XIX. MICROWAVE THEORY

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A. IMPEDANCE TRANSFORMATIONS BY THE ISOMETRIC CIRCLE METHOD

Impedance transformations through bilateral two terminal-pair networks are, for a given frequency, usually performed by the linear fractional transformation

$$Z' = \frac{aZ + b}{cZ + d} \qquad ad - bc = 1 \tag{1}$$

The complete locus of points in the neighborhood of which lengths are unaltered in magnitude by the transformation is obtained from

$$\frac{dZ'}{dZ} = \frac{1}{(cZ + d)^2}$$

The locus is the circle

$$|cZ + d| = 1 \qquad c \neq 0$$

which is called the isometric circle of the direct transformation (ref. 1). The inverse transformation

$$Z = \frac{-dZ' + b}{cZ' - a}$$

has the corresponding isometric circle

$$|cZ' - a| = 1$$

The isometric circle of the direct transformation C_d has its center at $O_d = -d/c$ and a radius $R_c = |1/c|$; the isometric circle of the inverse transformation C_i has its center at $O_i = a/c$ and the same radius.

The fixed points of transformation 1 are

$$\zeta_1, \zeta_2 = \frac{a - d \pm \left[(a+d)^2 - 4\right]^{1/2}}{2c} \tag{2}$$

The following classification of different types of transformations is valid:

Hyperbolic, if a + d is real and $|a + d| > 2$

Parabolic, if $a + d = \pm 2$

Elliptic, if a + d is real and $|a + d| < 2$

Loxodromic, if a + d is complex.

The distance between the centers of the two isometric circles is $|(a+d)/c|$, while the sum of the two radii is $2/|c|$. Therefore, if we follow the classification given above, we get the hyperbolic case, if the two circles are external; the parabolic case, if they are tangent; and the elliptic case, if they intersect. See Fig. XIX-1. In the loxodromic

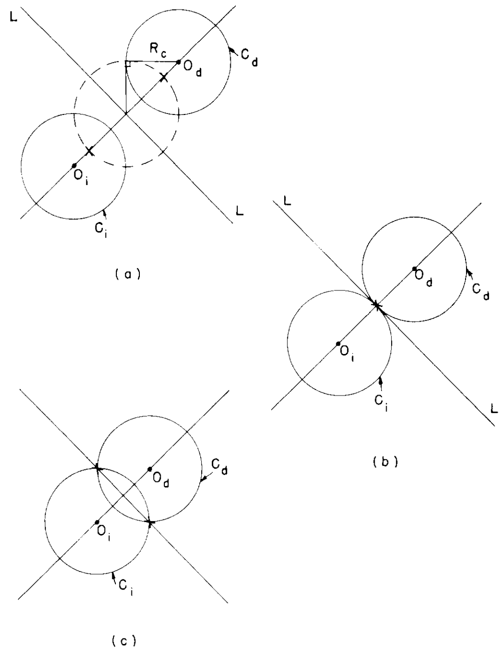

Fig. XIX-1. Classes of transformations: (a) hyperbolic; (b) parabolic; (c) elliptic.

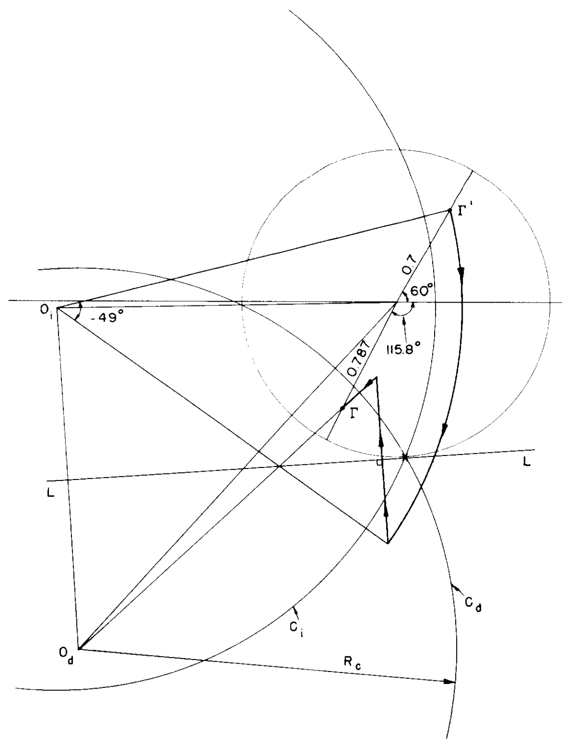

Fig. XIX-2. Example of loxodromic transformation.

transformation, each circle can have any relation to the other. The positions of the fixed points can easily be obtained from Eq. 2. The fixed points are marked as crosses in Fig. XIX-1.

Using the isometric circles, we can deduce the following graphical method (1) for the loxodromic case:

1. an inversion in the isometric circle of the direct transformation C_d;
2. a reflection in the symmetry line L to the two circles; and
3. a rotation around the center O_i of the isometric circle of the inverse transformation through an angle -2 arg (a+d).

An exchange can be made of the first two procedures, if the inversion is made in C_i instead of C_d. In the nonloxodromic cases the third operation is eliminated.

The graphical method is especially useful for transformations through lossless networks; in this case, the isometric circles are orthogonal to the imaginary axis in the complex impedance plane (Z-plane). The imaginary axis of the Z-plane, which corresponds to the unit circle in the complex-reflection-coefficient plane (Smith chart), constitutes the principal circle of a Fuchsian group. Therefore, the isometric circle method is of great use in connection with the Feldtkeller theory of symmetric networks, which was later extended to unsymmetric networks by Schulz, or with the circular geometric theory of Weissfloch, which was later extended by Lueg. Simple graphical proofs can be obtained for many circular geometric theorems; for example, for the well-known Weissfloch transformer theorem.

If we assume that the two terminal-pair networks contain negative resistances, then the lossy two terminal-pair networks form a Kleinian group.

An example of the use of the isometric method for a loxodromic transformation is shown in Fig. XIX-2. The example is the same as the one used by Storer, Sheingold, and Stein (2). The reflection coefficient is transformed by the formula

$$\Gamma' = \frac{\dfrac{s_{12}^2 - s_{11}\, s_{22}}{s_{12}}\,\Gamma + \dfrac{s_{11}}{s_{12}}}{-\dfrac{s_{22}}{s_{12}}\,\Gamma + \dfrac{1}{s_{12}}}$$

where $s_{11} = 0.331\, e^{j135.0°}$, $s_{12} = 0.808\, e^{j70.6°}$, and $s_{22} = 0.328\, e^{j132.8°}$. Hence

$$O_d = -\frac{d'}{c'} = 3.05\, e^{-j132.8°}$$

$$O_i = \frac{a'}{c'} = 2.20\, e^{-j178.5°}$$

$$R_c = \frac{1}{|c'|} = 2.47$$

$$-2 \arg (a' + d') = 49.0°$$

In this example Storer, Sheingold, and Stein use $\Gamma' = 0.70\ e^{j60°}$. Following the procedure scheduled above in reverse order, we rotate Γ' around O_i through an angle -49.0°, reflect in L, and invert in C_d. The value obtained approximates nicely the value $\Gamma = 0.787\ e^{-j115.8°}$ obtained by Storer et al.

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References

1. L. R. Ford, Automorphic Functions (Chelsea Publishing Co., New York, 1951).
2. J. E. Storer, L. S. Sheingold, and S. Stein, Proc. IRE 41, 1004-1013 (Aug. 1953).

B. IMPEDANCE TRANSFORMATIONS IN THE THREE-DIMENSIONAL HYPERBOLIC SPACE

A disadvantage in applying the isometric circle method in the complex plane is that the graphical constructions are not always limited to a certain finite practical space (see Fig. XIX-2). For nonloxodromic transformations this requirement can be fulfilled by using the Cayley-Klein diagram which has hyperbolic measure. This diagram, which is called "hyperbolic plane" by Klein, "Cayley-diagram" by Van Slooten, and "projective plane" by Deschamps, was introduced into network theory by Van Slooten (1) and into microwave theory by Deschamps (2, 3). For loxodromic transformations the same requirement can be fulfilled by using the three-dimensional hyperbolic space with the Riemann unit sphere as the absolute surface. The complex plane is stereographically mapped on the sphere. After the transformation has been performed in the three-dimensional hyperbolic space, a projection can be made on the complex impedance plane, the Smith chart, or the Cayley-Klein diagram.

It can be shown that, in the three-dimensional hyperbolic space, a loxodromic transformation corresponds to two non-Euclidean reflections in two lines non-Euclidean perpendicular to the line that transforms into itself by the transformation. The points at which the latter line cuts the absolute surface constitute the fixed points of the transformation. The non-Euclidean distance on the line transformed into itself, which is between the two perpendiculars, and the non-Euclidean angle between the two planes through the line transformed into itself and each of the perpendiculars determine the multiplier of the normal (canonic) form of the linear fractional transformation. If the distance mentioned is zero, the transformation is a pure rotation (elliptic case); if the angle is zero, the transformation is a pure stretching (hyperbolic case). If the line that transforms into itself by a nonloxodromic transformation is tangent to the sphere, the parabolic case is obtained. The general (loxodromic) case corresponds to a screw motion around the line transformed into itself. Analytically, the transformations on the sphere can be performed by 4×4 real matrices belonging to the G_+ subgroup of the proper Lorentz group.

Some important problems – analysis of a two terminal-pair network from three measurements or the cascading of networks, for example – can be treated in a graphic way in the three-dimensional hyperbolic space. In the loxodromic case, the analysis of a given network can be performed by the generalized three-dimensional Pascal theorem stated by Klein in 1873. The theorem yields a Pascal line which constitutes one of the two lines transformed into itself by the transformation. Thus the fixed points and the multiplier can be found, and the network parameters can be determined by standard methods. The analysis of lossless networks is a special case, in which the generalized Pascal theorem is exchanged for the ordinary two-dimensional Pascal theorem. This case has been thoroughly treated by Van Slooten (1).

Cascading of two terminal-pair networks of the loxodromic type can be performed by the Schilling figure, often described in connection with the geometric theory of the hypergeometric function or the Schwarzian s-function. The Schilling figure, introduced by Schilling in 1891, consists of three arbitrary lines in the three-dimensional hyperbolic space and three other lines that are each non-Euclidean perpendicular to two of the former lines. All lines cut the absolute surface. The Schilling figure can be used to obtain the Pascal line that belongs to the resultant network. The analytical tools for

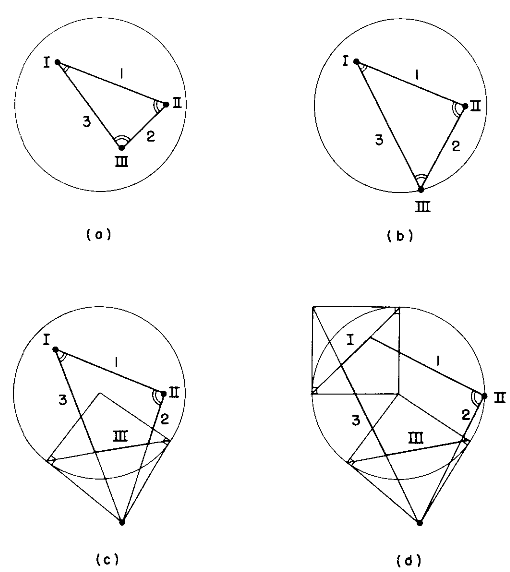

Fig. XIX-3. Use of the Schilling figure for cascading lossless networks:
(a) elliptic – elliptic – elliptic transformations;
(b) elliptic – elliptic – parabolic transformations;
(c) elliptic – elliptic – hyperbolic transformations;
(d) hyperbolic – parabolic – hyperbolic transformations.

obtaining the fixed points and the multiplier of the resultant network are furnished by the theory of invariance of quadratic forms and complex spherical trigonometry. In the special case of lossless networks, the Schilling figure simplifies to six lines or points I, II, III, 1, 2, and 3 in the Cayley-Klein diagram. Figure XIX-3 shows some illustrative examples of the use of the Schilling figure for cascading lossless networks.

The geometric theory outlined above will be published in Technical Report 312, Research Laboratory of Electronics, M. I. T., entitled, "Impedance Transformations in the Three-Dimensional Hyperbolic Space. The Isometric Circle Method." The part of the forthcoming report that deals with the three-dimensional hyperbolic space is to a large extent based on geometric works by Klein (4) and his pupil Schilling (5). The report constitutes the foundation for later research that has led, through a study of the three-dimensional elliptic space (6), to the investigation of the use of modern algebra, especially spinor algebra, in the microwave field.

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