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SHORTEST ROUTE MODELS FOR THE ALLOCATION OF INSPECTION EFFORT ON A PRODUCTION LINE

by

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ABSTRACT

Two shortest route models for determining where to allocate inspection effort on a production line are developed for the cases where this effort is unlimited or limited in its availability. A production line is defined as an ordered sequence of production stages, each stage consisting of a manufacturing operation followed by a potential inspection station. Items flow through the line in batches and may incur defects at any stage. Defects are assumed to be repairable or non-repairable. The defect generating process at any stage is taken to be an independent Bernoulli process with a known parameter. Two levels of inspection effort may be applied at any stage: no inspection or 100% inspection. Thus, both models are used to determine the stages at which batches are to be 100% inspected. A general cost structure is postulated which includes fixed and variable costs of inspection, a cost of repair, a cost of disposal, a cost of processing, and a cost of an undetected defect. The first three of these costs may depend on the most recent as well as the present inspection point. An expected cost per batch criterion is used to determine an optimal inspection plan. Examples are included.

Shortest Route Models for the Allocation of Inspection Effort on a Production Line*#

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1. Introduction

In this paper we develop a general deterministic model for the allocation of inspection effort on a production line. The model has the analytical form of a shortest route model; thus several very fast algorithms are available for generating computational results, [2], [3]. In addition, we show how the basic model may be extended to the case where the amount of available inspection effort is limited.

We define a production line as an ordered sequence of L production stages, each stage consisting of a manufacturing operation followed by a po-

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tential inspection station.[1] A five stage production line is illustrated in Figure 1. The produce being manufactured is assumed to enter stage 1 of the production line in batches of size $B \geq 1$. As the items within a batch move through the manufacturing operations they may incur defects. A defect created in any item at stage n will be called a type n defect. The defect generating process at the n^{th} stage is viewed as an independent Bernoulli process with a known parameter b_n, the probability of producing a defect of type n. The models to be developed allow for repairable and non-repairable defects. We let R(n) denote the subset of defects within the set {1,2,...,n} that are repairable at stage n. We also define

$$\bar{R}(n) = \{1,2,\ldots,n\} - R(n)$$

i.e., $\bar{R}(n)$ is the subset of defects within {1,2,...,n} that are not repairable at stage n.

If all defects 1,2,...,L are repairable, we assume that the net flow through each production stage is B items; in other words, we assume the existence of a repair facility that can supply non-defective items to replace any defectives found during the inspection of a batch at any stage. On the other hand, if some defects are taken to be non-repairable, then any items found to have such defects are scrapped, possibly with some salvage value. Items found to have only repairable defects are again assumed to be replaced with non-defective items supplied from a repair facility. Thus, for the situations where some defects are repairable and others are not, a batch that

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[1] In certain cases it may be convenient to think of the first manufacturing stage as the aggregation of all operations that come before the start of the production line, and of the first potential inspection station as an incoming inspection point.

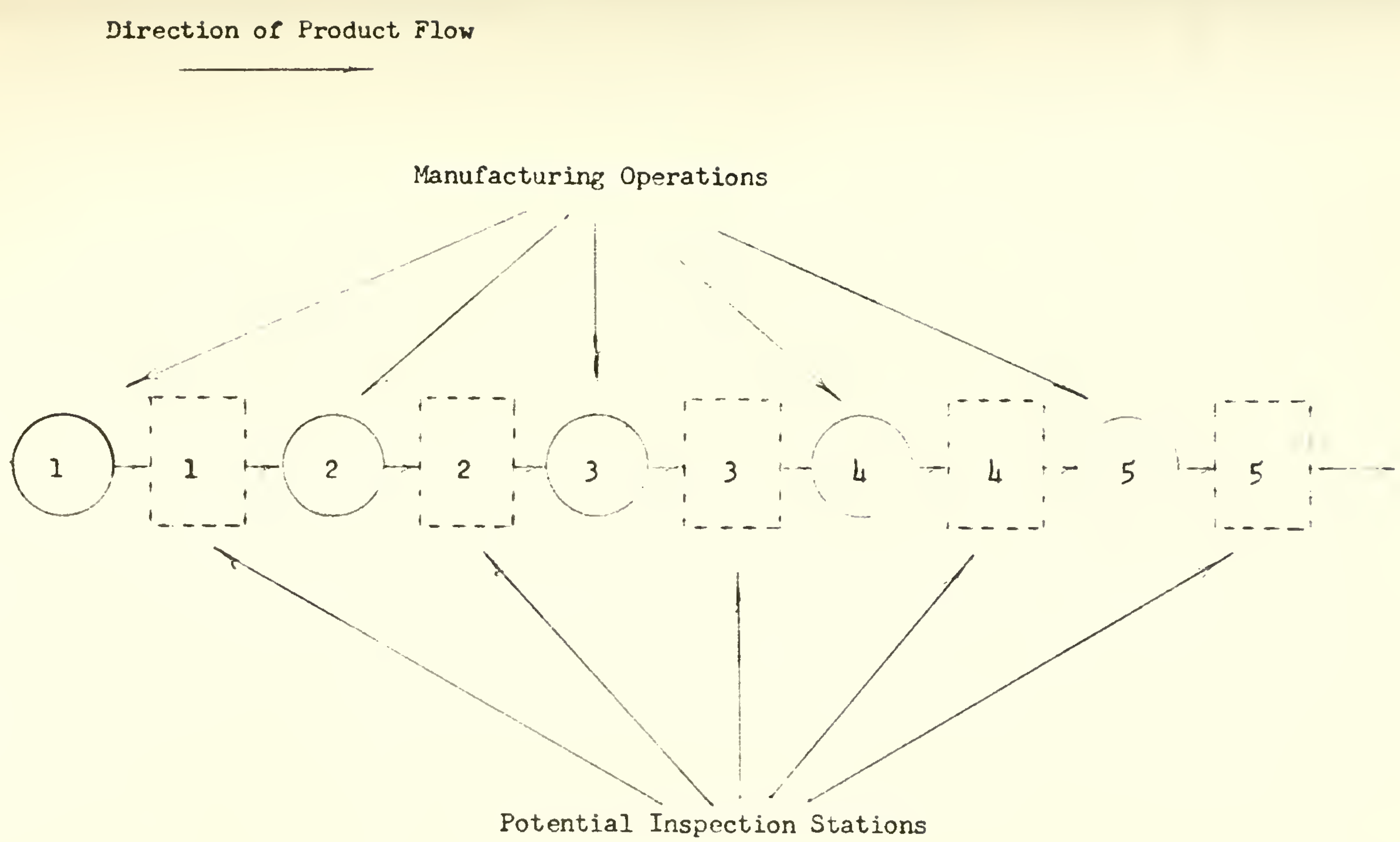

Direction of Product Flow
Manufacturing Operations
1
1
2
2
3
3
4
4
5
5
Potential Inspection Stations

Figure 1.

contains B items at the start of the line will be reduced in size at the inspection stations where non-repairable defective items are discarded. Models in which all defects are non-repairable have been considered by Lindsay and Bishop [4] and Pruzan and Jackson [5]. Beightler and Mitten [1] and White [6] have treated cases where all defects are repairable. The models presented in this paper include these extremes as special cases.

Consider now a class K of multistage inspection plans of the form $(k_1,k_2,...,k_L)$, where $k_n=0,1,...,B_n$ is the number of items to be inspected in each batch passing through inspection station n and B_n is the batch size. Lindsay and Bishop [4] and White [6] have shown for the cases of no repairable defects and all repairable defects, respectively, and for a fairly simple cost structure, that an optimal plan--one that minimizes the expected cost per processed batch--within the class K has the form $k_n=0$ or B_n, i.e., at any inspection point arriving batches are either passed without inspection or they are inspected 100%. In this paper we only consider inspection plans of the "all or nothing" form. The class of such plans will be denoted by K^*. The question as to whether K^* contains an optimal plan within K for our models is not investigated.

In what follows, we shall show that the problem of finding an optimal plan within K^* for a rather general cost structure may be formulated as a shortest route problem. We shall also show that a shortest route model may be formulated for the case where at most $T<L$ values of k_n may be greater than zero, i.e., at most a limited number T of the inspection stations may be active. To differentiate between these two models we shall refer to the unconstrained

model as Model I and the model in which inspection effort is limited as Model II. In both models, the costs to be considered at each stage include a fixed cost per batch inspected and a variable cost per item inspected, a cost of repairing defects, and an expected cost of wasted processing effort for scrapped items. All these costs may be made to depend on the most recent stage at which the batch had been inspected (if any) prior to the present inspection point. In addition, the models include an expected cost of defective items that pass the last inspection station undetected and an expected disposal cost per item scrapped. The disposal cost may be negative indicating a positive salvage value. The objective in the analysis of Model I is to find a plan in K^* which minimizes the expected cost per processed batch. In the case of Model II, the cost criterion is the same, but the search is restricted to those plans in K^* which satisfy the constraint on the number of active inspection stations.

Our point of departure in this paper is the work of Pruzan and Jackson [5] and of White [6]. Model I is viewed as a direct generalization of this previous work. With Model I we can now treat the more realistic situation where some defects are repairable and others are not, as well as the two extremes previously dealt with. In addition, the shortest route formulation allows for computational efficiencies over the dynamic programming formulations in [5] and [6]. As far as we know, Model II has not been previously considered.

2. Formulation of Model I

In this section we formulate the problem of where to make 100% inspections

in an L stage production line as a problem of finding the shortest route in a directed network. The network, denoted by N(L), consists of L+2 nodes numbered from 0 to L+1, and directed arcs between all nodes m and n for which $m<n$. The arc $(m,n)\varepsilon N(L)$, $m<n$, is directed towards n; thus node 0 is the source node and node L+1 the sink. The network N(5) is illustrated in Figure 2. (The numbers on the arcs are relevant to an example in section 5.)

The relation of N(L) to the unconstrained production-inspection problem is best described by an explanation of what it means to "travel" along any arc $(m,n)\varepsilon N(L)$. It is sufficient to consider three types of arcs: (i) arcs of the form (0,n), $n=1,2,\ldots,L$ have the interpretation that the first inspection takes place at stage n; (ii) arcs of the form (m,n), $m<n$, $m=1,2,\ldots,L-1$, $n=2,3,\ldots,L$, have the interpretation that inspection at stage m is followed by inspection at stage n and no inspection is performed between m and n; and (iii) arcs of the form (m,L+1), $m=0,1,\ldots,L$, have the interpretation that the last inspection is at stage m. Thus, for example, arc (0,L+1) implies that no inspection is performed at any stage.

Given the above interpretation of travelling through N(L) it remains to specify length of any arc. In our models the length of an arc will be measured in terms of the expected cost to traverse it. Once these costs are specified it will be shown that the Model I allocation of inspection effort problem is equivalent to the problem of finding the shortest (least expected cost) route from node 0 to node L+1 in N(L).

The arc costs, denoted by $c(m,n)$, $(m,n)\varepsilon N(L)$, are defined in terms of

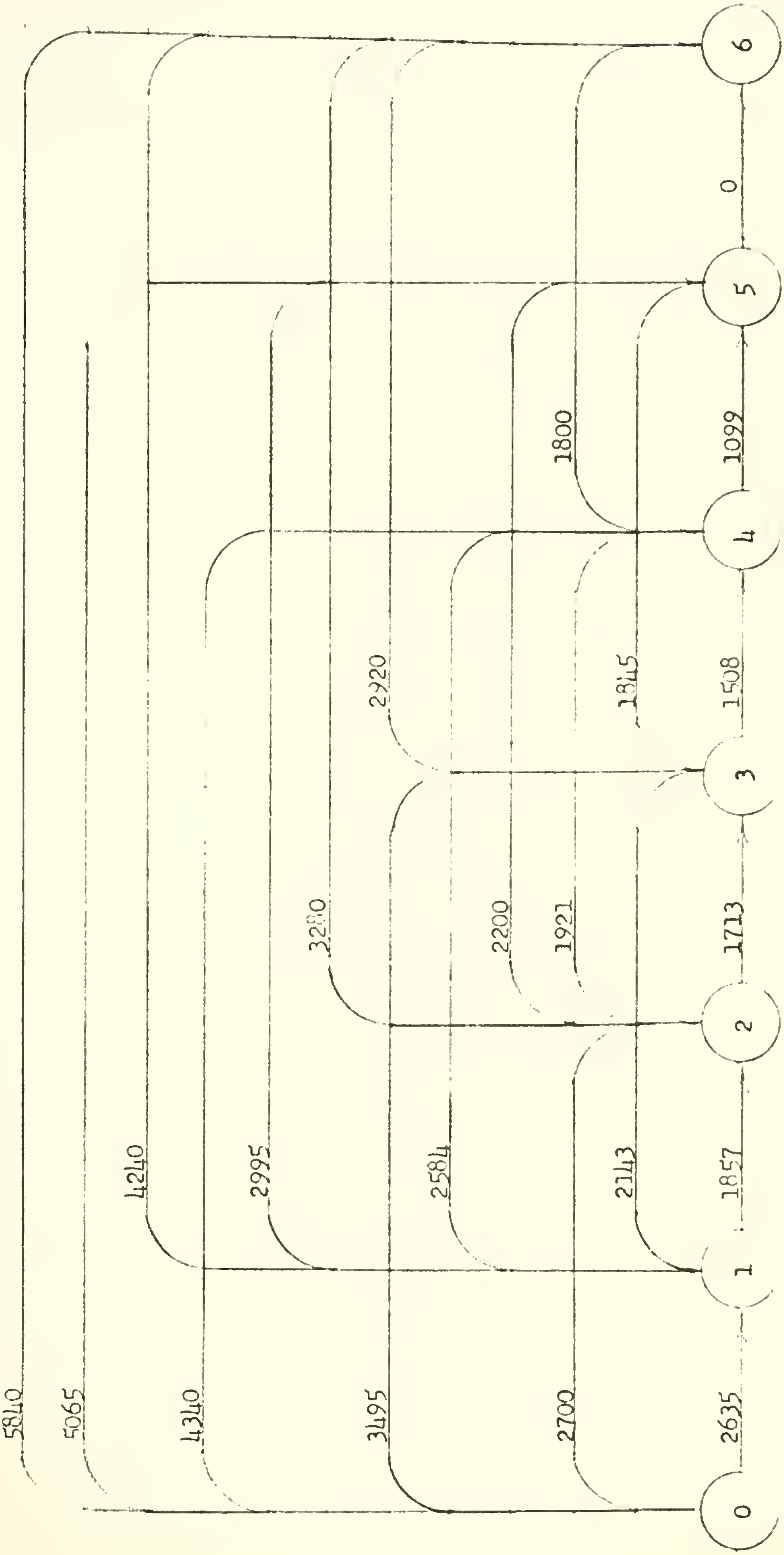

0
1
2
3
4
5
6
2635
1857
1713
1508
1099
0
5840
5065
4340
3495
2700
4240
2995
2584
2143
2200
1921
2920
1845
1800

Figure 2.

the following basic costs:

1. The costs of inspection:

   cf_{mn} = the fixed cost per batch inspected at stage n given that the batch was last inspected at stage m, m<n, m=0,1,...,L-1, n=1,2,...,L.

   cv_{mn} = the variable cost per unit inspected at stage n given that the unit was last inspected at stage m, m<n, m=0,1,...,L-1, n=1,2,...,L.

2. The cost of repair:

   cr_{mn} = the cost of repairing a type m defect discovered at stage n, $m \varepsilon R(n)$, n=1,2,...,L.

   $cr_{mn} \equiv 0$, $m \varepsilon \bar{R}(n)$, n=1,2,...,L.

3. The cost of processing:

   cp_n = the cost of processing a unit through manufacturing operation n, n=1,2,...,L.

4. The cost of disposal:

   cd_{mn} = the expected cost of disposing of a unit found to be non-repairable at stage n given that the unit was last inspected at stage m, m<n, m=0,1,...,L-1, n=1,2,...,L. (This cost may be negative indicating a salvage value.)

5. The cost of an undetected defect:

   cu_n = the cost of allowing a finished unit with a type n defect to leave the production line undetected, n=1, 2,...,L.

In the cases of cf_{0n}, cv_{0n}, and cd_{0n}, n=1,2,...,L, the corresponding defini-

tions above should be modified to read, in part, "given no previous inspection."

From these basic costs we develop expressions for the arc costs $c(m,n)$, $(m,n) \in N(L)$, as follows. First we define $B(n)$ as the expected number of items out of an original batch of size B that are either defect free or repairable after manufacturing operation n. Thus,

$$(2.1) \qquad \begin{aligned} B(n) &= B \cdot \prod_{j \in \bar{R}(n)} (1-b_j) \quad , \quad \bar{R}(n) \neq \phi \\ B(n) &= B \quad , \quad \bar{R}(n) = \phi \end{aligned}$$

$n=1,2,\ldots,L$, where ϕ denotes the empty set. For notational purposes it will be convenient to also define,

$$(2.2) \qquad B(0) \equiv B.$$

Then given (2.1) and (2.2) we can write the following expected costs per processed batch:

(i) The expected cost of inspection:

$$i(m,n) = cf_{mn} + B(m)cv_{mn} \quad , \quad m<n,\ m=0,1,\ldots,L-1,\ n=1,2,\ldots,L.$$

(ii) The expected cost of repairing defects:

$$r(m,n) = B(n) \sum_{j=m+1}^{n} cr_{jn} b_j \ , \ m<n,\ m=0,1,\ldots,L-1,\ n=1,2,\ldots,L.$$

(iii) The expected cost of scrapping items:

$$s(m,n) = \sum_{j=m+2}^{n} cp_j [B(m)-B(j-1)] + cd_{mn}[B(m)-B(n)],\ m<n,$$

$$m=0,1,\ldots,L-1,\ n=1,2,\ldots,L.$$

(Note that the first term in $s(m,n)$ expresses the expected cost of wasted manufacturing operations.)

(iv) The expected cost of undetected defective items:[2]

$$u(m,L+1) = B(m) \sum_{j=m+1}^{L} cu_j b_j \quad , \quad m=0,1,\ldots,L-1 \; ,$$

$$u(L,L+1) \equiv 0.$$

Finally, using the above expected costs we can write

$$c(m,n) = i(m,n)+r(m,n)+s(m,n) \quad , \quad m<n,\; m=0,1,\ldots,L-1,\; n=1,2,\ldots,L \; ,$$

(2.3)

$$c(m,L+1) = u(m,L+1) \quad , \quad m=0,1,\ldots,L \; .$$

With the arc costs (2.3) now defined, we can state the shortest route problem for Model I as follows. Let R(L) be the set of all routes from 0 to L+1 in N(L). Then the problem is to find a route $R^* \varepsilon R(L)$ such that

(2.4) $$C(R^*) = \min_{R\varepsilon R(L)} \sum_{(x,y)\varepsilon R} c(x,y) \; .$$

The route R^* found by solving (2.4) will consist of a set of arcs of the form $(0,n_1)$, (n_1,n_2), $\ldots(n_{z-1},n_z)$, $(n_z,L+1)$. The optimal multistage inspection plan within K^* corresponding to the route R^* is then given by

$$k_{n_i} = B_{n_i} \quad , \quad i=1,2,\ldots,z$$

$$k_n = 0 \quad , \quad \text{otherwise.}$$

---

[2]Alternatively, the cost of an undetected defective unit may be fixed independent of the number or types of defects. Then if cu is specified as the unit cost of an undetected defective item, u(m,L+1) can be written as,

$$u(m,L+1) = B(m)cu[1- \prod_{j=m+1}^{L} (1-b_j)] \; .$$

The value $C(R^*)$ is the minimum expected cost per process batch.

3. Solution Procedures for Model I

As is well known, any deterministic shortest route problem may be formulated as a dynamic program. Thus, letting f(n) be defined as the minimum expected cost of travelling from node 0 to node n, we can write the following dynamic program corresponding to the shortest route problem (2.4): Find f(L+1) where,

$$(3.1)\qquad \begin{aligned} f(0) &\equiv 0 \\ f(n) &= \min_{m=0,1,\ldots,n-1} \{f(m)+c(m,n)\},\ n=1,2,\ldots,L+1. \end{aligned}$$

The set of recurrence relations (3.1) can be solved directly to yield an optimal plan. However, for large networks, i.e., long production lines, it may pay to use an algorithm especially developed to exploit the particular structure of the shortest route model. Two of the fastest such algorithms are due to Dantzig [3] and Berge and Ghouila-Houri [2]. Berge and Ghourila-Houri's algorithm requires a maximum of

$$2(1+2+\ldots+[\tfrac{L+2}{2}]) \approx 1/2\frac{(L+2)(L+4)}{2}$$

comparisons to determine the shortest route between 0 and L+1 in the network N(L). Consequently, once the data has been prepared and the arc costs are generated, Model I may be evaluated for most production lines in a reasonable amount of computer time. Moreover, for small and medium size lines (say less than 25 stages) it will be possible to do a comprehensive sensitivity analysis

on any parameters of special interest.

4. An Analysis of Model II

Suppose now that the amount of available inspection effort is limited in the sense that an upper bound $T<L$ is put on the number of active inspection stations. If, for example, each active station were to require one inspector, then the constraint could be interpreted as an upper limit on the number of inspectors that may be assigned to the production line.

One approach to solving the constrained problem is to first solve the corresponding unconstrained problem and count the number of active inspection stations implied by the optimal inspection plan. If this number is less than or equal to T, the constrained problem is solved. However, if the number of active stations exceeds T, the solution is infeasible. And since an infeasible solution may occur, this approach will not always work.

An alternative approach that guarantees a feasible optimal solution can be developed by considering a modified version of the shortest route network developed for Model I. Let $N(T,L)$ denote a directed network with a source node 0_0, a sink node $L+1$, and additional nodes n_t, $n=t,t+1,\ldots,L$, $t=1,2,\ldots,T$. The arcs of $N(T,L)$ are directed from nodes m_t to nodes n_{t+1}, $0\leq t\leq m<n$, $n=1,2,\ldots,L$, $t=0,1,\ldots,T-1$, and from all nodes m_t, $t\leq m$, $m=0,1,\ldots,L$, to node $L+1$. The network $N(2,5)$ is depicted in Figure 3. (Again the numbers on the arcs are relevant to an example in section 5.)

The interpretation of travelling from node to node in $N(T,L)$ is similar to that given in section 2 for $N(L)$. The following correspondences are

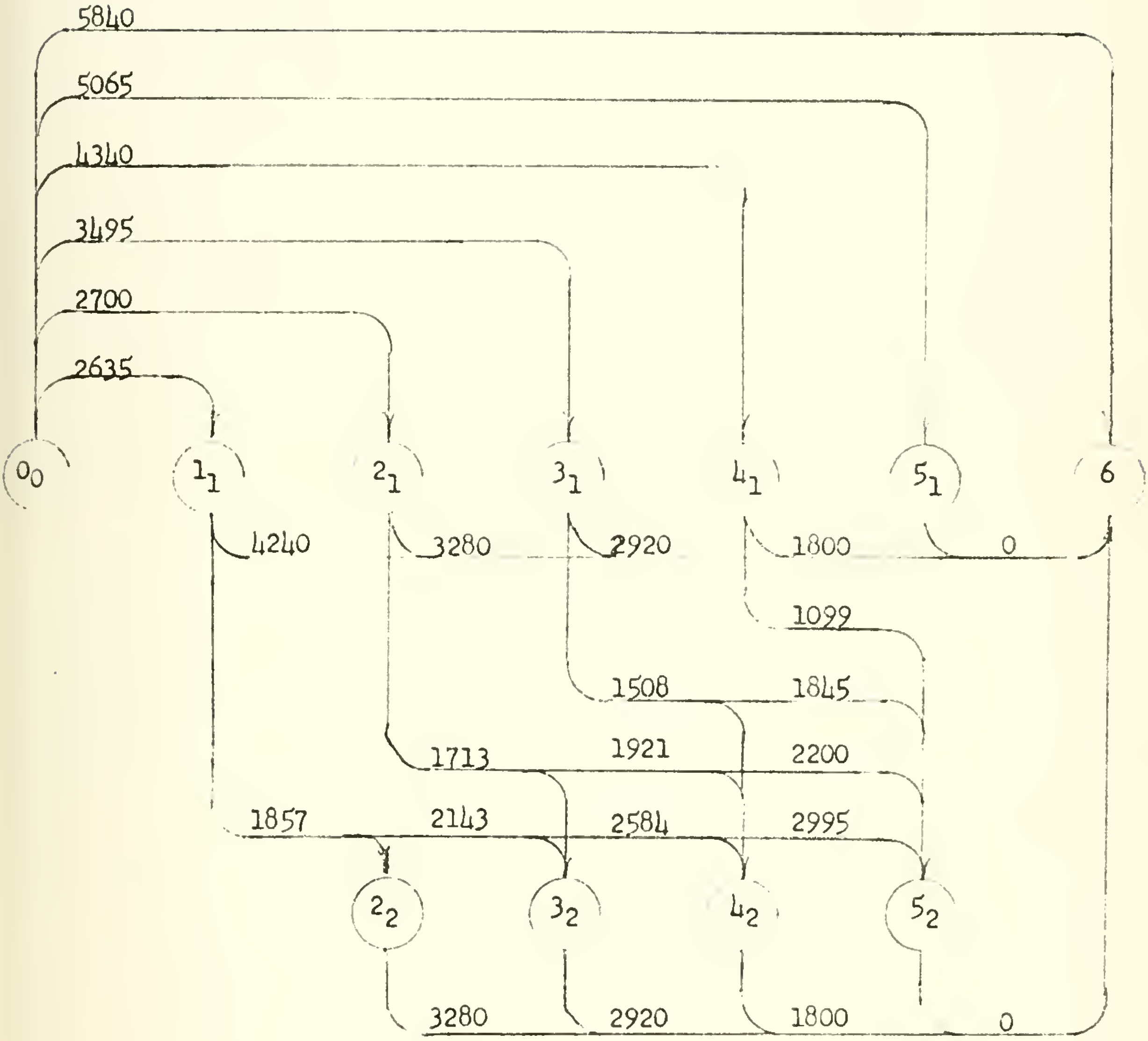

5840
5065
4340
3495
2700
2635
0_0
1_1
2_1
3_1
4_1
5_1
6
4240
3280
2920
1800
0
1099
1508
1845
1713
1921
2200
1857
2143
2584
2995
2_2
3_2
4_2
5_2
3280
2920
1800
0

Figure 3.

easily seen to hold: (i) travelling along arc $(m_t, n_{t+1}) \varepsilon N(T,L)$ has the same interpretation as travelling along $(m,n) \varepsilon N(L)$, $0 \le t \le m < n$, $n=1,2,\ldots,L$, $t=0,1,\ldots,T-1$, where t denotes the number of active inspection stations within the stages $1,2,\ldots,m$, and (ii) travelling along the arc $(m_t, L+1)$ $\varepsilon N(T,L)$ has the same interpretation as travelling along $(m,L+1) \varepsilon N(L)$, $0 \le t \le m$, $m=0,1,\ldots,L$, $t=0,1,\ldots,T$. Here t denotes the total number of active inspection stations. As a consequence of these correspondences, it immediately follows that the arc costs for N(T,L) can be expressed in terms of the arc costs already defined in (2.3) for N(L); specifically,

$$
(3.2) \quad \begin{aligned}
c(m_t, n_{t+1}) &= c(m,n) , \quad 0 \le t \le m < n , \quad n=1,2,\ldots,L , \quad t=0,1,\ldots,T-1, \\
c(m_t, L+1) &= c(m,L+1) , \quad 0 \le t \le m , \quad m=0,1,\ldots,L , \quad t=0,1,\ldots,T .
\end{aligned}
$$

Given the network N(T,L) and the arc costs as defined in (3.2) we can now state the constrained shortest route problem for Model II as follows. Let R(T,L) denote the set of all routes from 0_0 to L+1 in N(T,L). Then the problem is to find a route $R^{**} \varepsilon R(T,L)$ such that

$$
(3.3) \quad C(R^{**}) = \min_{R \varepsilon R(T,L)} \sum_{(x,y) \varepsilon R} c(x,y) .
$$

The dynamic program corresponding to (3.3) has the recurrence relations:

$$
(3.4) \quad \begin{aligned}
f(0) &\equiv 0 , \\
f(n_{t+1}) &= \min_{m=0,1,\ldots,n-1} \{f(m_t)+c(m,n)\} , \quad 0 \le t \le m < n , \quad n=1,2,\ldots,L-1, \\
& \qquad\qquad t=0,1,\ldots,T-1, \\
f(L+1) &= \min_{m_t} \{f(m_t)+c(m,L+1)\} .
\end{aligned}
$$

Computational results for Model II may be obtained by solving the dynamic program (3.4) directly or by applying one of the shortest route algorithms previously mentioned to the network N(T,L).

5. Examples

We use an example of Pruzan and Jackson [5] to illustrate Model I and then add constraints on the number of active inspection stations to give two illustrations of Model II. Pruzan and Jackson's example concerns a five stage production line such as the one that was illustrated in Figure 1. The cost characteristics of this line are the following:

1. The fixed costs of inspection, cf_{mn} are given by:

		n				
		1	2	3	4	5
	0	135.0	240.0	255.0	270.0	290.0
m	1		225.0	235.0	250.0	260.0
	2			215.5	240.0	240.0
	3				210.0	222.0
	4					200.0

cf_{mn}: the fixed cost of inspection

2. The variable costs of inspection, cv_{mn}, are given by:

		n				
		1	2	3	4	5
	0	0.25	0.23	0.26	0.27	0.29
	1		0.17	0.18	0.19	0.20
m	2			0.16	0.17	0.18
	3				0.14	0.16
	4					0.10

cv_{mn}: the variable cost of inspection

3. The costs of processing, cp_n, are given by:

$$cp_2 = 0.40 \text{ , } cp_3 = 0.75 \text{ , } cp_4 = 1.00 \text{ , } cp_5 = 0.50 \text{ .}$$

4. The cost of allowing a defective finished unit to leave the production line: cu = 4.00. (Notice that this specification means that we shall be using the alternative definition of u(m,L+1) given in footnote 2 when calculating the expected cost of undetected defectives.)

The other characteristics of the production-inspection process relevant to the examples are the batch size B = 10,000 and the probabilities of producing defects:

$$b_1 = 0.04 \text{ , } b_2 = 0.025 \text{ , } b_3 = 0.01 \text{ , } b_4 = 0.03 \text{ , } b_5 = 0.05 \text{ .}$$

It is assumed that all defects are non-repairable.

The Model I network corresponding to the example problem has been given in Figure 2. The numbers next to the arcs are the arc costs c(m,n) for the example; they are also summarized in Table 1 below. The arc costs were computed using the equations (2.3).

		n					
		1	2	3	4	5	6
	0	2635	2700	3495	4340	5065	5840
	1		1857	2143	2584	2995	4240
m	2			1713	1921	2200	3280
	3				1508	1845	2920
	4					1099	1800
	5						0

Table 1

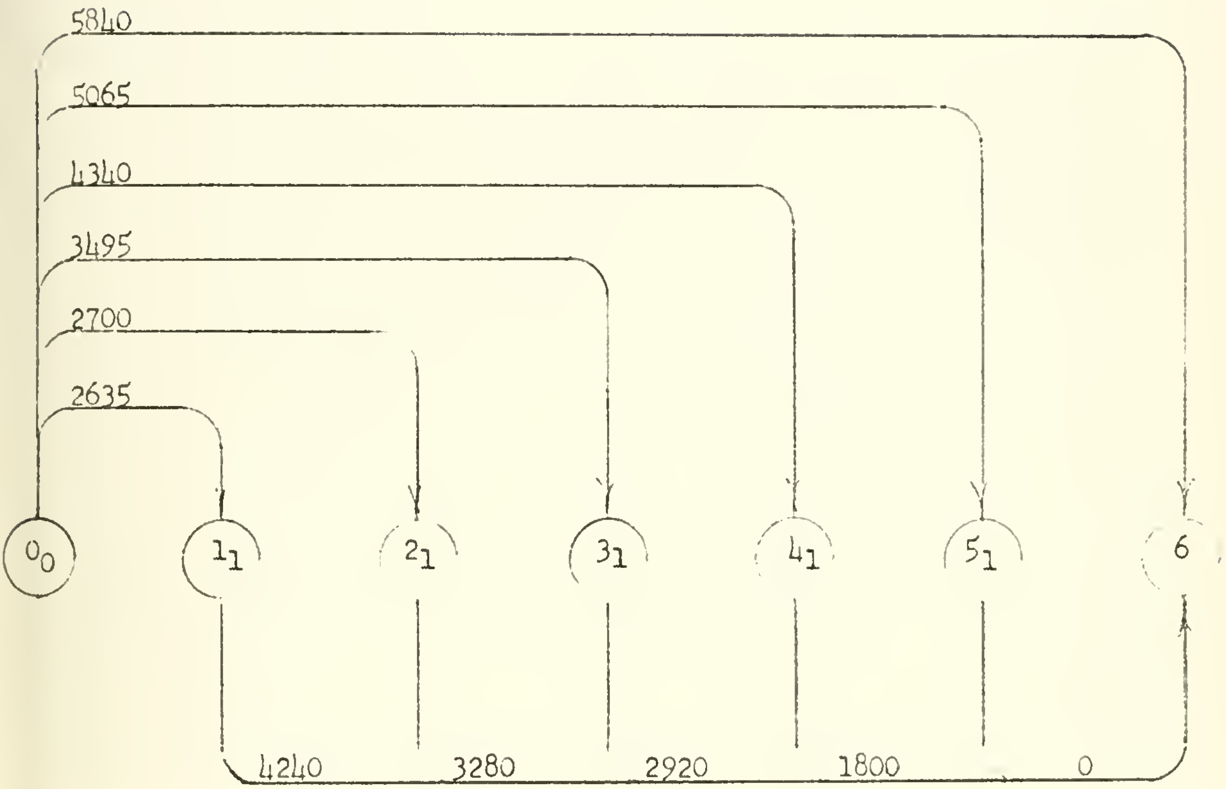
The optimal inspection plan can now be found by applying any shortest route algorithm to the network in Figure 2. If this is done it turns out that an optimal plan requires 100% inspection at stages 2 and 5 and has an expected total cost per processed batch of $4900.

Now suppose the Pruzan and Jackson example, but with a constraint that the number of active inspection be less than or equal to 2, i.e., T=2. The appropriate network is now N(2,5) which has been illustrated in Figure 3. The numbers on the arcs again specify the arc costs. If a shortest route algorithm is applied to N(2,5), clearly the optimal inspection rule will turn out to be the same as in the unconstrained problem. Thus, for this example the unconstrained solution remains feasible.

But now suppose we put the constraint T=1 on the initial example. In this case the unconstrained solution is infeasible and so we must evaluate the network N(1,5) to find an optimal solution. This network together with its arc costs is given in Figure 4. For this simple example it is easy to see that an optimal plan calls for inspection at the fifth stage. The expected cost of this policy is $5065 indicating a "shadow cost" of $165 per batch processed associated with the policy of using one inspector rather than two.

6. Concluding Remarks

We believe that the models presented in this paper offer the following advantages to the production systems analyst interested in solving a particular allocation of inspection effort problem. First, the models provide


5840
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4240
3280
2920
1800
0

Figure 4.

the analyst with a general cost structure within which to formulate his problem. Second, the models have a network representation, thus giving the analyst an essentially non-mathematical means for describing his problem to management. And third, the models may be solved efficiently with existing algorithms.

The models also retain certain assumptions found in previous formulations which may be considered disadvantageous to their application. Chief among these is the assumption that the Bernoulli process parameter b_n at each stage be known. We are at present developing computer programs for Models I and II which will be able to test the seriousness of this assumption by sensitivity analysis. Also we are working on Bayesian formulations of Models I and II which treat the b_n's as random variables. In the Bayesian context, inspection will have a dual purpose--to collect information related to the unknown quantities b_n, and to screen out defective items. Under these circumstances a class of inspection plans more general than K must be considered.

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