XV. MICROWAVE THEORY

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A. GEOMETRIC-ANALYTIC THEORY OF NOISY TWO-PORTS

Sections XV-A, B, and C are extensions of the geometric-analytic theory of noisy two-ports published in the Quarterly Progress Report of July 15, 1957, pages 163-169. This report will use the notation that was given in the previous report. In the presentation of the geometric-analytic theory voltages and currents were used. In many microwave applications, however, it is more convenient to use a wave representation.

We split the reflected wave at the output of the two-port network into two components

$$a_2 = a_{2n} + \Gamma_{cor}\, a_1 \tag{1}$$

where a_1 is the transmitted wave, and Γ_{cor} is a complex correlation reflection coefficient, defined by

$$\Gamma_{cor} = \frac{\overline{a_2 a_1^*}}{\overline{a_1 a_1^*}} \tag{2}$$

In Eq. 1, a_{2n} and $\Gamma_{cor}\, a_1$ are uncorrelated and thus

$$\overline{a_{2n} a_1^*} = 0 \tag{3}$$

If we set

$$\left.\begin{aligned} \overline{a_2 a_2^*} &= k\Delta f\, T_2 \\ \overline{a_{2n} a_{2n}^*} &= k\Delta f\, T_{2n} \\ \overline{a_1 a_1^*} &= k\Delta f\, T_1 \end{aligned}\right\} \tag{4}$$

where T_2, T_{2n}, and T_1 are noise temperatures, Eq. 1 yields

$$T_2 = T_{2n} + |\Gamma_{cor}|^2\, T_1 \tag{5}$$

The four-vector Q can now be written in the wave representation as

$$Q_w = \begin{pmatrix} Q_{w1} \\ Q_{w2} \\ Q_{w3} \\ Q_{w4} \end{pmatrix} = \begin{pmatrix} \overline{a_2 a_2^*} \\ \overline{a_2 a_1^*} \\ \overline{a_1 a_2^*} \\ \overline{a_1 a_1^*} \end{pmatrix} = k\Delta f\, T_1 \begin{pmatrix} \frac{T_{2n}}{T_1} + |\Gamma_{cor}|^2 \\ \Gamma_{cor} \\ \Gamma_{cor}^* \\ 1 \end{pmatrix} \tag{6}$$

The reflected and transmitted waves at the input of a noise-free two-port network are expressed in the transmitted and reflected waves at the output by the equation

$$\psi'_w = \begin{pmatrix} a'_2 \\ a'_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = T_w \psi_w \; ; \; AD - BC = 1 \tag{7}$$

By using the chain matrix, we obtain

$$T_w = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = S\, T\, S^{-1} \tag{8}$$

For a noise process, we have

$$Q'_w = \begin{pmatrix} \overline{a'_2 a'^*_2} \\ \overline{a'_2 a'^*_1} \\ \overline{a'_1 a'^*_2} \\ \overline{a'_1 a'^*_1} \end{pmatrix} = \begin{pmatrix} AA^* & AB^* & BA^* & BB^* \\ AC^* & AD^* & BC^* & BD^* \\ CA^* & CB^* & DA^* & DB^* \\ CC^* & CD^* & DC^* & DD^* \end{pmatrix} \begin{pmatrix} \overline{a_2 a^*_2} \\ \overline{a_2 a^*_1} \\ \overline{a_1 a^*_2} \\ \overline{a_1 a^*_1} \end{pmatrix} = L_w Q_w \tag{9}$$

where

$$L_w = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} aa^* & ab^* & ba^* & bb^* \\ ac^* & ad^* & bc^* & bd^* \\ ca^* & cb^* & da^* & db^* \\ cc^* & cd^* & dc^* & dd^* \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{10}$$

L_w is the product of the three Kronecker products $S \times S$, $T \times T^*$, and $S^{-1} \times S^{-1}$.

If we set

$$\left.\begin{aligned} V &= \frac{1}{\sqrt{2}}(a_1 + a_2) \\ I &= \frac{1}{\sqrt{2}}(a_1 - a_2) \end{aligned}\right\} \qquad \left.\begin{aligned} a_2 &= \frac{1}{\sqrt{2}}(V-I) \\ a_1 &= \frac{1}{\sqrt{2}}(V+I) \end{aligned}\right\} \tag{11}$$

we obtain

$$\left.\begin{aligned} P_{Z1} &= \tfrac{1}{2}(VI^* + V^*I) = -\tfrac{1}{2}\left(a_2a_2^* - a_1a_1^*\right) = -P_{\Gamma 3} \\ P_{Z2} &= -\tfrac{j}{2}(VI^* - V^*I) = -\tfrac{j}{2}\left(a_2a_1^* - a_1a_2^*\right) = P_{\Gamma 2} \\ P_{Z3} &= \tfrac{1}{2}(VV^* - II^*) = \tfrac{1}{2}\left(a_2a_1^* + a_1a_2^*\right) = P_{\Gamma 1} \\ P_{Z4} &= \tfrac{1}{2}(VV^* + II^*) = \tfrac{1}{2}\left(a_2a_2^* + a_1a_1^*\right) = P_{\Gamma 4} \end{aligned}\right\} \tag{12}$$

Thus, for bilateral, two-port networks, Eq. 9 can be interpreted geometrically as a movement in a Poincaré model of the three-dimensional hyperbolic space that has the complex reflection-coefficient plane for its absolute surface. The corresponding Cayley-Klein model, which has the unit sphere as the absolute surface, is obtained from the Cayley-Klein model of the last report by a 90° rotation around the y-axis.

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B. CASCADING OF NOISY TWO-PORT NETWORKS

With the notation of Fig. XV-1, on a voltage-current basis, we write

$$\psi'' = \psi^o + \psi' = \psi^o + T\psi \tag{1}$$

Similarly, on a power basis, we obtain

$$Q'' = Q^o + Q' = Q^o + T \times T^* Q \tag{2}$$

If we insert in Eq. 2 the Q-vectors expressed in the equivalent noise resistance r_n, the equivalent noise conductance g_n, and the complex correlation impedance Z_{cor} (1),

$$Q = 4k\Delta f \begin{pmatrix} r_n + g_n \left|Z_{cor}\right|^2 \\ g_n Z_{cor} \\ g_n Z_{cor}^* \\ g_n \end{pmatrix} \tag{3}$$

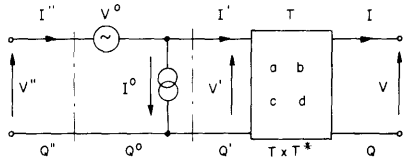

Fig. XV-1. Noisy two-port network.

we obtain

$$\left.\begin{aligned} r''_n &= r^o_n + r'_n + \frac{g^o_n g'_n \left| Z'_{cor} - Z^o_{cor} \right|}{g^o_n + g'_n} \\ g''_n &= g^o_n + g'_n \\ Z''_{cor} &= \frac{g^o_n Z^o_{cor} + g'_n Z'_{cor}}{g^o_n + g'_n} \end{aligned}\right\} \quad (4)$$

From Eqs. 2 and 3 we also obtain

$$\left.\begin{aligned} g'_n &= cc^* r_n + \left| c Z_{cor} + d \right|^2 g_n \\ r'_n &= \frac{\left| ad - bc \right|^2 r_n g_n}{cc^* r_n + \left| c Z_{cor} + d \right|^2 g_n} \\ Z'_{cor} &= \frac{ac^* r_n + (a Z_{cor} + b)(c^* Z^*_{cor} + d^*) g_n}{cc^* r_n + \left| c Z_{cor} + d \right|^2 g_n} \end{aligned}\right\} \quad (5)$$

Equations 4 and 5 are formulas obtained by Dahlke (2). In Dahlke's formulas $a_{11} = d$, $a_{12} = c$, $a_{21} = b$, and $a_{22} = a$.

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References

1. H. Rothe, W. Dahlke, Proc. IRE 44, 811-818 (1956); AEÜ 9, 117-121 (1955); cf. E. F. Bolinder, Quarterly Progress Report, July 15, 1957, p. 163.
2. W. Dahlke, AEÜ 9, 391-401 (1955).

C. GEOMETRIC-ANALYTIC THEORY OF PARTLY POLARIZED ELECTROMAGNETIC WAVES

The geometric-analytic theory of noisy two-ports is converted into a theory of partly polarized electromagnetic waves, if the voltages and currents are exchanged for two complex electric field-strength quantities E_y and E_x, perpendicular in space to each other and to the direction of propagation of the wave. The complex correlation impedance Z_{cor} is replaced by a complex correlation polarization ratio p_{cor}, where

$$p_{cor} = \frac{\overline{E_y E_x^*}}{\overline{E_x E_x^*}} \tag{1}$$

For a polarized wave (beam),

$$p = \frac{E_y}{E_x} \tag{2}$$

(Some authors (1) use -jp instead of p.) The mapping of the complex polarization ratio p on the Riemann unit sphere was introduced into optics by Poincaré (2), the "Poincaré sphere," and into antenna theory by Deschamps (3).

The transformations of elliptically polarized waves through lossless transducers form a unitary group. Therefore,

$$\psi_e' = \begin{pmatrix} E_y' \\ E_x' \end{pmatrix} = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix} \begin{pmatrix} E_y \\ E_x \end{pmatrix} = T_e \psi_e \ ; \ |a|^2 + |c|^2 = 1 \tag{3}$$

In the p-plane

$$p' = \frac{ap - c^*}{cp + a^*} \tag{4}$$

A simple example of a lossless p-transformation ($a = (\sqrt{6}/3)e^{j60°}$, $c = (\sqrt{3}/3)e^{j30°}$, $p = 0$, and $p' = (\sqrt{2}/2)e^{j210°}$) is performed by the isometric circle method (4) and shown

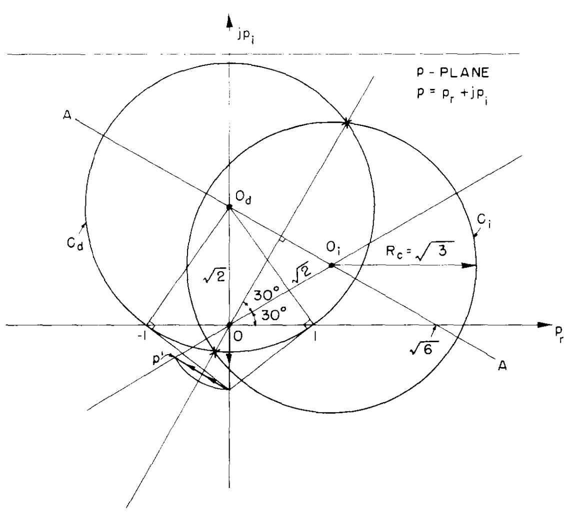

Fig. XV-2. Example of a lossless polarization-ratio transformation.

in Fig. XV-2. The isometric circles intersect, so that the transformation is elliptic. The fixed points of the transformation (marked by crosses) fall, by a stereographic mapping, on a diameter of the unit sphere. The transformation consists of a rotation of the sphere through an angle 2ϕ around the diameter, where ϕ is given by one of the eigenvalues of the matrix T_e. Thus we have

$$e^{\pm j\phi} = \frac{1}{2}\left[(a + a^*) \pm \sqrt{(a + a^*)^2 - 4}\right] \tag{5}$$

An elegant and convenient tool to use in connection with unitary groups is Hamilton's theory of quaternions (5). Partly polarized waves can be represented by the Q_e- or P_e-power four-vectors (Stokes vectors). Points, representing power ratios, are obtained both in a three-dimensional space that is formed by an extension of the polarization-ratio plane, and inside the unit sphere. An unpolarized wave (natural light) is represented by the center of the sphere or by the point (0, 0, 1) in the Q_e-space.

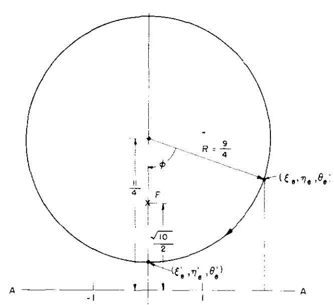

Fig. XV-3. Example of a lossless power-ratio transformation in the Q_e-space.

Figure XV-3 shows a cross section through A-A of Fig. XV-2 extended to three dimensions. The point $(\xi_e, \eta_e, \theta_e) = (\sqrt{6}, 0, 2)$ is transformed into $(\xi_e', \eta_e', \theta_e') = (\sqrt{6}/4, 3\sqrt{2}/4, 1/2)$ by a rotation through the angle ϕ. This rotation may be considered as a non-Euclidean rotation around the point F, the crossover point between the plane and a semicircle through the fixed points. The semicircle is orthogonal to the ξ_e- η_e-plane. For a unitary transformation, the semicircle passes through (0, 0, 1).

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References

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2. H. Poincaré, Théorie Mathématique de la Lumière (G. Carré, Paris, Vol. I, 1889; Vol. II, 1892).
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5. W. R. Hamilton, Lectures on Quaternions (Hodges and Smith, Dublin, 1853).