0), & \text{if } C(W, S, t) = 0. \end{cases} \tag{16}$$

where we have used the first order conditions (7). For $t < T$, since $C_W > 0$ for all $C > 0$, we can write, for all $x > 0$ in the range of $C(W, S, t)$:

$$\ln \mathcal{U}(C^{-1}(x, S, t), S, t) = \ln u_c(x, t) - \ln J_W(\underline{W}, \underline{S}, 0), \tag{17}$$

where C^{-1} denotes the inverse of C with respect to its first argument. From this relation, we conclude that its left-hand side must be independent of S when $C(W, S, t) > 0$ as the utility function is state independent. Furthermore, by the concavity of $u(x, t)$ in x and (7), we must have, for all $W > 0$, $S > 0$, and $t \in (0, T)$ so that $C(W, S, t) = 0$,

$$\mathcal{U}(W, S, t) \geq \lim_{x \downarrow 0} \mathcal{U}(C^{-1}(x, \hat{S}, t), \hat{S}, t) \quad \forall \hat{S} > 0. \tag{18}$$

Relations (16)–(18) involve J and its derivatives in the interior of their domain. We now proceed to derive some conditions at the boundary of their domain. For this purpose, we use a result of

Cox and Huang (1989, section 2.3), which states that, under the square-integrability assumption of Condition A2, along the optimally invested wealth process, there exists a scalar $\lambda > 0$ so that[12]

$$J_W(W(t), S(t), t) \begin{cases} = & \lambda\xi(t)e^{-rt}, & t \in [0, T), \\ \leq & \lambda\xi(T)e^{-rT} & t = T, \end{cases} \tag{19}$$

where the inequality holds as an equality when $W(T) > 0$. Note that the λ above is a Lagrangian multiplier and $\xi(t)e^{-rt}$ is the Arrow-Debreu state price at time 0 for time t consumption per unit of probability P. Since W, S, and ξ are processes with continuous paths, if the optimally invested wealth reaches zero at T, (19) implies that J_W is continuous except possibly when $W = 0$ at T and

$$\lim_{\substack{W \downarrow 0 \\ t \uparrow T}} J_W(W, S, t) \geq J_W(0, S, T) = V'(0), \quad \forall S > 0, \tag{20}$$

where the equality follows from (4).[13]

Given the above discussion, we now impose one more regularity condition on (C, A):

A6. $R(W, S, t)$, $H(W, S, t)$, $N(W, S, t)$, and their derivatives are continuous functions of t at $t = T$ except possibly for $W = 0$.

This condition together with Condition A3 accomplishes two things. First, (16)–(18) can be extended to $t = T$ for $W > 0$ and $S > 0$, and we have

$$\begin{aligned} \ln \mathcal{U}(W, S, T) &= \ln V'(W) - \ln J_W(\underline{W}, \underline{S}, 0), \quad W > 0, S > 0, \\ \lim_{\substack{W \downarrow 0 \\ t \uparrow T}} \mathcal{U}(W, S, t) &\geq \ln V'(0) - \ln J_W(\underline{W}, \underline{S}, 0) = \lim_{W \downarrow 0} \ln \mathcal{U}(W, \hat{S}, T), \quad \forall S > 0, \hat{S} > 0, \end{aligned} \tag{21}$$

where we note that the first relation indicates that $\ln \mathcal{U}(W, S, T)$ is a function of W only, and the equality in the second relation follows from the fact that V is continuously differentiable. In addition, Q, X, and N are twice continuously differentiable in (W, S) and continuously differentiable in t for $W > 0$, $S > 0$, and $t \in [0, T]$.

Second, we conclude by continuity and $H(W, S, T) = 0$ that

$$\lim_{t \uparrow T} H(W, S, t) = 0, \quad W > 0, \ S > 0, \tag{22}$$

and

$$\begin{aligned} A(W, S, T)\left(\frac{\mu(S,T)-r}{\sigma^2(S,T)}\right)^{-1} &= \lim_{t \uparrow T}\left(R(W, S, t) + SH(W, S, t)\left(\frac{\mu(S,t)-r}{\sigma^2(S,t)}\right)^{-1}\right) \\ &= R(W, S, T) = -\frac{V'(W)}{V''(W)}, \quad W > 0, \ S > 0, \end{aligned} \tag{23}$$

---

[12]The conditions stated below and henceforth implicitly assume that the optimization problem starts at $t = 0$. Similar conditions hold for other starting times t and other starting states $W(t) = x$ and $S(t) = y$.

[13]The possible discontinuity of J_W at T when $W = 0$ is the reason that in Condition A3 N is not assumed to be well-defined there.

which is a function of W only.

We will term relations (9)-(11) *consistency conditions*, since they basically require that high order derivatives of the indirect utility function J exist and that J can be differentiated consistently with respect to any ordering of W, S and t. Relations (17), (18), (21), (22) and (23) will be termed state independency conditions, as they follow from our assumption that u and V are both state-independent. Note that if we defined efficiency more broadly to include state-dependent utility functions, then obviously the state independency conditions need not be satisfied. We will denote henceforth the set of efficient policies satisfying Conditions A1–A6 by $\mathcal{E}$. For brevity, (C, A) is said to be efficient if it is an element of $\mathcal{E}$.

Before leaving this section, we record one well-known fact about $(C, A) \in \mathcal{E}$, namely, that the current wealth plus the cumulative past consumption, both in units of the bond, is a martingale under Q.

Proposition 1 *Let $(C, A) \in \mathcal{E}$. Then $W(t)e^{-rt} + \int_0^t C(W(s), S(s), s)e^{-rs}ds$ is a martingale under Q.*

PROOF. See, for example, Dybvig and Huang (1988). ∎

3 Necessary Conditions for Efficiency

In this section we derive necessary conditions for a given (C, A) to be efficient. Our derivation proceeds as follows. First, we use the first order conditions to express the function N in terms of R, H, C, A, and the derivatives of R and H. Second, we write R and H in terms of C, A and their derivatives. As a result, R, H and N are explicit functions of C, A and their derivatives. Necessary conditions derived in Section 2 about R, H, and N are then brought to bear on the given functions C and A.

In this process, we also derive some characterizations of $(C, A) \in \mathcal{E}$ that are of independent interest. For example, we show that when there is no intermediate consumption, the risk tolerance process along the optimally invested wealth in units of the riskless asset must be a local martingale under the risk neutral probability. This implies that when the hedging demand normalized by the bond price is a decreasing process and the risk premium per unit of variance is positive and increasing over time, one expects an efficient portfolio policy to invest less in the stock over time in present value terms normalized by the risk premium per unit of variance. When the hedging demand normalized by the bond price is increasing and the risk premium per unit of variance is decreasing over time, the opposite is true under an additional condition.

Hereafter, we will use $\kappa(S,t)$, or simply $\kappa(t)$ to denote $(\mu(S,t) - r)/\sigma^2(S,t)$, which is the risk premium on the stock per unit of the variance on its rate of return. Assume $\kappa(S,t) \neq 0$ except possibly on a set of S and t that is of Lebesgue (product) measure zero.

We begin by giving a lemma and a proposition. The lemma expresses the function N in terms of R, H, C, A, and the derivatives of R and H. Then we show in the proposition that R must satisfy a linear partial differential equation.

Lemma 1 *Let* $(C, A) \in \mathcal{E}$. *Then, for* $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T]$,

$$
\begin{aligned}
N &= \frac{1}{2}\sigma^2 A^2 \left(\frac{1}{R}\right)_W + \sigma^2 A S \left(\frac{1}{R}\right)_S - \frac{1}{2}\sigma^2 S^2 \left(\frac{H}{R}\right)_S \\
&\quad + (rW + A(\mu - r) - C)\frac{1}{R} - \mu S \frac{H}{R} - \frac{\kappa^2 \sigma^2}{2} - r.
\end{aligned}
\tag{24}
$$

That is, N can be expressed in terms of R, H, C, A, and the derivatives of R and H.

PROOF. We will prove (24) for $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T)$. The assertion for $t = T$ then follows from continuity.

Differentiating the Bellman's equation with respect to W and simplifying the resulting equation using the first order conditions (7) and (8), we get

$$
0 = J_{Wt} + rJ_W + J_{WW}(rW + A(\mu - r) - C) + J_{WS}\mu S + \frac{1}{2}\sigma^2 A^2 J_{WWW} + A\sigma^2 S J_{WWS} + \frac{1}{2}\sigma^2 S^2 J_{WSS},
$$

where we have used the fact that $C_W = 0$ on the set $\{(W, S, t) : C(W, S, t) = 0\}$. This equation implies that the drift of dJ_W is $-rJ_W$. Since (8) implies that the diffusion term of dJ_W is $-\kappa\sigma J_W$, we conclude that

$$
dJ_W = -rJ_W\,dt - \kappa\sigma J_W\,dw.
\tag{25}
$$

Hence, the drift of $d\ln J_W$ must be equal to $-r - \frac{\kappa^2\sigma^2}{2}$. However, the drift of $d\ln J_W$ is

$$
\begin{aligned}
&\frac{1}{2}\sigma^2 A^2 \left(\frac{J_{WW}}{J_W}\right)_W + \sigma^2 A S \left(\frac{J_{WW}}{J_W}\right)_S + \frac{1}{2}\sigma^2 S^2 \left(\frac{J_{WS}}{J_W}\right)_S + (rW + A(\mu - r) - C)\frac{J_{WW}}{J_W} \\
&\quad + \mu S \frac{J_{WS}}{J_W} + \frac{J_{Wt}}{J_W} \\
&= -\frac{1}{2}\sigma^2 A^2 \left(\frac{1}{R}\right)_W - \sigma^2 A S \left(\frac{1}{R}\right)_S + \frac{1}{2}\sigma^2 S^2 \left(\frac{H}{R}\right)_S - (rW + A(\mu - r) - C)\frac{1}{R} + \mu S \frac{H}{R} + N.
\end{aligned}
$$

We get (24) and this completes our proof. ∎

Proposition 2 *For $(W, S, t) \in (0, \infty) \times (0, \infty) \times [0, T]$, the function R must satisfy the following partial differential equation:*

$$\frac{1}{2}\sigma^2 A^2 R_{WW} + A\sigma^2 S R_{WS} + \frac{1}{2}\sigma^2 S^2 R_{SS} + (rW - C)R_W + rS R_S + R_t + C_W R - rR = 0. \quad (26)$$

Consequently, $R(W(t), S(t), t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ (along the optimally invested wealth) is a positive local martingale[14], and thus a supermartingale[15], under the equivalent martingale measure Q on $[0, T]$.

PROOF. Differentiating (24) with respect to W yields

$$\begin{aligned}
N_W &= \frac{1}{2}\sigma^2 A^2 \left(\frac{1}{R}\right)_{WW} + \sigma^2 AS \left(\frac{1}{R}\right)_{WS} - \frac{1}{2}\sigma^2 S^2 \left(\frac{H}{R}\right)_{WS} \\
&\quad + (rW + A(\mu - r) - C)\left(\frac{1}{R}\right)_W - \mu S\left(\frac{H}{R}\right)_W + \sigma^2 A A_W \left(\frac{1}{R}\right)_W \\
&\quad + \sigma^2 S A_W \left(\frac{1}{R}\right)_S + (r + A_W(\mu - r) - C_W)\frac{1}{R}
\end{aligned}$$

Using (9) and (11), we can rewrite the above equation as

$$\begin{aligned}
\frac{1}{R^2}R_t &= \frac{1}{2}\sigma^2 A^2 \left(\frac{1}{R}\right)_{WW} + \sigma^2 AS \left(\frac{1}{R}\right)_{WS} + \frac{1}{2}\sigma^2 S^2 \left(\frac{1}{R}\right)_{SS} \\
&\quad + (rW + A(\mu - r) - C)\left(\frac{1}{R}\right)_W + \mu S\left(\frac{1}{R}\right)_S + \sigma^2 A A_W \left(\frac{1}{R}\right)_W \\
&\quad + \sigma^2 S A_W \left(\frac{1}{R}\right)_S + (r - C_W)\frac{1}{R} + A_W(\mu - r)\frac{1}{R}
\end{aligned}$$

Since

$$\left(\frac{1}{R}\right)_{XY} = -\frac{R_{XY}}{R^2} + 2\frac{R_X R_Y}{R^3}$$

for $X, Y = W, S$, we obtain

$$\begin{aligned}
&\frac{1}{2}\sigma^2 A^2 R_{WW} + A\sigma^2 S R_{WS} + \frac{1}{2}\sigma^2 S^2 R_{SS} + (rW - C)R_W + rS R_S + R_t + C_W R - rR \\
&= \sigma^2 A^2 \frac{R_W^2}{R} + 2\sigma^2 AS \frac{R_W R_S}{R} + \sigma^2 S^2 \frac{R_S^2}{R} + (r - \mu)S R_S \\
&\quad + A(\mu - r)R^2\left(\frac{1}{R}\right)_W + + \sigma^2 A A_W R^2\left(\frac{1}{R}\right)_W + \sigma^2 S A_W R^2\left(\frac{1}{R}\right)_S + A_W(\mu - r)R^2\frac{1}{R} \\
&= \frac{\sigma^2}{R}(A R_W + S R_S)^2 + (r - \mu)S R_S + (\mu - r)(A_W R - A R_W) \\
&\quad - \sigma^2 A_W(A R_W + S R_S). \quad (27)
\end{aligned}$$

---

[14]The process X is a local martingale under Q if there exists a sequence of stopping times τ_n with $\tau_n \to T$ Q-a.s., so that $\{X(t \wedge \tau_n), t \in [0, T]\}$ is a martingale under Q for all n. For the definition of a stopping time see, for example, Liptser and Shiryayev (1977, p.25).

[15]The process X is a supermartingale under Q if $\mathbf{E}^*[X(s)|\mathcal{F}_t] \leq X(t)$ Q-a.s. for $s \geq t$.

It remains to show that the right-hand side of the above equation is zero. We note by (8) and (9),

$$\frac{A}{R} = \kappa + S\frac{H}{R}.$$

Hence,

$$\left(\frac{A}{R}\right)_W = S\left(\frac{H}{R}\right)_W = S\frac{R_S}{R^2},$$

which implies

$$A_W = (SR_S + AR_W)/R. \tag{28}$$

Substituting this expression into the right-hand side of (27), we confirm that the right-hand side is indeed zero.

For the second assertion, we first show that R being a local martingale is implied by (26). From Condition A5, we know that the wealth never reaches zero before T. For any $t \in [0, T)$, apply Itô's lemma to $R(t)e^{-rt}$ and use (26) to get

$$R(W(t), S(t), t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$$

$$= R(W(0), S(0), 0) + \int_0^t e^{-rs}(R_W(s)A(s)\sigma(s) + R_S(s)\sigma(s)S(s))dw^*(s), \quad t \in [0, T).$$

Since $R > 0$ for all $W > 0$ and $C_W \geq 0$, the left-hand side is strictly positive. The right-hand side is a local martingale under Q on $[0, T)$ since it is an Itô integral. By the definition of a local martingale (see Footnote 14), a local martingale on $[0, T)$ is automatically a local martingale on $[0, T]$. In addition, it is well-known that a positive local martingale is a supermartingale; see, for example, Dybvig and Huang (1988, lemma 2). Thus $R(W(t), S(t), t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is a supermartingale on $[0, T]$. ∎

Cox and Leland (1982) are the first to point out this local martingale property of the risk tolerance process in the context where the stock price process is a geometric Brownian motion. Proposition 2 shows that this is in fact a general property of an efficient policy. In the case with no intermediate consumption, it states that the risk tolerance implied by an optimal policy must be a local martingale under the risk neutral probability measure. This is certainly true for the utility functions that have constant relative risk aversion, since the risk tolerance in this case is a linear function of the wealth and the optimally invested wealth in units of the bond is a martingale under the risk neutral probability. In general, given $C_W \geq 0$, the above proposition states that an efficient (C, A) must make the risk tolerance in units of the bond a supermartingale under Q on $[0, T]$ – one expects the risk tolerance in units of the bond, on the average according to Q, to go down in the future.

Remark 1 *In Proposition 2 we have stated the local martingale result on the whole of* $[0,T]$. *This implicitly assumes that the dynamic consumption-portfolio problem of (3) starts at* $t = 0$. *This local martingale result is of course true starting from any* $t \in [0,T)$, $W(t) = x$ *and* $S(t) = y$ *and the proof is identical. Our implicit hypothesis is for notational simplicity and will be made henceforth unless otherwise noted.*

We record an immediate corollary of Proposition 2.

Corollary 1 *Let* $(C, A) \in \mathcal{E}$. *Then*

$$\frac{A(t) - S(t)H(t)}{\kappa(t)}e^{-rt} + \int_0^t C_W(s)\frac{A(s) - S(s)H(s)}{\kappa(t)}e^{-rs}ds$$

is a positive local martingale and thus is a supermartingale under Q *on* $[0,T]$.

PROOF. The assertion follows directly from (8) and Proposition 2. ∎

Several implications of Corollary 1 deserve attention. First, consider the special case of a geometric Brownian motion stock price with $\mu > r$. Note that in this case the hedging demand is zero as $H = 0$, κ is a constant, $A = \kappa R > 0$ when $W > 0$, and Condition A5 is satisfied. Corollary 1 implies that

$$A(t)e^{-rt} + \int_0^t C_W(s)A(s)e^{-rs}ds$$

is a local martingale under Q on $[0,T]$. In addition, by Proposition 2, for all $(W,t) \in (0,\infty) \times [0,T]$,

$$\frac{1}{2}\sigma^2 A^2 A_{WW} + (rW - C)A_W - rA + C_W A + A_t = 0.$$

This is just proposition 3 of Cox and Leland (1982).[16] Since $C_W \geq 0$, these imply that $A(t)e^{-rt}$ is a positive supermartingale. That is, the present value of the dollar amount invested in stock in the future is less than the current amount invested in the stock. This, however, does not necessarily mean that one expects to shift value over time from the stock to the bond. To see this we recall from Proposition 1 that

$$W(t)e^{-rt} + \int_0^t C(s)e^{-rs}ds \quad t \in [0,T]$$

is a martingale under Q. This implies that

$$(W(t) - A(t))e^{-rt} + \int_0^t (C(s) - C_W(s)A(s))e^{-rs}ds \quad t \in [0,T)$$

---

[16]The reader familiar with Cox and Leland will note a difference between the result reported here and that in Cox and Leland. Here $A(t)e^{-rt} + \int_0^t C_W(s)A(s)e^{-rs}ds$ is shown to be a supermartingale under Q instead of a martingale. Indeed, with additional regularity conditions, $A(t)e^{-rt} + \int_0^t C_W(s)A(s)e^{-rs}ds$ becomes a martingale; see Corollary 2 below.

is a submartingale under Q. Note that the difference between $W(t)$ and $A(t)$ is the dollar amount invested in the bond. If $C - C_W A < 0$, $(W - A)e^{-rt}$ is also a submartingale under Q on $[0, T]$, and the optimal policy shifts value away from the stock to the bond and to consumption. When $C - C_W A > 0$, however, $(W(t) - A(t))e^{-rt}$ can indeed be a supermartingale under Q. In such a case, the policy shifts value away from both the stock and the bond to consumption.

The interpretation of Corollary 1 in the general case is a bit more complicated as there is now hedging demand. Assume for example that $\kappa(t)$ is a positive increasing process; that is, the risk premium per unit of variance increases over time, and $C_W = 0$. When the hedging demand normalized by the bond price, $H \cdot Se^{-rt}$, is a decreasing process (given that $H(W, S, T) = 0$ and S is strictly positive, this implies that the hedging demand is positive), Ae^{-rt}/κ is a supermartingale under Q. The present value of one's optimal investment in the stock in the future, per unit of κ, is lower than one's current investment in the stock, per unit of κ. Interpretations similar to the special case above can be made when $C_W \neq 0$ but with everything normalized by κ.

Under certain regularity conditions, Proposition 2 can be strengthened so that $R(t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is not only a local martingale but is indeed a martingale under Q. We record this result below in the second corollary of the proposition:

Corollary 2 *Suppose that $W(T) > 0$ a.s. and R and C_W satisfy a polynomial growth condition.*[17] *Then $R(t)e^{-rt} + \int_0^t C_W(s)R(s)e^{-rs}ds$ is a martingale on $[0, T]$ under Q.*

PROOF. The assertion is a consequence of the Feyman-Kac representation; see, for example, Karatzas and Shreve (1988, theorem 5.7.6). ∎

With the conditions of Corollary 2, we have a sharper interpretation of the intertemporal behavior of R as well A. For example, in the geometric Brownian motion case discussed above, when $C_W = 0$ and $W(T) > 0$ a.s., $A(t)e^{-rt}$ becomes a martingale under Q, and thus there is no shift of value over time away from or into the stock. This is the Cox-Leland result. Other interpretations are left to the reader. We now proceed to complete our derivation of the necessary conditions for (C, A) to be efficient.

First, consider the special case that $C(W, S, t) > 0$ for all $(W, S, t) \in (0, \infty) \times (0, \infty) \times (0, T)$. Since the first order condition (7) holds as an equality, the chain rule of differentiation and Conditions A2 and A3 imply that

$$H(W, S, t) = -\frac{C_S(W, S, t)}{C_W(W, S, t)} \quad \forall W > 0, S > 0, t \in [0, T). \tag{29}$$

---

[17] See Footnote 9.

Conditions A2 and A6 immediately necessitate that

$$H(W, S, T) = \lim_{t \to T} -\frac{C_S(W, S, t)}{C_W(W, S, t)} = 0, \quad W > 0. \tag{30}$$

Note that (30) places nontrivial restrictions on consumption policies. For example, any consumption policy that is time separable, in that $C(W, S, t) = c(W, S)f(t)$ for all $W > 0$, $S > 0$, and $t \in (0, T)$ for some functions c and f, can never be an efficient consumption policy unless it is independent of the stock price.

Now substituting (29) into (8) and using Conditions A2, A3, and A6 give

$$R = \left[A + S\frac{C_S}{C_W} \right] / \kappa, \quad \forall W > 0, S > 0, t \in [0, T]. \tag{31}$$

We have thus expressed R and H solely in terms of C, A and their derivatives. Define N by (24). The R, H, and N so defined must satisfy the necessary conditions stipulated in Proposition 2 and the consistency conditions and the state-independency conditions derived in Section 2.

The following theorem summarizes the above discussion.

Theorem 1 *Let $(C, A) \in \mathcal{E}$ with $C(W, S, t) > 0$ for all $W > 0$, $S > 0$, and $t \in (0, T)$. Define R, H, N as in (31), (29), (30), and (24), respectively, and Q, X, Y, and $\mathcal{U}$ as in (12), (13), (14), and (15), respectively. We must have*

(1) $A(0, S, t) = C(0, S, t) = 0$, $S > 0$, $t \in [0, T]$;

(2) $R(W, S, t) > 0$ for $W > 0$;

(3) $C_W(W, S, t) \geq 0$, and $C_W(W, S, t) > 0$ for $W > 0$;

(4) the state-independency conditions hold: $\mathcal{U}(C^{-1}(x, S, t), S, t)$ is independent of S for all $x > 0$ in the range of $C(W, S, t)$ and $t < T$, $\mathcal{U}(W, S, T)$ is independent of S for all $W > 0$, $\lim_{W \downarrow 0} \mathcal{U}(W, S, t) \geq \lim_{W \downarrow 0} \mathcal{U}(W, \hat{S}, T)$ for all $S > 0$ and $\hat{S} > 0$, $\lim_{t \uparrow T} \frac{C_S(W, S, t)}{C_W(W, S, t)} = 0$, and $A(W, S, T)/\kappa(S, T)$ is a function of W only for all $W > 0$;

(5) the consistency conditions (9), (10), and (11) hold; and

(6) (C, A) satisfies the PDE (26).

Second, consider the general case that C can be zero at strictly positive wealth levels. For example, when $u = 0$, then $C = 0$ for all wealth levels. In this case H cannot be expressed in terms of C and A directly using (7) in the region where $C = 0$. The following proposition is instrumental for expressing R and H in terms of C, A, and their derivatives generally.

Proposition 3 *Let* $(C, A) \in \mathcal{E}$. *For* $W > 0$, $S > 0$, *and* $t \in [0, T]$,

$$\Gamma_1 R + \Gamma_2 H = \sigma K(W, S, t), \tag{32}$$

where

$$\Gamma_1 = -\frac{1}{2}\left[\hat{\kappa}_{SS}\sigma^2 S^2 + 2\hat{\kappa}_S \mu S + 2\hat{\kappa}_t\right],$$

$$\Gamma_2 = \frac{1}{2}S\left[\sigma_{SS}\sigma^2 S^2 + 2\sigma_S\sigma^2 S + 2\sigma_S rS + 2\sigma_t\right],$$

$$
\begin{aligned}
K(W, S, t) &= \frac{1}{2}\sigma^2 A^2 A_{WW} + \sigma^2 SA A_{WS} + \frac{1}{2}\sigma^2 S^2 A_{SS} + (\sigma_S \sigma SA + rW)A_W \\
&\quad + (\sigma_S \sigma S^2 + Sr)A_S + \frac{1}{2}(\sigma_{SS}\sigma S^2 + 2r\sigma_S S/\sigma - 2r + 2\sigma_t/\sigma)A \\
&\quad + A_t + (C_W A - CA_W) + SC_S.
\end{aligned}
$$

PROOF. We will prove the assertion for $t \in (0, T)$. At $t = 0$ and T, the assertion follows from Conditions A2, A3, and A6 by continuity.

For any function f of W, S and t that is twice continuously differentiable in (W, S) and continuously differentiable in t, define the differential generator $\mathcal{L}$ under Q:

$$\mathcal{L}(f) = \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 f}{\partial W^2} + \sigma^2 SA \frac{\partial^2 f}{\partial W \partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (rW - C)\frac{\partial f}{\partial W} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t}.$$

Direct computation shows, for $W > 0$, $S > 0$, and $t \in (0, T)$,

$$\mathcal{L}\left(\frac{1}{R}\right) = -\left[\sigma^2 A A_W \left(\frac{1}{R}\right)_W + \sigma^2 A_W S\left(\frac{1}{R}\right)_S + (r - C_W)\left(\frac{1}{R}\right)\right], \tag{33}$$

$$\mathcal{L}\left(\frac{H}{R}\right) = \hat{\kappa}_S\hat{\kappa} - C_S\left(\frac{1}{R}\right) - r\left(\frac{H}{R}\right) + \frac{1}{2}(\sigma^2 A^2)_S\left(\frac{1}{R}\right)_W + (\sigma^2 AS)_S\left(\frac{1}{R}\right)_S - \frac{1}{2}(\sigma^2 S^2)_S\left(\frac{H}{R}\right)_S, \tag{34}$$

where we have used the fact that

$$\sigma A_W R = R_S \sigma S + R_W A\sigma,$$

which is a consequence of (28), and (26) for (33), and (8), (10), (11), and (24) for (34). Following the definition of K, we have

$$\mathcal{L}(\sigma A) = \sigma K + r\sigma A - \sigma A C_W - \sigma S C_S. \tag{35}$$

Since $\frac{\sigma A}{R} = -\hat{\kappa} + \sigma S\frac{H}{R}$,

$$\mathcal{L}\left(\frac{\sigma A}{R}\right) = -\mathcal{L}(\hat{\kappa}) + \mathcal{L}\left(\sigma S\frac{H}{R}\right).$$

Applying $\mathcal{L}(XY) = Y\mathcal{L}(X) + X\mathcal{L}(Y) + \sigma^2 A^2 X_W Y_W + \sigma^2 AS(X_S Y_W + X_W Y_S) + \sigma^2 S^2 X_S Y_S$ and substituting (33), (34) and (35) into the above equation, we get (32). ∎

Proposition 3 plays an important role. It allows us to express the unknown functions R and H in terms of A and C and their derivatives. This is done as follows. If $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$, we can solve from (8) and (32) for R and H as functions of A and its derivatives, except when $W = 0$ at T:

$$R = \frac{\sigma^2 SK - \sigma A\Gamma_2}{\sigma S\Gamma_1 + \hat{\kappa}\Gamma_2}, \tag{36}$$

$$H = \frac{\sigma A\Gamma_1 + \hat{\kappa}\sigma K}{\sigma S\Gamma_1 + \hat{\kappa}\Gamma_2}. \tag{37}$$

Once this is done, (26) becomes a PDE in A and C. In addition, substituting (36) and (37) into (24) expresses N solely in terms of C and A and their derivatives.

If $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0$, we cannot solve for R, H, and N in terms of C and A. Nevertheless, we still get a PDE that C and A must satisfy. We take two cases. First, $\Gamma_1 = \Gamma_2 = 0$ except possibly when $\hat{\kappa} = 0$. (Here we note that if $\kappa \neq 0$, $\Gamma_1 = 0$ only if $\Gamma_2 = 0$. In addition, $\Gamma_2 = 0$ implies $\Gamma_1 = 0$.) Then we have $K = 0$ except possibly when $\hat{\kappa} = 0$. Second, $\Gamma_1 \neq 0$ and $\Gamma_2 \neq 0$ except possibly when $\hat{\kappa} = 0$. Then (8) and (32) imply that

$$\frac{-\Gamma_1}{\hat{\kappa}} = \frac{\Gamma_2}{\sigma S} = \frac{K}{A} \tag{38}$$

except possibly when $\hat{\kappa} = 0$. In both cases, we have an equation solely in terms of A and C to verify. We now collect all the necessary conditions any $(C, A) \in \mathcal{E}$ has to satisfy:

Theorem 2 *For $(C, A) \in \mathcal{E}$, we must have*

(1) $A(0, S, t) = C(0, S, t) = 0$, $S > 0$ and $t \in [0, T]$; and $C_W(W, S, t) > 0$ when $C > 0$, $W > 0$, $S > 0$, $t \in [0, T]$;

(2) Suppose $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$. Define R, H, and N as in (36), (37), and (24), respectively; and define Q, X, and Y as in (12), (13), and (14), respectively. Then

    (a) $R(W, S, t) > 0$ for $W > 0$;

    (b) the state-independency conditions hold: $\mathcal{U}(C^{-1}(x, S, t), S, t)$ is independent of S for all $x > 0$ and $t < T$, $\mathcal{U}(W, S, t) \geq \lim_{x\downarrow 0} \mathcal{U}(C^{-1}(x, \hat{S}, t), \hat{S}, t)$ for all $W > 0$, $S > 0$, and $t \in [0, T)$ such that $C(W, S, t) = 0$ and all $\hat{S} > 0$, $\lim_{W\downarrow 0} \mathcal{U}(W, S, t) \geq \lim_{W\downarrow 0} \mathcal{U}(W, \hat{S}, T)$ for all $S > 0$ and $\hat{S} > 0$, $H(W, S, T) = 0$ for all (W, S), and $A(W, S, T)/\kappa(S, T)$ is a function of W only;

(c) *the consistency conditions (9), (10), and (11) hold; and*

(d) *(C, A) satisfies the PDE (26);*

(3) *Suppose that $\Gamma_1 = \Gamma_2 = 0$ except possibly when $\hat{\kappa} = 0$. Then $K = 0$ except possibly when $\hat{\kappa} = 0$; and*

(4) *Suppose that $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0$ and $\Gamma_1 \neq 0$ and $\Gamma_2 \neq 0$ except possibly when $\hat{\kappa} = 0$. Then (38) holds except possibly when $\hat{\kappa} = 0$.*

Note that in Theorems 1 and 2, (26) is the most substantive necessary condition. Other conditions are either consistency conditions or the state-independency conditions.

We now present two examples, one to demonstrate the necessary condition for efficiency and another to demonstrate a price process where $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0$. More examples can be found in Section 5.

Example 1 *It has been asserted in the literature that the pair*

$$
\begin{aligned}
C(W, S, t) &= f(t)W, \\
A(W, S, t) &= \frac{\mu(S, t) - r}{\sigma^2(S, t)} W,
\end{aligned}
$$

with $f(t) > 0$, is the optimal consumption-portfolio policy for the log *utility function with certain time preferences captured by $f(t)$; see Merton (1973).[18] Since $C > 0$ for all $W > 0$ and $S > 0$, this policy must satisfy the conditions of Theorem 1.*

First, the consumption policy implies that $H(W, S, t) = 0$ and thus the portfolio policy implies $R(W, S, t) = W$. Direct computation using (24) gives $N(t) = -f(t)$. One can then easily verify that all the conditions of Theorem 1 are satisfied.

Next we turn our attention to Theorem 2. Note that this theorem applies generally independently of whether $C > 0$ for all $W > 0$. Direct computation shows that $K(W, S, t) = \Gamma_1(S, t)W/\sigma(S, t)$. We take cases.

Case 1. Suppose that $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$. By (36) and (37), we have

$$
\begin{aligned}
R &= \frac{\sigma^2 S \Gamma_1 / \sigma + \hat{\kappa} \Gamma_2}{\sigma S \Gamma_1 + \hat{\kappa} \Gamma_2} W = W, \\
H &= \frac{-\hat{\kappa} \Gamma_1 + \hat{\kappa} \sigma \Gamma_1 / \sigma}{\sigma S \Gamma_1 + \hat{\kappa} \Gamma_2} W = 0.
\end{aligned}
$$

---

[18] This assertion was supported by the fact that $J(W, S, t) = g(S, t) + h(t) \ln W$ solves the Bellman's equation for some functions g and h. However, this J function fails to satisfy the polynomial growth condition, which is sufficient to derive the Bellman's equation. Thus, to our knowledge, it has not been actually shown that an optimal policy exists for the log utility function using dynamic programming. However, this policy can be verified to be optimal using other arguments such as those of Cox and Huang (1989, 1991).

These are consistent with our calculation above while using Theorem 1. Then it is straightforward to verify that (2a)–(2d) of Theorem 2 are satisfied.

Case 2. Suppose that $\Gamma_1 = \Gamma_2 = 0$. Then $K = 0$ and (3) of Theorem 2 is satisfied.

Case 3. Suppose that $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0$ and $\Gamma_1 \neq 0$ and $\Gamma_2 \neq 0$. Then $K/A = -\Gamma_1/\hat{\kappa}$ and (4) of Theorem 2 is satisfied.

The following example gives a scenario where $\Gamma_1 = \Gamma_2 = 0$.

Example 2 *Suppose that $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} = 0$ and σ is a constant. Then $\Gamma_2 = 0$. This implies that $\Gamma_1 = 0$. Note that Γ_1 is the drift term of $d\hat{\kappa}$. Thus $\hat{\kappa}$ must be a local martingale. Since σ and r are constant, this implies that μ is a local martingale.[19]*

Conversely, given that σ is a constant and μ is a local martingale, $\Gamma_1 = \Gamma_2 = 0$. In this case,

$$K = \frac{1}{2}\sigma^2 A^2 A_{WW} + \sigma^2 SAA_{WS} + \frac{1}{2}\sigma^2 S^2 A_{SS} + rWA_W + rSA_S - rA + A_t + C_W A + SC_S - CA_W = 0.$$

Using the same arguments as in the proof of Proposition 2 we have that

$$A(t)e^{-rt} + \int_0^t (C_W(s)A(s) + S(s)C_S(s) - C(s)A_W(s))ds, \quad t \in [0, T)$$

is a local martingale under Q and is a martingale under Q with similar regularity conditions as in Corollary 2. In particular, if $C = 0$, $W(T) > 0$ a.s., and A satisfies the growth condition stipulated in Corollary 2, we know $A(t)e^{-rt}$ must be a martingale under Q. Thus the present value of the future investment in the stock must be equal to the current investment, a property obeyed by any optimal policy satisfying the same regularity conditions in the geometric Brownian motion case.

Besides checking whether a given policy (C, A) satisfies the necessary conditions for efficiency for a fixed stock price process, Theorems 1 and 2 can also be used to answer a more general question: For a given policy to satisfy the necessary conditions for efficiency, what should be the price process? The following familiar example demonstrates this.

Before presenting our example, we note that when the stock price follows a geometric Brownian motion, and when the (direct) utility function exhibits a constant Arrow-Pratt measure of relative risk aversion, the optimal portfolio policy is a constant mix policy; that is, $A(t) = \alpha W$ for some α, and the optimal consumption policy is a linear policy $C(t) = f(t)W$ for some strictly positive function $f(t)$.[20]

---

[19] He and Leland (1992) show that when σ is a constant and μ is a local martingale, ξ is path-independent. Hence, any optimal policy must be path-independent.

[20] The astute reader will question whether one needs to restrict the coefficient of relative risk aversion to be less than one since otherwise the indirect utility function fails to satisfy a polynomial growth condition. The linear policy can however be shown to be optimal using a different technique; see Cox and Huang (1989).

On the other hand, it seems quite likely that for this pair of linear policies to be optimal, it would be necessary that the stock price process be a geometric Brownian motion, and the utility function exhibit a constant Arrow-Pratt measure of relative risk aversion. Using Theorem 1, we show in the following example that this pair of linear policies only necessitates that κ be a constant, and μ and σ be independent of S, except in the case where $\alpha = \kappa$ (which is Example 1).

Example 3 *Let $C(W, S, t) = f(t)W$ and $A(W, S, t) = \alpha W$, where $f(t) > 0$ and $\alpha \neq 0$. Since the consumption policy is independent of S, $H = 0$. This implies that $R(W, S, t) = \alpha W / \kappa(S, t)$ and*

$$N = (\mu - r)\frac{\kappa - \alpha}{2} + \frac{r\kappa}{\alpha} - f(t)\frac{\kappa}{\alpha} - r + \sigma^2 S \kappa_S.$$

Thus

$$\ln \mathcal{U}(x/f(t), S, t) = \frac{\kappa(S, t)}{\alpha}[-\ln(x/f(t)) + \ln \underline{W}] + Y(\underline{S}, t). \tag{39}$$

By the state-independence of the utility function, the right-hand side must be independent of S. Thus κ is a function only of t, and hence, $N = (\mu - r)(\kappa - \alpha)/2 + (r - f(t))\kappa/\alpha - r$.

Assume without loss of generality that $\kappa(t) \neq 0$. Then $R(W, S, t) = \alpha W / \kappa(t)$ must satisfy (26). This implies $\kappa'(t)\alpha W = 0$ for all $W > 0$, and $\kappa(t)$ must be independent of t and is a constant. Relation (39) thus shows that the utility function exhibits a constant relative risk aversion equal to κ/α.

The fact that κ is a constant does not necessarily mean that μ and σ are constants. Now note that the consistency condition (10) implies that

$$N_S = \mu_S \frac{\kappa - \alpha}{2} = 0.$$

Suppose $\kappa \neq \alpha$. Then $\mu_S = 0$ and μ is independent of S. Consequently, σ is independent of S.

In summary, for (C, A) to be efficient, it is necessary that κ is a constant, and the utility function must exhibit a relative risk aversion κ/α. In addition, μ and σ are functions of time if $\kappa \neq \alpha$.

4 Sufficient Conditions for Efficiency and Recoverability

In this section we first show that with some minor regularity conditions, the necessary conditions derived in the previous section are also sufficient. We then discuss how one can recover the utility function that supports a given efficient policy. Specifically, we provide an integral formula to recover the utility function.

We give two sets of sufficient conditions for a given (C, A) that satisfies Conditions A1, A2, and A5 of Section 2 to be efficient. First, for a C such that $C(W, S, t) > 0$ for all $W > 0$ and $S > 0$, the necessary conditions recorded in Theorem 1 together with the hypothesis that the R, H, and N defined in (31), (29), and (24), respectively, satisfy Conditions A3 and A6 are sufficient for (C, A) to be efficient. Second, in the case where consumption is not always strictly positive for strictly positive wealth and where $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$, the necessary conditions of Theorem 2 together with the same conditions on R, H, and N are also sufficient. Our proof for these two sets of sufficient conditions are through construction; we construct a pair of utility functions (u, V) so that the policy (C, A) solves (3).

Since any efficient policy must be such that $C(0, S, t) = 0$ and $A(0, S, t) = 0$, that is, whenever the wealth reaches zero there will be neither investment nor consumption afterwards, we will restrict our attention to this kind of policies. For any one of these policies, it follows from Dybvig and Huang (1988) that we must have

$$E^* \left[\int_0^T C(W(t), S(t), t) e^{-rt} dt + W(T) e^{-rT} \right] \leq W(0), \tag{40}$$

that is, the present value of future consumption and final wealth must be less than the current wealth. By concavity of the utility functions, sufficient conditions for (C, A) to be a solution to (3) are that (40) holds with equality and there exists a strictly positive scalar $\lambda > 0$ so that, for all $t \in [0, T]$,

$$u_c(C(W(t), S(t), t), t) \quad \begin{cases} = \lambda \xi(t) e^{-rt} & \text{if } C(W(t), S(t), t) > 0, \\ \leq \lambda \xi(t) e^{-rt} & \text{if } C(W(t), S(t), t) = 0; \end{cases} \tag{41}$$

$$V'(W(T)) \quad \begin{cases} = \lambda \xi(T) e^{-rT}, & \text{if } W(T) > 0, \\ \leq \lambda \xi(T) e^{-rT}, & \text{if } W(T) = 0, \end{cases} \tag{42}$$

where we recall the definition of $\xi(t)$ in (1) and its interpretation as the Arrow-Debreu price at time 0 for time t consumption per unit of probability P.[21] The following is our first set of sufficient conditions:

Theorem 3 *Let $S\sigma$ satisfy a linear growth condition and let (C, A) satisfy Conditions A1, A2, and A5, and $C > 0$ for all $W > 0$ and $S > 0$. Define R, H, N as in (31), (29), and (24), respectively, and Q, X, Y, and $\mathcal{U}$ as in (12), (13), (14), and (15) respectively. Suppose that*

(1) R, H, N satisfy the continuity and differentiability conditions of Conditions A3 and A6; and

---

[21] Here we remind the reader of our implicit hypothesis that the starting date of the optimization is $t = 0$ but our arguments apply to other starting dates t with arbitrary starting states $W(t)$ and $S(t)$.

(2) conditions (1)–(6) of Theorem 1 are satisfied.

Then $(C, A) \in \mathcal{E}$ and the utility functions correspond to (C, A) are

$$\ln u_c(x, t) = \begin{cases} \mathcal{U}(C^{-1}(x, S, t), S, t) & x > 0, \\ \lim_{z \downarrow 0} \mathcal{U}(C^{-1}(z, S, t), S, t) & x = 0; \end{cases}$$

$$\ln V'(x) = \begin{cases} \mathcal{U}(x, S, T) & x > 0, \\ \lim_{z \downarrow 0} \mathcal{U}(z, S, T) & x = 0. \end{cases}$$

PROOF. See Appendix. ∎

When C is not always strictly positive for strictly positive wealth, R cannot be defined through (31), and we have the second set of sufficient conditions. Note however that this set of conditions applies generally whenever $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$, independently of whether $C(W, S, t) > 0$ for all $W > 0$.

Theorem 4 *Let $S\sigma$ satisfy a linear growth condition and let (C, A) satisfy Conditions A1, A2, and A5, and $\Gamma_1 \sigma S + \Gamma_2 \hat{\kappa} \neq 0$. Define R, H, and N as in (36), (37), and (24), respectively; and define Q, X, Y, and $\mathcal{U}$ as in (12), (13), (14), and (15), respectively. Suppose that*

(1) R, H, and N satisfy Conditions A3 and A6; and

(2) conditions (1) and (2a)–(2d) of Theorem 2 are satisfied.

Then $(C, A) \in \mathcal{E}$ and the utility functions that correspond to (C, A) are

$$\ln u_c(x, t) = \begin{cases} \mathcal{U}(C^{-1}(x, S, t), S, t) & x > 0, \\ \lim_{z \downarrow 0} \mathcal{U}(C^{-1}(z, S, t), S, t) & x = 0; \end{cases}$$

$$\ln V'(x) = \begin{cases} \mathcal{U}(x, S, T) & x > 0, \\ \lim_{z \downarrow 0} \mathcal{U}(z, S, T) & x = 0. \end{cases}$$

PROOF. See Appendix. ∎

Using Theorem 3 one easily shows that the linear policies in Example 3 are indeed optimal for the stock price process identified there and the utility function exhibiting a constant relative risk aversion equal to κ/α. Of course, this result is well-known from solving the dynamic consumption and portfolio problem using dynamic programming.

The general procedure to recover the utility function should be obvious from the proofs of Theorems 3 and 4. First, define R, H, N as in Theorems 3 and 4. Second, define Q, X, Y, and $\mathcal{U}$ as in (12), (13), (14), and (15), respectively. Now, define $\mathbf{U}(x, t)$ and $\mathbf{V}$ such that

$$\ln \mathbf{U}(x, t) = \begin{cases} \mathcal{U}(C^{-1}(x, S, t), S, t) & x > 0, \\ \lim_{z \downarrow 0} \mathcal{U}(C^{-1}(z, S, t), S, t) & x = 0; \end{cases}$$

$$\ln \mathbf{V}(x) = \begin{cases} \mathcal{U}(x, S, T) & x > 0, \\ \lim_{z \downarrow 0} \mathcal{U}(z, S, T) & x = 0. \end{cases}$$

Then, the utility functions that support (C, A) must be

$$u(x,t) = \int^x \mathbf{U}(x,t)dx,$$

$$V(x) = \int^x \mathbf{V}(x)dx.$$

These are the integral formulae which recover the utility functions that support a given efficient policy (C, A).

Before leaving this section, we point out that the necessary and sufficient conditions established in Sections 3 and 4 can be readily extended to infinite horizon problems, i.e., $T = \infty$ and $V = 0$. In this case, in Theorems 1 and 2, all conditions relating to time T should be removed. In addition, in Theorems 3 and 4, one needs to add that the present value of the future wealth goes to zero when the future extends to infinity; that is, $\mathbf{E}^*[W(t)e^{-rt}] \to 0$ as $t \to \infty$.[22]

5   Further Examples

In this section we present two more examples to demonstrate our results.

Example 4 *Consider a pair of consumption and investment functions*

$$C(W, S, t) = f(t)W^{\alpha(t)}S^{1-\alpha(t)}$$

$$A(W, S, t) = \left(\frac{\mu(S,t) - r}{\sigma(S,t)^2 \rho} W^{\beta(t)} S^{1-\beta(t)} - \frac{1 - \alpha(t)}{\alpha(t)} W \right)$$

where f is a strictly positive deterministic function, α and β are deterministic functions with values between 0 and 1 satisfying $\lim_{t \to T} \alpha(t) = 1$, and and ρ is a strictly positive constant. The parameters of the wealth process generated by (C, A) may not satisfy a linear growth and a local Lipschitz condition for some functions α and β. We will ignore this problem for now and proceed with other necessary conditions.

We will show that for this pair of policies to be efficient, it is necessary that $\alpha = \beta = 1$ for all t, and either $\rho = 1$ or $(\mu - r)/\sigma$ is independent of S.

To begin, the consumption policy determines the hedging demand function

$$H(W, S, t) = -\frac{C_S(W, S, t)}{C_W(W, S, t)} = -\frac{1 - \alpha(t)}{\alpha(t)} \frac{W}{S},$$

which in turn determines the risk tolerance function

$$R(W, S, t) = \frac{1}{\rho} W^{\beta(t)} S^{1-\beta(t)}.$$

---

[22] See Huang and Pagès (1992).

Since $\alpha(t) \to 1$ as $t \to T$, we have $H(W, S, t) \to 0$ as $t \uparrow T$, which is (30) and is part of the state-independency condition.

Next, for (9) to be satisfied, we need

$$\rho(\beta(t) - 1)W^{-\beta(t)}S^{\beta(t)-2} = -\rho(\beta(t) - 1)\frac{1 - \alpha(t)}{\alpha(t)}W^{-\beta(t)}S^{\beta(t)-2}.$$

This implies that $\beta(t) = 1$ for all t and thus

$$R(W, S, t) = \frac{1}{\rho}W, \quad A(W, S, t) = \left(\frac{\mu(S, t) - r}{\sigma(S, t)^2\rho} - \frac{1 - \alpha(t)}{\alpha(t)}\right)W.$$

Now, define N according to (24):

$$\begin{aligned}
N &= \frac{1}{2}\sigma^2 A^2\left(-\frac{\rho}{W^2}\right) - \frac{1}{2}\sigma^2 S^2\left(\frac{1 - \alpha(t)}{\alpha(t)}\right)\frac{\rho}{S^2} \\
&\quad + (rW + A(\mu - r) - C(t)W^{\alpha(t)}S^{1-\alpha(t)})\frac{\rho}{W} + \mu S\left(\frac{1 - \alpha(t)}{\alpha(t)}\frac{\rho}{S}\right) - \frac{\kappa^2\sigma^2}{2} - r.
\end{aligned}$$

Since $N_W = R_t/R^2 = 0$ by (11), we deduce that

$$-f(t)\rho(\alpha(t) - 1)W^{\alpha(t)-1}S^{1-\alpha(t)} = 0$$

for all $W > 0$, $S > 0$, and $t \in [0, T]$. This cannot be true unless $\alpha(t) = 1$ for all t. Consequently, the hedging demand must be zero and

$$N = \frac{1}{2}\left(1 - \frac{1}{\rho}\right)\kappa^2\sigma^2 + r\rho - r - f(t)\rho.$$

Equation (10) then implies that

$$N_S = \frac{1}{2}\left(1 - \frac{1}{\rho}\right)(\kappa^2\sigma^2)_S = (H/R)_t = 0,$$

and we must either have $\rho = 1$ or $(\kappa^2\sigma^2)_S = 0$. Note that in the former case the parameters of the wealth process satisfy a local Lipschitz and a growth condition if $(\mu - r)^2/\sigma^2$ also satisfies these conditions, and in the latter case the parameters of the wealth process are purely deterministic.

In summary, in order for the pair of policies to be efficient for some stock price process, it must be the case that α and β be constant and equal to one, and either $\rho = 1$ or $(\mu - r)/\sigma$ is independent of S. Indeed, when $\rho = 1$, the log utility function supports (C, A), and when $\kappa^2\sigma^2$ is independent of S, the utility function that exhibits a constant coefficient of relative risk aversion equal to ρ supports (C, A).

Note that in the case where $\kappa^2\sigma^2$ is independent of S, $\hat{\kappa}$ is independent of S. From Cox and Huang (1989), there will be no hedging demand. The optimal consumption policy will be a function

only of wealth, and the optimal portfolio policy can be calculated as an explicit integral and can be represented generally as

$$A(W, S, t) = \frac{\hat{\kappa}}{\sigma(S, t)} g(W(t), t)$$

for some function g.

Example 5 *Consider a pair of consumption and investment functions for an infinite horizon problem,*

$$
\begin{aligned}
C(W, S, t) &= \gamma SW, \\
A(W, S, t) &= f(S)W,
\end{aligned}
$$

where $f(S) = A_3 - r/(\alpha\sigma^2) - \sqrt{A_1 + A_2 S}$, where $\alpha > 0$, $\gamma > 0$, $\sigma > 0$, $A_1 \geq 0$, $A_2 > 0$, and A_3 are constants. Note that the marginal propensity to consume is proportional to the stock price, and the proportion of the wealth invested in the stock is a decreasing function of the stock price. Note also that with (C, A) defined above, the parameters of the wealth dynamics may not satisfy a linear growth condition and the consumption policy may not satisfy the integral condition of Condition A2. We will ignore this problem and proceed with other necessary conditions. We ask: Can (C, A) be a pair of optimal policy for some utility function and for some stock price process? We will see that the answer is affirmative for the following price process:

$$dS(t) = \alpha\sigma^2(A_3 + 1 - \sqrt{A_1 + A_2 S(t)})S(t)dt + \sigma S(t)dw(t),$$

provided that 1) there exists a solution to this stochastic differential equation, 2) the equivalent martingale measure exists for this price process on any finite interval $[0, t]$, 3) the parameters of the price process satisfy certain restrictions, and 4) the wealth never reaches zero.[23]

Clearly, the consumption policy implies that $H = -W/S$. Thus

$$R = \frac{A - SH}{\kappa} = \frac{f + 1}{\kappa}W = \frac{1}{\alpha}W.$$

Using relation (24) we write

$$N = -\frac{\alpha\sigma^2}{2}\left(A_3 - \frac{r}{\alpha\sigma^2} - \sqrt{A_1 + A_2 S}\right)^2 - \frac{1}{2}\alpha\sigma^2$$

---

[23] We ignore the condition that the expected discounted wealth under the equivalent martingale measure goes to zero. This condition may be checked by simulation. Also, using the boundary conditions discussed in Chapter 5, Gihman and Skorohod (1972), one can check that zero is inaccessible for this price process.

$$+ \alpha \left(r + \alpha \sigma^2 \left(A_3 - \frac{r}{\alpha \sigma^2} - \sqrt{A_1 + A_2 S} \right) \left(A_3 - \frac{r}{\alpha \sigma^2} - \sqrt{A_1 + A_2 S} + 1 \right) - \gamma S \right)$$

$$+ \alpha^2 \sigma^2 \left(A_3 + 1 - \sqrt{A_1 + A_2 S} \right) - \frac{\alpha^2 \sigma^2}{2} \left(A_3 - \frac{r}{\alpha \sigma^2} + 1 - \sqrt{A_1 + A_2 S} \right)^2 - r.$$

$$= \quad -\frac{\alpha \sigma^2}{2} \left(A_3 - \frac{r}{\alpha \sigma^2} - \sqrt{A_1 + A_2 S} \right)^2 + \frac{\alpha^2 \sigma^2}{2} \left(A_3 - \frac{r}{\alpha \sigma^2} + 1 - \sqrt{A_1 + A_2 S} \right)^2$$

$$- \alpha \gamma S - \frac{1}{2} \alpha \sigma^2 + 2\alpha r - r.$$

We now choose γ and α as follows.

$$\gamma = \frac{1}{2} A_2 \sigma^2 (\alpha - 1),$$

$$\alpha = \frac{A_3 + r/\sigma^2 + \sqrt{(A_3 + r/\sigma^2)^2 - 4(A_3 + 1)r/\sigma^2}}{2(A_3 + 1)}.$$

Since γ has to be strictly positive, we choose A_3 so that $\alpha > 1$. This can be achieved if r/σ^2 is large, or if $A_3 + 1$ is positive and sufficiently small. For the γ and α defined above, N is independent of S and is a constant. The consistency and the state-independency conditions are easily seen to be satisfied.

Next, one verifies that (26) is satisfied and, by Theorem 3, (C, A) is efficient provided that all the regularity conditions stated above can be verified. The utility function that supports (C, A) exhibits a constant coefficient of relative risk aversion equal to α, which is strictly greater than one.

Note that if $A_3 > r/(\alpha \sigma^2) + \sqrt{A_1}$, then the proportion of the wealth invested in the stock is positive when the stock price is low and is negative when the stock price is high. By Itô's lemma,

$$d\ln S(t) = \alpha \sigma^2 \left(A_3 + 1 - \frac{1}{2\alpha} - \sqrt{A_1 + A_2 S(t)} \right) dt + \sigma dw(t).$$

Thus the log price process follows a mean reversion process if $A_3 + 1 - 1/2\alpha - \sqrt{A_1} > 0$.

6 Concluding Remarks

We have studied the "inverse" of the classical optimal consumption-portfolio problem of Merton (1971). This inverse optimal problem should be viewed as a dynamic recoverability problem in financial markets with continuous trading. We have derived the necessary and sufficient conditions for a given consumption-portfolio policy to be optimal for some agent with an increasing, concave, time-additive, and state independent utility function in an economy with one risky and one riskless asset. The risky asset price follows a general diffusion process, and the riskless interest rate is a constant. Using identical arguments, we can generalize the results reported to cases where there

are more than one risky asset, and the interest rate is stochastic. We leave this generalization to the interested reader. We can also generalize our results to allow state-dependent utility functions by simply removing all the state-independency conditions. The technique we have exploited to derive our results is dynamic programming. It seems plausible that our method might also be generalized to allow for non-time-additive utility functions as long as the optimal consumption-portfolio problem for these utility functions can be analyzed by dynamic programming. This may be a fruitful direction for future research given the increasing interest in non-time-additive utility functions.

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8   Appendix

PROOF OF THEOREM 3. Without loss of generality, we assume that the optimization starts from $t = 0$ with $W(0) = \underline{W}$ and $S(0) = \underline{S}$. First we show that (40) holds as an equality. Under Q, the discounted wealth process and the discounted stock price process become

$$
\begin{aligned}
d(W(t)e^{-rt}) &= -C(t)e^{-rt}dt + A(t)\sigma(t)e^{-rt}dw^*(t), \quad t \in [0, T], \\
d(S(t)e^{-rt}) &= S(t)e^{-rt}\sigma(t)dw^*(t),
\end{aligned}
$$

where we recall that w^* is a Brownian motion under Q. By the hypothesis that $A\sigma$ and $S\sigma$ satisfy a linear growth condition (see Condition A1 for the former), Friedman (1975, theorem 5.2.3) shows that there exist constants L_m so that $\mathbf{E}^*[|W(t)|^{2m}] \le (1 + |\underline{W}|^{2m})e^{L_m t}$ and $\mathbf{E}^*[|S(t)|^{2m}] \le (1 + |\underline{S}|^{2m})e^{L_m t}$, for all integers $m = 1, 2, \ldots,$. Given this, we can easily show that

$$
\mathbf{E}^*\left[\int_0^T |A(t)\sigma(t)e^{-rt}|^2 dt\right] < \infty.
$$

Hence $W(t)e^{-rt} + \int_0^t C(s)e^{-rs}ds$ is a square integrable martingale under Q; see, for example, Liptser and Shiryayev (1977, §4.2). Consequently, (40) holds as an equality.

Next, by condition (4) of Theorem 1, u and V are well-defined and are state-independent; and by condition (2) of Theorem 1, u and V are strictly increasing concave. We remain to verify that there exists a $\lambda > 0$ such that the first order conditions (41) and (42) hold. For any function f, define the operator $\bar{\mathcal{L}}$:

$$
\begin{aligned}
\bar{\mathcal{L}}(f) = {}& \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 f}{\partial W^2} + \sigma^2 S A \frac{\partial^2 f}{\partial W \partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (rW - C)\frac{\partial f}{\partial W} + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \\
& -rf + C_W f.
\end{aligned}
$$

Condition (5) of Theorem 1 can be written as $\bar{\mathcal{L}}(R) = 0$. This implies that

$$
\begin{aligned}
\bar{\mathcal{L}}(\frac{1}{R}) &= \sigma^2 A^2 \frac{R_W^2}{R^3} + 2\sigma^2 S A \frac{R_W R_S}{R^3} + \sigma^2 S^2 \frac{R_S^2}{R^3} - \frac{2r}{R} + \frac{2C_W}{R} \\
&= \frac{\sigma^2 A_W^2}{R} - \frac{2r}{R} + \frac{2C_W}{R},
\end{aligned} \tag{43}
$$

where the second equality follows from (28), which is a consequence of the definition of R and H in (31) and (29), respectively, and the consistency condition (9). By Condition A5, the wealth never

reaches zero before T. For any $t \in [0, T)$, Itô's lemma implies that

$$
\begin{aligned}
dO(W(t), S(t), t) &= (\mathcal{L}(O) + O_t)dt + \frac{1}{R(W(t), S(t), t)}\sigma A dw(t) \\
&+ \left(\int_{\underline{W}}^{W(t)} \left(\frac{1}{R(z, S(t), t)} \right)_S dz \right) \sigma S(t) dw(t),
\end{aligned}
$$

where $\mathcal{L}$ is the differential generator of W and S under P. By the consistency condition (9), the diffusion term of dO becomes

$$
\begin{aligned}
&\frac{1}{R(W(t), S, t)}\sigma A - \left(\int_{\underline{W}}^{W(t)} \left(\frac{H(z, S, t)}{R(z, S, t)} \right)_z dz \right) \sigma S \\
&= \left(\frac{1}{R(W(t), S, t)}\sigma A - \frac{H(W(t), S, t)}{R(W(t), S, t)}\sigma S + \frac{H(\underline{W}, S, t)}{R(\underline{W}, S, t)}\sigma S \right) \\
&= \left(-\hat{\kappa}(S, t) + \frac{H(\underline{W}, S, t)}{R(\underline{W}, S, t)}\sigma S \right).
\end{aligned}
$$

The drift of dO is

$$
\begin{aligned}
\mathcal{L}(O) + O_t &= \mu S \int_{\underline{W}}^{W(t)} \left(\frac{1}{R} \right)_S dz + (rW(t) + A(\mu - r) - C)\frac{1}{R(W(t), S, t)} \\
&+ \frac{1}{2}\sigma^2 S^2 \int_{\underline{W}}^{W(t)} \left(\frac{1}{R(z, S, t)} \right)_{SS} dz + \frac{1}{2}\sigma^2 A^2 \left(\frac{1}{R(W(t), S, t)} \right)_W \\
&+ \sigma^2 AS \left(\frac{1}{R(W(t), S, t)} \right)_S + \int_{\underline{W}}^{W(t)} \left(\frac{1}{R(z, S, t)} \right)_t dz.
\end{aligned}
$$

Using integration by parts, we have

$$
\begin{aligned}
\mathcal{L}(O) + O_t &= \int_{\underline{W}}^{W(t)} \mu S \left(\frac{1}{R(W(t), S, t)} \right)_S dz + \int_{\underline{W}}^{W(t)} (rz + A(\mu - r) - C) \left(\frac{1}{R(z, S, t)} \right)_z dz \\
&+ \int_{\underline{W}}^{W(t)} (r + A_z(\mu - r) - C_z)\frac{1}{R}dz + \int_{\underline{W}}^{W(t)} \frac{1}{2}\sigma^2 S^2 \left(\frac{1}{R(z, S, t)} \right)_{SS} dz \\
&+ \int_{\underline{W}}^{W(t)} \frac{1}{2}\sigma^2 A^2 \left(\frac{1}{R(z, S, t)} \right)_{zz} dz + \int_{\underline{W}}^{W(t)} \sigma^2 AA_z \left(\frac{1}{R(z, S, t)} \right)_z dz \\
&+ \int_{\underline{W}}^{W(t)} \sigma^2 AS \left(\frac{1}{R(z, S, t)} \right)_{zS} dz + \int_{\underline{W}}^{W(t)} \sigma^2 A_z S \left(\frac{1}{R(z, S, t)} \right)_S \\
&+ \int_{\underline{W}}^{W(t)} \left(\frac{1}{R(z, S, t)} \right)_t dz + h_1(S, t),
\end{aligned}
$$

where

$$
\begin{aligned}
h_1(S, t) &= (r\underline{W} + A(\underline{W}, S, t)(\mu - r) - C(\underline{W}, S, t))\frac{1}{R(\underline{W}, S, t)} \\
&+ \frac{1}{2}\sigma^2 A^2(\underline{W}, S, t) \left(\frac{1}{R(\underline{W}, S, t)} \right)_W + \sigma^2 A(\underline{W}, S, t)S \left(\frac{1}{R(\underline{W}, S, t)} \right)_S.
\end{aligned}
$$

Using the definition of $\bar{\mathcal{L}}$, we get

$$
\begin{aligned}
\mathcal{L}(O) + O_t &= \int_{\underline{W}}^{W(t)} \bar{\mathcal{L}} \left(\frac{1}{R(z,S,t)} \right) dz + \int_{\underline{W}}^{W(t)} (\mu - r) S \left(\frac{1}{R(z,S,t)} \right)_S dz \\
&+ \int_{\underline{W}}^{W(t)} (\mu - r) A \left(\frac{1}{R(z,S,t)} \right)_z dz + \int_{\underline{W}}^{W(t)} (\mu - r) A_z \frac{1}{R(z,S,t)} dz \\
&+ \int_{\underline{W}}^{W} \sigma^2 A_z A \left(\frac{1}{R(z,S,t)} \right)_z dz + \int_{\underline{W}}^{W(t)} \sigma^2 A_z S \left(\frac{1}{R(z,S,t)} \right)_S dz \\
&+ \int_{\underline{W}}^{W(t)} \frac{2r}{R(z,S,t)} dz - \int_{\underline{W}}^{W(t)} \frac{2C_z(z,S,t)}{R(z,S,t)} dz + h_1(S,t) \\
&= \int_{\underline{W}}^{W(t)} \sigma^2 A_z^2 \frac{1}{R(z,S,t)} dz + \int_{\underline{W}}^{W(t)} \sigma^2 A A_z \left(\frac{1}{R(z,S,t)} \right)_z dz \\
&+ \int_{\underline{W}}^{W(t)} \sigma^2 A_z S \left(\frac{1}{R(z,S,t)} \right)_S dz + h_1(S,t) \\
&= h_1(S,t)
\end{aligned}
$$

where both the second and the third equalities follow from (28), which as we mentioned before is a consequence of the definition of R and H in (31) and (29), respectively, and the consistency condition (9). We can thus write

$$
dO(W(t), S(t), t) = h_1(S(t),t)dt - \frac{H(\underline{W}, S(t),t)}{R(\underline{W}, S(t),t)} \sigma S(t) dw(t) - \hat{\kappa} dw(t).
$$

Similarly, Itô's lemma implies that

$$
dX(S(t),t) = \frac{H(\underline{W}, S(t),t)}{R(\underline{W}, S(t),t)} \sigma S(t) dw(t) + h_2(S(t),t)dt,
$$

where

$$
h_2(S,t) = \frac{H(\underline{W}, S,t)}{R(\underline{W}, S,t)} \mu S + \frac{1}{2}\sigma^2 S^2 \left(\frac{H(\underline{W}, S,t)}{R(\underline{W}, S,t)} \right)_S + \int_{\underline{S}}^{S} \left(\frac{H(\underline{W}, \eta, t)}{R(\underline{W}, \eta, t)} \right)_t d\eta.
$$

Using the consistency condition (10) and the definition of N in (24), we have

$$
-h_1(S,t) + h_2(S,t) = -N(\underline{W}, \underline{S}, t) - \frac{\hat{\kappa}(S,t)^2}{2} - r.
$$

Finally, since $dY(t) = N(\underline{W}, \underline{S}, t)dt$,

$$
\ln \mathcal{U}(W(t), S(t), t) = -O(W(t), S(t), t) + X(S(t), t) + Y(t) = \ln \xi(t) - rt.
$$

Since $C(W, S, t) > 0$ for all $W > 0$, and the wealth never reaches zero before T by the hypothesis, we thus have

$$
\ln u_c(C(W(t), S(t), t), t) = \ln \lambda + \ln \xi(t) - rt, \quad a.s. \ t \in [0, T)
$$

with $\lambda = 1$. At T, on the set where $W(T) > 0$, by the continuity of $\ln \mathcal{U}(W, S, t)$ except when $W = 0$ at T, we have

$$\ln \xi(T) - rT = \lim_{t \uparrow T} \ln \mathcal{U}(W(t), S(t), t) = \ln \mathcal{U}(W(T), S(T), T) = \ln V'(W(T)).$$

On the other hand, on the set where $W(T) = 0$, we have

$$\ln \xi(T) - rT = \lim_{t \uparrow T} \ln \mathcal{U}(W(t), S(t), t) \geq \lim_{W \downarrow 0} \ln \mathcal{U}(W, S(T), T) = V'(0).$$

We have thus shown that (C, A) is efficient.   ∎

PROOF OF THEOREM 4. Again take $S(0) = \underline{S}$ and $W(0) = \underline{W}$ and start from $t = 0$. The only thing different in this case is that the consumption may not be strictly positive at nonzero wealth levels. But we still have

$$\ln \mathcal{U}(W(t), S(t), t) = \ln \lambda + \ln \xi(t) - rt, \quad a.s. \ t \in [0, T)$$

with $\lambda = 1$. Naturally, if $C > 0$, $\ln u_c(C(W(t), S(t), t), t) = \ln \xi(t) - rt$. If $C = 0$, condition (2b) shows that

$$\ln u_c(0, t) \leq \ln \xi(t) - rt.$$

Finally, arguments identical to those used in the proof of Theorem 3 show that

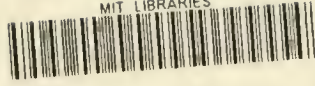$$\ln V'(W(T)) \leq \ln \xi(T) - rT$$

with equality holding for $W(T) > 0$. These prove the assertion.   ∎

Date Due