XIX. PHYSICAL ACOUSTICS*

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RESEARCH OBJECTIVES

The research objectives of the Physical Acoustics group were described in Quarterly Progress Report No. 51 (p. 95). In connection with the program on nonlinear problems mentioned there, a new experiment on scattering of sound by sound, and measurements of nonlinear acoustic effects in liquid helium are in preparation.

U. Ingard

A. SIMPLE EXAMPLE OF MAGNETOMECHANICAL WAVE MOTION

By means of simple mechanical systems, such as a periodic one-dimensional line of masses coupled magnetically, some of the elementary features of magnetohydrodynamic waves can be demonstrated. Such a simple mechanical system is illustrated in Fig. XIX-1. It consists of a series of coils that are free to move in a magnetic field (loud-speaker coils and magnets). Each coil consists of two separate windings. The individual coils are connected electrically with each other to form a periodic line. If one of these coils is set in motion, the voltage produced gives rise to a current that flows to the neighboring coil and sets it in motion. The motion of this coil, in turn, generates a new current that is sent to the next neighbor, and so on. The phase velocity of the wave disturbance which is produced in this way can readily be calculated and shown to be analogous to the phase velocity of a transverse (Alfvén) wave in a conducting, incompressible fluid.

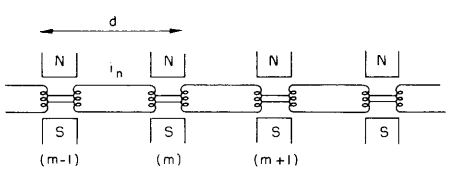

Fig. XIX-1.

If the particle velocity of coil number n is denoted by u_n, the mass and inductance of each coil by M and L, respectively, the magnetic field by B, and the current flowing into coil number n by i_n, we obtain the following equations of motion:

$$M \frac{du_n}{dt} = B(i_n - i_{n+1})$$

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$$L \frac{di_n}{dt} = B(u_{n-1} - u_n)$$

That is,

$$\frac{d^2 u_n}{dt^2} = \frac{B^2}{ML}(u_{n-1} + u_{n+1} - 2u_n)$$

In these equations we have assumed that the length of the coil is unity. If this is not the case, the magnetic field B in these equations should be replaced by B multiplied by the length of the coil.

If the phase shift in one period is described by $u_{n-1}/u_n = e^{i\Gamma}$, by inserting this expression in the last equation, we obtain

$$\cos \Gamma - 1 = -\frac{\omega^2 ML}{2B^2}$$

At low frequencies the expression for Γ is then

$$\Gamma^2 \approx \frac{\omega^2 ML}{B^2}$$

If the distance between two coils in the lattice is d, we can introduce the mass and inductance per unit length as $m = M/d$ and $\ell = L/d$, respectively. Furthermore, if we get $\Gamma = \omega d/V$, where V is the phase velocity of the wave, we obtain

$$V = \frac{B}{(m\ell)^{1/2}}$$

It is interesting to notice that this expression for the phase velocity in the magneto-mechanical system is analogous to the phase velocity of the well-known Alfvén wave (1) in an incompressible, conductive fluid of infinite conductibility,

$$V_a = \frac{B}{(\rho\mu)^{1/2}}$$

where ρ is the density of the medium, corresponding to our mass per unit length, and μ is the magnetic permeability of the medium, corresponding to our inductance per unit length. (Note that the coil winding in the model was assumed to have unit length.)

This mechanical lattice, of course, can be generalized to include "compressibility" of the medium by linking the coils with mechanical couplings of various kinds. Whatever mechanical lattice we produce can always be represented by an electrical network analog. Such an analog may prove quite valuable, particularly in the study of magneto-mechanical wave motion in an inhomogeneous magnetic field.

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References

1. H. Alfvén, Cosmical Electrodynamics (Oxford University Press, London, 1950), p. 80.

B. MODE COUPLING ON A STRING IN A MAGNETIC FIELD

In Section XIX-A a magnetomechanical system involving a number of discrete masses coupled magnetically (and mechanically) was studied. As another simple example of the magnetomechanics of a continuous elastic system, we shall select a conducting string oscillating in a magnetic field. The string is mounted in a magnetic field and made to oscillate in a direction perpendicular to the magnetic field, as shown in Fig. XIX-2.

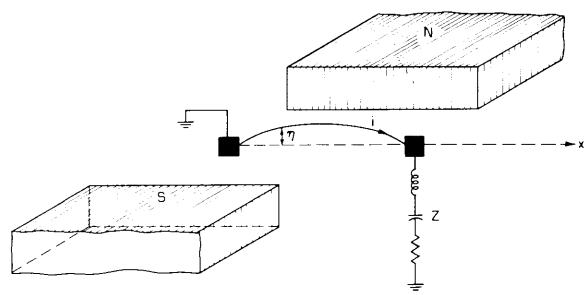

Fig. XIX-2.

The terminations of the string are clamped and connected electrically through an electric circuit that may be a combination of inductive, capacitive, and resistive elements.

If we introduce an x-coordinate axis, as shown in Fig. XIX-2, and let the magnetic field be described by a function $B = B(x)$, and if we let the electric load be a pure resistance R, the small amplitude equation of motion of the string can be written

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2} - \frac{B(x)}{\rho R} \int_0^{\ell} \frac{\partial \eta}{\partial t} B(x)\, dx$$

In this equation η represents the vertical displacement of the string; ρ is the mass per unit length; $c = (T/\rho)^{1/2}$, the natural wave speed on the string (with $B = 0$); and T is the tension in the string.

Now, if we consider free motions of the string without any external driving forces, we can express the displacement of the string as the sum of normal modes with time-variant amplitudes as follows:

$$\eta(x, t) = \sum \eta_m(t) \sin k_m x$$

$$k_m = \frac{m\pi}{d} \qquad (d = \text{length of string})$$

We shall consider a magnetic field of the form

$$B(x) = \sum B_n \sin k_n x$$

When the magnetic field is zero, these mode amplitudes η_m are time-independent and their values depend only on the initial conditions of the string. In the presence of the magnetic field, these amplitudes will not only be time-dependent but they will be coupled with each other. As the amplitude of one mode decreases, the corresponding energy is partly transferred into other modes. The quantitative description of this coupling is immediately obtained from the equation of motion and found to be

$$\ddot{\eta}_m = -\omega_m^{02} \eta_m - \sum_n \alpha_{mn} \dot{\eta}_n$$

The mode coupling coupling factor α_{mn} is found to be

$$\alpha_{mn} = \frac{d}{2\rho R} B_m B_n$$

which, with a constant magnetic field, becomes

$$\alpha_{mn} = \frac{8d}{\pi^2} \frac{B^2}{\rho R} \frac{1}{mn}$$

If we can approximate the magnetic field by one single mode $B \approx B_m \sin k_m x$, only the mth mode of the string will be affected by the field, and it will decay at a rate given by $\exp - (\alpha_{mn} t/2)$, where $\alpha_{mn} = B_m^2 \, d/2\rho R$. The decay will be aperiodic when $B_m d \geqslant 2(\pi\rho R)^{1/2}$.

Similar equations can readily be obtained for other types of electric load, each with its own special characteristics. We shall not give a detailed discussion of these characteristics but merely set down the general equation of motion which corresponds to an inductance, a capacitance, and a resistance in series. If this resonant circuit is specified by its characteristic frequency Ω and damping constant $\delta (\delta = R/2L)$, the differential equation governing the coupling between the various modes can be shown to be

$$\left(\frac{d^2}{dt^2} + \omega_{om}^2\right)\left(\frac{d^2}{dt^2} + \delta\frac{d}{dt} + \Omega^2\right)\eta_m = -\sum_n \beta_{mn}\frac{d^2\eta_n}{dt^2}$$

The solution to this equation has been studied in the special case in which there is coupling only between two modes and the load is inductive (all magnetic field Fourier coefficients equal zero except B_1 and B_2). The string is started from $\eta = \eta_1 \sin(\pi x/d)$ and released. The growth of the second harmonic with time and the continuous change of shape of the string can then easily be found.

If the string is made to oscillate in a plane that makes an angle with the plane of the magnetic field, it is clear that only the component that is perpendicular to the field will be affected. Consequently, this component can be damped out by making the electric load a small resistance. This feature might be of interest in eliminating whirling of a string that is vibrating at large amplitudes. It also offers the possibility of turning the plane of "polarization" of a wave passing through the magnetic field into the plane of the magnetic field lines.

The scattering of a wave on a string by a magnetic field (the portion of the string in the magnetic field is part of a closed electric circuit) has the feature that not only are the shapes of the scattered and transmitted waves different from the incident pulse but the transmitted pulse will start as soon as the incident pulse enters the magnetic field. Further details regarding this scattering problem will be described in a later note.

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C. ACOUSTIC NOISE SOURCE DISTRIBUTION IN A TURBULENT JET

A recent report by Ribner (1) presents an analysis of the noise source distribution in a jet, in which Lighthill's general equations of sound production by turbulence are used. It has been found that the source distribution in the fully developed turbulent region of the jet should decrease as the seventh power of the distance from the jet nozzle. The purpose of this note is to give an alternative, simple derivation of this result. The derivation is based on the following observation.

In a perfectly laminar flow, there is no "loss" of energy, apart from that caused by the comparatively small viscous stresses. Under such conditions a loss of kinetic energy shows up as a gain in potential energy in the fluid. On the other hand, whenever the fluid breaks into turbulence, energy is "lost." Energy is taken from the ordered motion of the stream and is converted into the random motion of turbulence, which sooner or later is converted into heat. Whenever such a conversion of energy into turbulence occurs, a certain portion of this energy is transferred into sound. The efficiency of this sound production increases with the velocity of the turbulent

fluctuations (and hence with the velocity of the stream) and is proportional to the fifth power of the Mach number, which is a well-known result from the theory of turbulence.

Let us now apply these general considerations to the jet. In the fully developed turbulent region of the jet, it is well known that a jet from a circular nozzle spreads out in conical fashion in such a way that the diameter of the jet increases in proportion to the distance x from the jet. Consequently, the area of the jet increases as x^2. Experiments show that the average flow velocity decreases as $1/x$. Therefore, the kinetic energy flux E through an area of the jet a distance x from the nozzle is proportional to x^2u^3. Since $u \approx 1/x$, we then have $E \approx 1/x$, and the loss of energy per unit length becomes $|dE/dx| \approx 1/x^2$. A certain fraction of this mechanical energy loss is converted into sound, and because the efficiency η of generation is proportional to the fifth power of the velocity it follows that the sound power output per unit length of the jet is $W \approx \eta\,|dE/dx| \approx u^5\,|dE/dx| = x^{-5}x^{-2} = x^{-7}$, which is Ribner's result.

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References

1. H. S. Ribner, On the strength distribution of noise sources along a jet, Report 51, Institute of Aerophysics, University of Toronto, 1958.

D. ACOUSTIC AMPLIFIER

The nonlinearities of the acoustic wave equation can manifest themselves in a great number of ways. A study is being made to determine the systematic behavior of some of these manifestations of nonlinearity. In particular, we consider the equation

$$\frac{\partial^2\rho}{\partial t^2} + \frac{\omega_o}{Q}\frac{\partial\rho}{\partial t} - c_o^2\nabla^2\rho = \frac{\partial^2\left(\rho u^2 + \frac{\rho^2}{2}\left[\frac{d^2\rho}{d\rho^2}\right]_{\rho=\rho_o}\right)}{\partial x^2} = f(\rho) \tag{1}$$

(The damping term $(\omega_o/Q\cdot\partial\rho/\partial t)$ is strictly phenomenological and includes effects of viscosity, and so forth, so that Q is frequency-dependent.) Considering a resonant system with frequency ω_o and $Q \gg 1$, we see that Eq. 1 will show instability if $f(\rho)$ has the same time phase as $\partial\rho/\partial t$ and

$$f(\rho) > \frac{\omega_o}{Q}\frac{\partial\rho}{\partial t} \tag{2}$$

In other words, the driving term will be greater than the damping term.

One interesting example of this situation is an acoustic parametric amplifier. We

consider three sound frequencies ω_1, ω_2, ω_3 ($\omega_1 + \omega_2 = \omega_3$) and a cavity tuned at both ω_1 and ω_2. If three sound fields, ρ_1 at ω_1, ρ_2 at ω_2, and ρ_3 at ω_3, are considered as satisfying the conditions $|\rho_3| \gg |\rho_1|$, $|\rho_2|$, then ρ_3 can be considered as given, and Eq. 1 for ρ_1 and ρ_2 becomes

$$\begin{aligned} \frac{\partial^2 \rho_2}{\partial t^2} + \frac{\omega_2}{Q_2}\frac{\partial \rho_2}{\partial t} + \omega_2^2 \rho_2 &= \rho_o \frac{\partial^2 (u_1 u_3)}{\partial x^2} + \frac{(\gamma-1)\, c_o^2}{2\rho_o}\frac{\partial^2(\rho_1\rho_3)}{\partial x^2} \\ \frac{\partial^2 \rho_1}{\partial t^2} + \frac{\omega_1}{Q_1}\frac{\partial \rho_1}{\partial t} + \omega_1^2 \rho_1 &= \rho_o \frac{\partial^2 (u_2 u_3)}{\partial x^2} + \frac{(\gamma-1)\, c_o^2}{2\rho_o}\frac{\partial^2(\rho_2\rho_3)}{\partial x^2} \end{aligned} \tag{3}$$

(Note that the terms on the right-hand side contain both sum and difference frequencies, but only one frequency will drive the resonance on the left-hand side. The equations have been taken as one-dimensional to make them clearer.) If we find solutions with the terms on the right-hand side of Eq. 3 in phase with the damping terms on the left-hand side, it is immediately apparent that the condition for instability of ρ_1 in the presence of ρ_3, with $|\rho_3| \gg |\rho_1|$, is

$$\frac{|\rho_3|^2}{\rho_o^2} \geqslant \frac{64}{Q_1 Q_2 (\gamma+1)^2} \tag{4}$$

This means that such a device will act as an oscillator that produces sound of frequency ω_1 and ω_2, provided that sound of frequency ω_3 is pumped in.

The cavity can be analyzed as a circuit element coupled to waveguides. The ordinary microwave methods and terminology can be used, and this analysis shows that for values of ρ_3 which do not satisfy Eq. 4 the device acts as an amplifier with gain given by

$$G = \frac{4}{Q_o Q_e \left(\frac{1}{Q_1} - \frac{Q_2\, \rho_3^2 (\gamma+1)^2}{64\rho_o^2}\right)^2}$$

where Q_o is the exit Q for ρ_1, and Q_e is the entrance Q for ρ_1.

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