XVIII. MICROWAVE THEORY

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A. USE OF NON-EUCLIDEAN GEOMETRY MODELS

Non-Euclidean geometry models have been used in Electrical Engineering for about ten years. Gradually, engineers and physicists are becoming more interested in using this convenient tool; therefore it seems worth while to try to answer the following questions: "Which non-Euclidean geometry models are available?", "How are they interconnected?", and "Where are these models described in mathematical literature?"

Non-Euclidean geometry is, to use Klein's terminology, divided into hyperbolic geometry, or Gauss-Bolyai-Lobachevsky geometry, and elliptic geometry, or Riemann geometry. Hyperbolic geometry is characterized by the properties that through a point outside a straight line two parallel lines can be drawn; that the sum of the angles of a triangle is less than π; and that this geometry is valid on a surface of constant negative curvature. Elliptic geometry is characterized by the properties that through a point outside a straight line no parallel line can be drawn; that the sum of the angles of a triangle is greater than π; and that this geometry is valid on a surface of constant positive curvature. The limiting case between the two geometries, parabolic geometry, is conventional Euclidean geometry.

A discussion of some of the non-Euclidean geometry models follows.

1. Hyperbolic Geometry Models

1.1 Two-dimensional models

a. The pseudosphere

The pseudosphere, which originates if a tractrix is rotated around its asymptote (see Fig. XVIII-1), is the simplest surface with constant negative curvature. This important adjunct was combined with hyperbolic geometry by Beltrami in 1868 (1). The pseudosphere has recently been thoroughly studied by Schilling (2, 3). It has a singular line. It was shown by Hilbert (4) that in Euclidean space there is no analytic surface of constant negative curvature which does not have a singular line.

b. The hyperboloid

Another surface of constant negative curvature that is imbedded in a three-dimensional space is the hyperboloid, which will be further discussed below. It has been studied by Schilling (3) and M. Riesz (5).

c. The Cayley-Klein diagram

This model was introduced by Beltrami (1) in 1868. However, it did not come into practical use until 1871, when, Klein (6) introduced it as a projective model with a conic as the absolute or fundamental curve. Klein based his work, to a large extent, on the work of Cayley (7) and, therefore, the model is usually called the "Cayley-Klein diagram." An example of this model is shown in Fig. XVIII-2. The unit circle in the figure corresponds to the absolute curve (infinity). A straight line is a chord. The hyperbolic distance between two points, P_1 and P_2, is defined as half of the logarithm of the cross ratio between P_1 and P_2 and the two points P_a and P_b that are cut out of the absolute curve by a straight line through P_1 and P_2.

d. The Poincaré model

Although it was known to Beltrami (1), this model has been called after Poincaré, because of his extensive use of it in his investigations on automorphic functions (8). Examples of this model are shown in Fig. XVIII-3 and Fig. XVIII-4. In Fig. XVIII-3 the unit circle is the absolute curve (infinity); in Fig. XVIII-4 it is a straight line. In both figures a straight line is represented by an arc of a circle which cuts the absolute curve orthogonally.

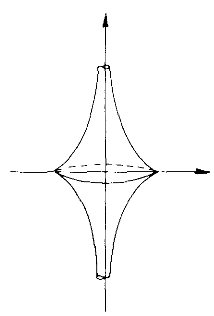

Fig. XVIII-1. Tractrix.

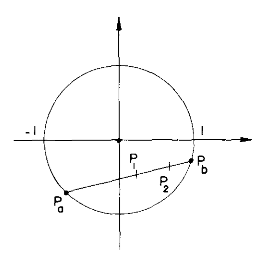

Fig. XVIII-2. Two-dimensional Cayley-Klein model.

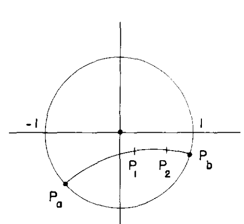

Fig. XVIII-3. Two-dimensional Poincaré model.

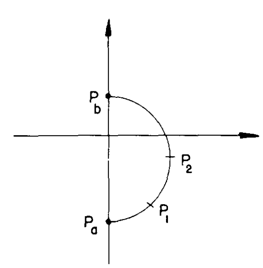

Fig. XVIII-4. Two-dimensional Poincaré model.

1.2 Three-dimensional models

a. Hyperhyperboloid

The hyperhyperboloid is imbedded in four dimensions, and therefore it is of little interest from a practical point of view. However, in many cases the calculations and constructions can be made in three dimensions and formally extended to four dimensions (5).

b. The Cayley-Klein model

This model is a direct generalization of the two-dimensional model to three dimensions. It has been thoroughly studied by Schilling (9), who, for simplicity, uses the (Riemann) unit sphere as an absolute surface.

c. The Poincaré model

This model is a direct generalization of the two-dimensional model to three dimensions.

2. Elliptic Geometry Models

2.1 Two-dimensional models

a. The sphere

The sphere is the simplest surface of constant positive curvature. However, the important assumption that two points on a diameter together form a single "point" has to be made in order that the sphere (hemisphere) constitute an ideal model of two-dimensional elliptic geometry.

b. The projective plane

A central projection of the hemisphere on a plane parallel to the plane that limits the hemisphere and is tangent to it yields a model of elliptic geometry in the form of a one-sided projective plane.

c. Closed surfaces

Two closed surfaces without singularities were found by Boy (10) and Schilling (11). These surfaces possess all of the properties of the projective plane.

2.2 Three-dimensional models

a. The hypersphere

The hypersphere is imbedded in four dimensions.

b. Three-dimensional elliptic space

This space has been thoroughly studied by Schilling (12), among others.

c. Two Riemann spheres

An interesting model of three-dimensional elliptic space consists of two Riemann spheres imbedded in Euclidean space. The natural analytic tool to use is the theory of quaternions. See the work of Study (13) and of others (14).

3. Interconnections of the Non-Euclidean Geometry Models

3.1 Two-dimensional models

a. Geometric treatment

Let us study a simple example, an ideal transformer transformation:

$$Z' = k^2 Z = e^{2\psi} Z, \qquad k = 2 \tag{1}$$

If we use the well-known transformation,

$$\Gamma' = \frac{Z' - 1}{Z' + 1}, \qquad \Gamma = \frac{Z - 1}{Z + 1} \tag{2}$$

where Γ' and Γ are complex reflection coefficients, Eq. 1 transforms into

$$\Gamma' = \frac{\Gamma \cosh \psi + \sinh \psi}{\Gamma \sinh \psi + \cosh \psi} = \frac{\frac{5}{4}\Gamma + \frac{3}{4}}{\frac{3}{4}\Gamma + \frac{5}{4}} \tag{3}$$

If the complex-impedance plane, $Z = R + jX$, is stereographically mapped on the Riemann unit sphere, the reflection-coefficient plane, $\Gamma = \Gamma_z + j\Gamma_y$, falls in the yz-plane, because it can easily be shown that Eq. 2 corresponds to a rotation of the projection center from (0, 0, 1) to (-1, 0, 0) on the sphere. In Fig. XVIII-5 the point $Z = 1$ corresponds to $\Gamma = 0$; and $Z' = 4$, which is obtained from Eq. 1, corresponds to $\Gamma'_z = 0.6$. The ideal transformer corresponds to a stretching (hyperbolic transformation) of the surface of the sphere directed from the fixed point (0, 0, -1) toward the fixed point (0, 0, 1), so that (1, 0, 0) is transformed into (0.47, 0, 0.88).

We now consider a unit hyperboloid with the x-axis as the rotation axis. In Fig. XVIII-5, therefore, we obtain a unit hyperbola. M. Riesz (5) has shown that if, in Fig. XVIII-5, the point $\Gamma'_z = P_p$, which may be considered to lie in a two-dimensional Poincaré model in the yz-plane, is stereographically projected on the hyperboloid from (-1, 0, 0) so that P'_h is obtained, then a central line $\overline{OP'_h}$ cuts the vertical plane parallel to the yz-plane through (1, 0, 0) in P'_{CK} which may be considered a point of a two-dimensional Cayley-Klein model. The Cayley-Klein diagram may also be obtained by an orthographic projection of the sphere on the same plane. An orthographic projection

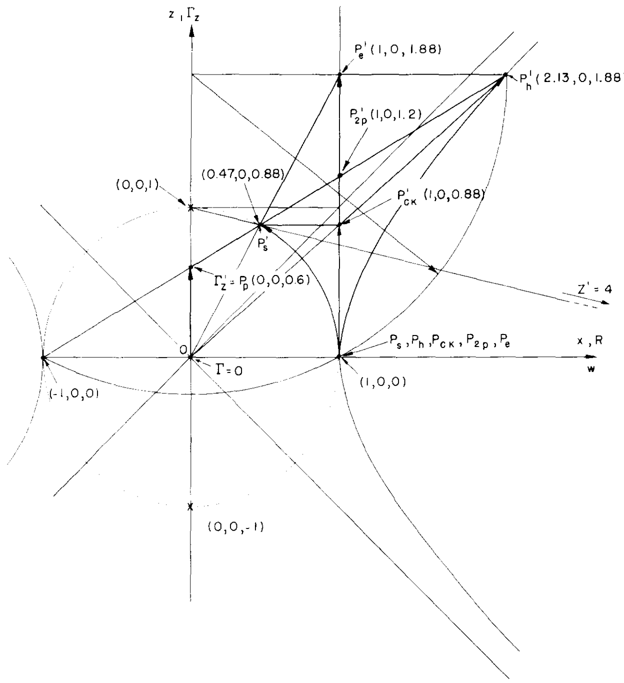

Fig. XVIII-5. Interconnections of different non-Euclidean models.

of P_h' on the same plane yields P_e', which can also be obtained by a central projection of P_s'. Therefore, P_e' may be considered as lying in an elliptic projective plane. Thus we have the hyperbolic Cayley-Klein model (P_{CK}'), the parabolic (Euclidean) plane (P_{2p}'), and the elliptic projective plane (P_e') coalescing in the same vertical plane. By varying $Z'(\Gamma')$ we obtain a geometric picture of how the points P_s', P_{CK}', P_{2p}', P_e', and P_h' vary. The example selected, $Z = 1 \rightarrow Z' = 4$, is indicated by arrows in Fig. XVIII-5.

b. Analytic treatment

A point Γ in the Γ-plane (Poincaré model) is

$$\Gamma = \Gamma_z + j\Gamma_y$$

If Γ is expressed in tetracyclic coordinates (14), we obtain

$$\left.\begin{aligned} \sigma x_4 &= 1 + \Gamma_z^2 + \Gamma_y^2 \\ \sigma x_3 &= 1 - \Gamma_z^2 - \Gamma_y^2 \\ \sigma x_2 &= 2\Gamma_y \\ \sigma x_1 &= 2\Gamma_z \end{aligned}\right\}$$

The point P_s on the unit (hemi) sphere is

$$\left.\begin{aligned} z_s &= \frac{2\Gamma_z}{1+\Gamma_z^2+\Gamma_y^2} \\ y_s &= \frac{2\Gamma_y}{1+\Gamma_z^2+\Gamma_y^2} \\ x_s &= \frac{1-\Gamma_z^2-\Gamma_y^2}{1+\Gamma_z^2+\Gamma_y^2} \end{aligned}\right\}$$

P_s is obtained by a stereographic projection on the sphere.

The point P_{CK} in the Cayley-Klein diagram is

$$\left.\begin{aligned} \Gamma_{zCK} &= \frac{2\Gamma_z}{1+\Gamma_z^2+\Gamma_y^2} = z_s \\ \Gamma_{yCK} &= \frac{2\Gamma_y}{1+\Gamma_z^2+\Gamma_y^2} = y_s \end{aligned}\right\}$$

In homogeneous coordinates the point P_{CK} is expressed as

$$\rho\begin{pmatrix} w \\ z \\ y \end{pmatrix} = \rho\begin{pmatrix} \dfrac{1}{(1-\Gamma_{zCK}^2-\Gamma_{yCK}^2)^{1/2}} \\ \dfrac{\Gamma_{zCK}}{(1-\Gamma_{zCK}^2-\Gamma_{yCK}^2)^{1/2}} \\ \dfrac{\Gamma_{yCK}}{(1-\Gamma_{zCK}^2-\Gamma_{yCK}^2)^{1/2}} \end{pmatrix} = \rho\begin{pmatrix} \dfrac{1+\Gamma_z^2+\Gamma_y^2}{1-\Gamma_z^2-\Gamma_y^2} \\ \dfrac{2\Gamma_z}{1-\Gamma_z^2-\Gamma_y^2} \\ \dfrac{2\Gamma_y}{1-\Gamma_z^2-\Gamma_y^2} \end{pmatrix} = \rho\begin{pmatrix} \dfrac{1}{x_s} \\ \dfrac{z_s}{x_s} \\ \dfrac{y_s}{x_s} \end{pmatrix}$$

The point P_h on the surface of the hyperboloid, $w^2 - z^2 - y^2 = 1$, is

$$\left.\begin{aligned} w &= \frac{1}{x_s} \\ z &= \frac{z_s}{x_s} \\ y &= \frac{y_s}{x_s} \end{aligned}\right\}$$

The transformation of $P_h \to P'_h$ on the hyperboloid is expressed as

$$\begin{pmatrix} w' \\ z' \\ y' \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} w \\ z \\ y \end{pmatrix}$$

The transformation by the real 3×3 matrix belongs to the G_+ subgroup of the proper Lorentz group.

The point P'_{CK} in the Cayley-Klein diagram in homogeneous coordinates is

$$\rho \begin{pmatrix} w' \\ z' \\ y' \end{pmatrix}$$

The point P'_s on the sphere is

$$\left.\begin{aligned} z'_s &= \frac{z'}{w'} = \frac{z'}{(1+z'^2+y'^2)^{1/2}} \\ y'_s &= \frac{y'}{w'} = \frac{y'}{(1+z'^2+y'^2)^{1/2}} \\ x'_s &= \frac{1}{w'} = \frac{1}{(1+z'^2+y'^2)^{1/2}} \end{aligned}\right\}$$

The point P'_{CK} in the Cayley-Klein model is

$$\left.\begin{aligned} \Gamma'_{zCK} &= z'_s \\ \Gamma'_{yCK} &= y'_s \end{aligned}\right\}$$

The point Γ'_z in the Γ-plane (Poincaré model) is

$$\left.\begin{aligned} \Gamma'_z &= \frac{z'_s}{1+x'_s} = \frac{z'}{1+(1+z'^2+y'^2)^{1/2}} = \frac{\Gamma'_{zCK}}{1+(1-\Gamma'^2_{zCK}-\Gamma'^2_{yCK})^{1/2}} \\ \Gamma'_y &= \frac{y'_s}{1+x'_s} = \frac{y'}{1+(1+z'^2+y'^2)^{1/2}} = \frac{\Gamma'_{yCK}}{1+(1-\Gamma'^2_{zCK}-\Gamma'^2_{yCK})^{1/2}} \end{aligned}\right\}$$

The rather complicated procedure, which is presented above, for transforming $\Gamma \to \Gamma'$ was performed in order to show analytically how some of the two-dimensional non-Euclidean geometry models can be used. In the presentation we have proceeded from the Γ-plane (Poincaré model) to the hyperboloid by means of the sphere and the Cayley-Klein model. Instead of the Cayley-Klein model we might just as well have used the elliptic projective plane.

3.2 Three-dimensional models

The interconnections of the three-dimensional models can be shown by a direct generalization of the two-dimensional case. We start out with a point Q in three dimensions (Poincaré model) which we express, this time, in pentaspheric coordinates:

$$\left.\begin{aligned} \sigma x_5 &= 1 + Q_z^2 + Q_y^2 + Q_x^2 \\ \sigma x_4 &= 1 - Q_z^2 - Q_y^2 - Q_x^2 \\ \sigma x_3 &= 2Q_x \\ \sigma x_2 &= 2Q_y \\ \sigma x_1 &= 2Q_z \end{aligned}\right\}$$

We proceed from the Poincaré model to the hyperhyperboloid by means of the hypersphere and either the three-dimensional Cayley-Klein model or the three-dimensional elliptic space. The transformation on the hyperhyperboloid is performed by a real 4×4 matrix which is the same matrix that was discussed in the Quarterly Progress Report of July 15, 1956, pages 83-85.

While the two-dimensional models are suited for impedance transformations through lossless, bilateral, two terminal-pair networks, the three-dimensional models are equally suited to impedance transformations through lossy, bilateral, two terminal-pair networks.

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References

1. E. Beltrami, Giorn. Mat. 6, 284-312 (1868); Opere Matematiche di E. Beltrami, Vol. I (Ulrico Hoepli, Milano, 1902), pp. 374-405.
2. F. Schilling, Die Pseudosphäre und die nichteuklidische Geometrie (B. G. Teubner Verlag, Leipzig and Berlin, 1935).
3. F. Schilling, Pseudosphärische, hyperbolisch-spärische und elliptisch-sphärische Geometrie (B. G. Teubner Verlag, Leipzig and Berlin, 1937).
4. D. Hilbert, Trans. Am. Math. Soc. 2, 87-99 (1901).
5. M. Riesz, Lunds Univ. Årsskrift, N. F. Avd. 2, 38, No. 9 (1943); Kungl. Fysiograf. Sällsk. Handl., N. F. 53, No. 9 (1943), 76 pp. (In Swedish).
6. F. Klein, Math. Ann. 4, 573-625 (1871); F. Klein Gesammelte Mathematische Abhandlungen. Herausgegeben von R. Fricke und A. Ostrowski, Vol. I (Springer Verlag, Berlin, 1921), pp. 254-305.

7. A. Cayley, Trans. Roy. Soc. (London) 149, 61-90 (1859); The Collected Mathematical Papers of A. Cayley, Vol. II (Cambridge University Press, London, 1889), pp. 561-592.
8. H. Poincaré, Acta. Mat. 1, 1-62 (1882).
9. F. Schilling, Die Bewegungstheorie im nichteuklidischen hyperbolischen Raum, Vols. I, II (Leibniz-Verlag, München, 1948).
10. W. Boy, Math. Ann. 57, 151-184 (1903).
11. F. Schilling, Math. Ann. 92, 69-79 (1924).
12. F. Schilling, Die Bewegungstheorie im nichteuklidischen elliptischen Raum (Danzig, 1943).
13. E. Study, Am. J. Math. 29, 101-167 (1907).
14. F. Klein, Vorlesungen über höhere Geometrie. Bearbeitet und herausgegeben von W. Blaschke (Springer Verlag, Berlin, 1926).

B. ELEMENTARY NETWORK THEORY FROM AN ADVANCED GEOMETRIC STANDPOINT

In the early days of network theory two different methods of transforming impedances by the linear fractional transformation

$$Z' = \frac{aZ + b}{cZ + d}, \qquad ad - bc = 1 \tag{1}$$

were developed. These methods were called the "iterative impedance method" and the "image impedance method."

In the first method the input impedance Z' is expressed in the output impedance Z by the linear fractional transformation

$$Z' = \frac{Z \dfrac{Z_{f1} e^{\lambda} - Z_{f2} e^{-\lambda}}{Z_{f1} - Z_{f2}} - \dfrac{Z_{f1} Z_{f2}}{Z_{f1} - Z_{f2}} (e^{\lambda} - e^{-\lambda})}{Z \dfrac{e^{\lambda} - e^{-\lambda}}{Z_{f1} - Z_{f2}} + \dfrac{Z_{f1} e^{-\lambda} - Z_{f2} e^{\lambda}}{Z_{f1} - Z_{f2}}} \tag{2}$$

where the iterative impedances Z_{f1} and Z_{f2}, the fixed points of the transformation, are expressed as

$$\left.\begin{matrix} Z_{f1} \\ Z_{f2} \end{matrix}\right\} = \frac{a - d \pm [(a+d)^2 - 4]^{1/2}}{2c}, \qquad ad - bc = 1 \tag{3}$$

and

$$e^{\lambda} = \frac{a + d - [(a+d)^2 - 4]^{1/2}}{2} \tag{4}$$

In the Quarterly Progress Report of April 15, 1956, pages 126-8, it was shown that

Eq. 2, written in the canonic form

$$\frac{Z' - Z_{f1}}{Z' - Z_{f2}} = e^{2\lambda} \frac{Z - Z_{f1}}{Z - Z_{f2}} \tag{5}$$

can be interpreted as a spiral (loxodromic) motion of the surface of the unit Riemann sphere around an inner axis through the fixed points Z_{f1} and Z_{f2}. This motion can be split into a stretching and a rotation, given by the real part λ' and the imaginary part λ'' of λ, respectively.

A similar geometric interpretation is valid for the image impedance method. For that case Eq. 1 can be written:

$$Z' = \frac{Z \cosh\theta \left(\frac{Z_i'}{Z_i}\right)^{1/2} + \sinh\theta\,(Z_i\ Z_i')^{1/2}}{Z \sinh\theta \frac{1}{(Z_i\ Z_i')^{1/2}} + \cosh\theta \left(\frac{Z_i}{Z_i'}\right)^{1/2}} \tag{6}$$

with

$$Z_i = (Z_o\ Z_s)^{1/2} \tag{7}$$

$$Z_i' = (Z_o'\ Z_s')^{1/2} \tag{8}$$

$$\tanh\theta = \left(\frac{Z_s}{Z_o}\right)^{1/2} = \left(\frac{Z_s'}{Z_o'}\right)^{1/2} \tag{9}$$

where Z_i and Z_i' are the image impedances at the output and at the input, Z_o and Z_o' are open-circuit impedances, Z_s and Z_s' are short-circuit impedances, and θ is the image propagation function, also called the "image transfer constant." A comparison of Eqs. 6 and 1 yields:

$$Z_i = \left(\frac{bd}{ac}\right)^{1/2}, \quad Z_i' = \left(\frac{ab}{cd}\right)^{1/2}, \quad \tanh\theta = \left(\frac{bc}{ad}\right)^{1/2} \tag{10}$$

Let us study a simple example consisting of the unsymmetric resistive network shown in Fig. XVIII-6. The six impedances $Z_{f1} = 2$, $Z_{f2} = -1$, $Z_o' = 3$, $Z_s' = 1$, $Z_o = -2$, and $Z_s = -2/3$ are stereographically mapped on the unit sphere (unit circle), as shown in Fig. XVIII-7. The sphere (circle) is considered as the absolute surface (curve) of a three- (two-) dimensional hyperbolic space with hyperbolic metric. In this space a hyperbolic distance is defined as half the logarithm of the cross ratio between the given points and the two points on the unit sphere (circle) which are cut out by a straight line through the given points. See, for example, references 1 and 2.

Equations 7 and 8 yield simple constructions for the image impedances $Z_i = 2\sqrt{3}/3$

and $Z_i' = \sqrt{3}$. See Fig. XVIII-7. Equation 9 is essentially the equation for an ideal transformer. Therefore, the hyperbolic distances $\overline{os}$ and $\overline{o's'}$ in Fig. XVIII-7 are equal; here $\tanh\theta = \sqrt{3}/3$. From Eqs. 7, 8, 9, 3, and 1 we obtain

$$(-Z_s Z_o')^{1/2} = (-Z_s' Z_o)^{1/2} = (Z_i Z_i')^{1/2} = (-Z_{f1} Z_{f2})^{1/2} \ (= \sqrt{2}\,) \tag{10}$$

Therefore, the lines $\overline{Z_s Z_o'}$, $\overline{Z_s' Z_o}$, $\overline{(-Z_i) Z_i'}$, and $\overline{Z_{f1} Z_{f2}}$ must all pass through a point f in Fig. XVIII-7, yielding, for example, a simple construction for finding Z_o, when Z_s', Z_o', and Z_s are known.

Equation 6 can be split as follows:

$$Z_a = \frac{Z_i'}{Z_i} Z \tag{11a}$$

$$Z' = \frac{Z_a \cosh\theta + Z_i' \sinh\theta}{Z_a \sinh\theta/Z_i' + \cosh\theta} \tag{11b}$$

$$Z_b = \frac{1}{Z_i} Z \tag{12a}$$

$$Z_c = \frac{Z_b \cosh\theta + \sinh\theta}{Z_b \sinh\theta + \cosh\theta} \tag{12b}$$

$$Z' = Z_i' Z_c \tag{12c}$$

$$Z_d = \frac{Z \cosh\theta + Z_i \sinh\theta}{Z \sinh\theta/Z_i + \cosh\theta} \tag{13a}$$

$$Z' = \frac{Z_i'}{Z_i} Z_d \tag{13b}$$

Equations 11a, 12a, 12c, and 13b represent ideal transformers that correspond to hyperbolic transformations (stretchings) along the vertical axis through the fixed points zero and infinity. Equations 11b, 12b, and 13a represent symmetric networks (a=d) that correspond to hyperbolic transformations (a+d > 2) along horizontal axes through the fixed points $\pm Z_i'$, ± 1, and $\pm Z_i$. The equivalent networks corresponding to Eqs. 11, 12, and 13 are shown in Figs. XVIII-8a, XVIII-9a, and Fig. XVIII-10a.

Finally, let us perform a transformation of the impedance $Z = 0.5$ to $Z' = 1.4$ by the iterative impedance method and the image impedance method. We know that the point $Z = 0$ is transformed into $Z' = Z_s'$ so that, according to the first method, f is transformed into g in Fig. XVIII-7. The hyperbolic distance $\overline{fg}$ is $\lambda' = \ln\frac{1}{2}$. Therefore, Z' is obtained from Z by two perspectivities with f and g as centers (1, 2).

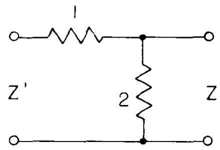

Fig. XVIII-6. Simple example of a resistive network.

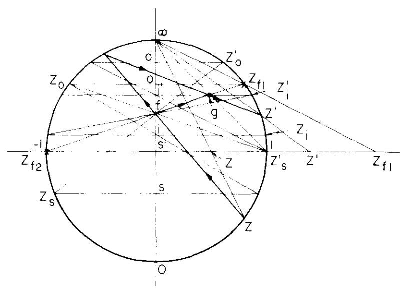

Fig. XVIII-7. Iterative impedance method.

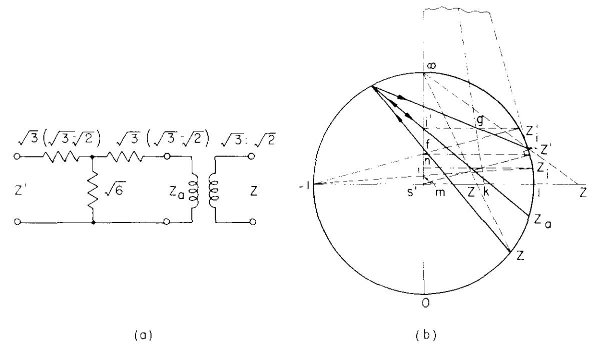

Fig. XVIII-8. Image impedance method. Example 1.

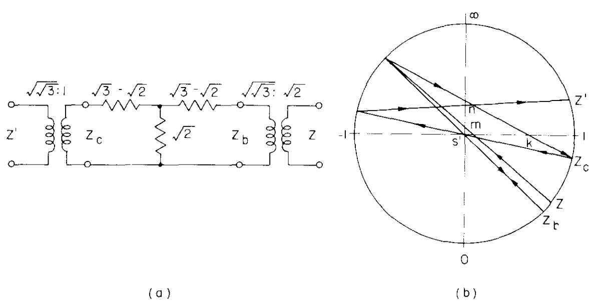

Fig. XVIII-9. Image impedance method. Example 2.

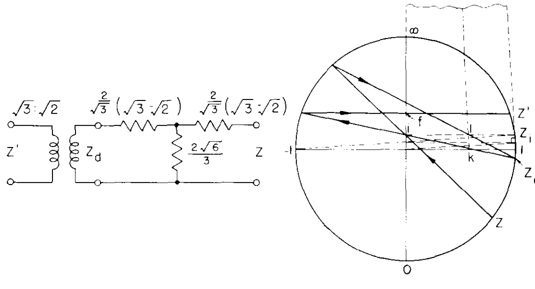

Fig. XVIII-10. Image impedance method. Example 3.

Transformations by the image impedance method are shown in Figs. XVIII-8b, XVIII-9b, and XVIII-10b. According to Fig. XVIII-8a and b, $Z \rightarrow Z'$ by two perspectivities through f and i', corresponding to the ideal transformer, followed by two perspectivities through i' and g, corresponding to the symmetric network. From Fig. XVIII-9a and b, we obtain six perspectivities through m and s' (first ideal transformer), s' and k (symmetric network = ideal reflection-coefficient transformer), and s' and n (second ideal transformer). From Fig. XVIII-10a and b, we obtain four perspectivities through i and l (symmetric network), and i and f (ideal transformer). Obviously, if the network studied is symmetric, $Z_{f1} = Z_i = Z_i'$ and $Z_{f2} = -Z_i = -Z_i'$, and the two methods coalesce.

If the network is composed of both lossy and lossless components the geometric constructions have to be performed in three dimensions.

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References

1. J. Van Slooten, Meetkundige Beschouwingen in Verband met de Theorie der Electrische Vierpolen (W. D. Meinema, Delft, 1946).
2. E. F. Bolinder, Acta Polytechnica, Electrical Engineering Series, Vol. 7, No. 5, 1956.