IX. DETECTION AND ESTIMATION THEORY*

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A. DETECTION OF SIGNALS TRANSMITTED OVER DOUBLY SPREAD CHANNELS

State-variable techniques have provided solutions to several problems in communication theory.[1-3] In particular, the optimum receiver for the detection of Gaussian signals in Gaussian noise can be realized and its performance conveniently analyzed when the signal is modeled as the output of a finite-state system driven by white Gaussian noise.[2] When a known waveform is transmitted over the Gaussian dispersive channel that is commonly classified as delay-spread, Doppler-spread, or doubly spread, a lumped-parameter state-variable model can be specified for the Doppler-spread case.[2] The concept of time and frequency duality relates the performance of the delay-spread and the Doppler-spread models.[2] For the doubly spread channel an exact finite-state representation is not possible.

This report considers the detection problem when a distributed parameter state-variable model is used for the doubly spread channel. First, we shall present the model and specialize it to the wide-sense stationary uncorrelated scatter (WSSUS) channel case. Second, we shall review the detection problem and obtain a realizable estimator for the distributed-parameter state variables. Third, we shall outline the derivation of a differential equation for the covariance of the estimation error. Finally, we shall compare the distributed-parameter model with a tapped delay line model of the doubly spread channel. Complex distributed-parameter state variables are used throughout.

1. Distributed-Parameter State-Variable Model for the Doubly Spread Channel

For the detection problem considered in this report the narrow-band transmitted signal can be expressed as[1]

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$$f(t) = \begin{cases} \sqrt{2}\ \mathrm{Re}\left[\tilde{f}(t)\, e^{j\omega_c t}\right] & 0 \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

The complex envelope of the received signal is

$$\tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t) \qquad T_i \leq t \leq T_f, \tag{2}$$

where w(t) is bandpass white noise with

$$E[\tilde{w}(t)\tilde{w}^*(u)] = N_o \delta(t-u). \tag{3}$$

(The star denotes conjugation.) For the doubly spread channel the signal component in (2) is given by[2]

$$\tilde{s}(t) = \int_{-\infty}^{\infty} \tilde{f}(t-x)\, \tilde{Y}(x,t)\, dx. \tag{4}$$

The complex distributed parameter Gaussian process $\tilde{Y}(x,t)$ represents the effect of the doubly spread channel on the transmitted signal.

The distributed parameter state-variable model for $\tilde{Y}(x,t)$ presented here is a special case of the model given by Tzafestas and Nightingale,[4] with the complex formulation added according to Van Trees and co-workers.[1] It is

$$\begin{aligned} \frac{\partial \underline{\tilde{X}}(x,t)}{\partial t} &= \underline{\tilde{F}}(x,t)\, \underline{\tilde{X}}(x,t) + \underline{\tilde{G}}(x,t)\, \underline{\tilde{U}}(x,t) \\ \tilde{Y}(x,t) &= \underline{\tilde{C}}(x,t)\, \underline{\tilde{X}}(x,t), \end{aligned} \tag{5}$$

where $\underline{\tilde{X}}(x,t)$ is the n-dimensional distributed state vector, $\underline{\tilde{F}}(x,t)$, $\underline{\tilde{G}}(x,t)$ and $\underline{\tilde{C}}(x,t)$ are known gain matrices, and $\underline{\tilde{U}}(x,t)$ is the p-dimensional, temporally white, Gaussian noise input

$$\begin{aligned} &E[\underline{\tilde{U}}(x,t)\, \underline{\tilde{U}}^{\dagger}(y,\tau)] = \underline{\tilde{Q}}(x,y,t)\, \delta(t-\tau) \\ &E[\underline{\tilde{U}}(x,t)\, \underline{\tilde{U}}^{T}(y,\tau)] = \underline{0} \\ &E[\underline{\tilde{U}}(x,t)] = \underline{0}, \end{aligned} \tag{6}$$

where superscript T denotes transpose, and superscript dagger denotes conjugate transpose. The state at time t can be written

$$\underline{\tilde{X}}(x,t) = \underline{\tilde{\Psi}}(x,t,t_o)\, \underline{\tilde{X}}(x,t_o) + \int_{t_o}^{t} \underline{\tilde{\Psi}}(x,t,\tau)\, \underline{\tilde{G}}(x,\tau)\, \underline{\tilde{U}}(x,\tau)\, d\tau, \qquad t > t_o, \tag{7}$$

where $\tilde{\Psi}(x,t,\tau)$ is the solution to

$$\frac{\partial \underline{\tilde{\Psi}}(x,t,\tau)}{\partial t} = \underline{\tilde{F}}(x,t)\, \underline{\tilde{\Psi}}(x,t,\tau)$$
$$\underline{\tilde{\Psi}}(x,t,t) = \underline{I} \tag{8}$$

The covariance matrix of the state vector is

$$E[\underline{\tilde{X}}(x,t)\underline{\tilde{X}}^{\dagger}(y,\tau)] = \underline{\tilde{K}}_x(x,t;y,\tau). \tag{9}$$

From (7), it can be written

$$\underline{\tilde{K}}_x(x,t;y,\tau) = \begin{cases} \underline{\tilde{\Psi}}(x,t,\tau)\, \underline{\tilde{K}}_x(x,\tau;y,\tau) & t > \tau \\ \underline{\tilde{K}}_x(x,t;y,t)\, \underline{\tilde{\Psi}}^{\dagger}(y,\tau,t) & t < \tau. \end{cases} \tag{10}$$

Note that

$$\underline{\tilde{K}}_x^{\dagger}(x,t;y,\tau) = \underline{\tilde{K}}_x(y,\tau;x,t). \tag{11}$$

Now

$$\frac{\partial \underline{\tilde{K}}_x(x,t;y,t)}{\partial t} = E\left[\frac{\partial \underline{\tilde{X}}(x,t)}{\partial t}\, \underline{\tilde{X}}^{\dagger}(y,t)\right] + E\left[\underline{\tilde{X}}(x,t)\, \frac{\partial \underline{\tilde{X}}^{\dagger}(y,t)}{\partial t}\right] \tag{12}$$

and, from (7),

$$E[\underline{\tilde{X}}(x,t)\underline{\tilde{U}}^{\dagger}(y,t)] = \frac{1}{2}\, \underline{\tilde{G}}(x,t)\, \underline{\tilde{Q}}(x,y,t). \tag{13}$$

Substitution of (5) in (12) plus the relation (13) gives the differential equation

$$\frac{\partial \underline{\tilde{K}}_x(x,t;y,t)}{\partial t} = \underline{\tilde{F}}(x,t)\, \underline{\tilde{K}}_x(x,t;y,t) + \underline{\tilde{K}}_x(x,t;y,t)\, \underline{\tilde{F}}^{\dagger}(y,t) + \underline{\tilde{G}}(x,t)\, \underline{\tilde{Q}}(x,y,t)\, \underline{\tilde{G}}^{\dagger}(y,t) \tag{14}$$

with

$$\underline{\tilde{K}}_x(x,T_i;y,T_i) = \underline{\tilde{P}}_i(x,y). \tag{15}$$

From the assumption that

$$E\left[\underline{\tilde{X}}(x,T_i)\underline{\tilde{X}}^{T}(y,T_i)\right] = \underline{0} \tag{16}$$

it follows by an argument similar to that of Van Trees et al.[1] that

$$E[\tilde{\underline{X}}(x,t)\tilde{\underline{X}}^T(y,\tau)] = 0 \tag{17}$$

for all x, y, t, and τ. From (5),

$$\begin{aligned}\tilde{K}_Y(x,t;y,\tau) &= E[\tilde{Y}(x,t)\tilde{Y}^*(y,\tau)] \\ &= \tilde{\underline{C}}(x,t)\,\tilde{\underline{K}}_x(x,t;y,\tau)\,\tilde{\underline{C}}^\dagger(y,\tau) \\ E[\tilde{Y}(x,t)\tilde{Y}(y,\tau)] &= 0.\end{aligned} \tag{18}$$

The distributed-parameter state-variable model for the doubly spread channel is specified by (5) and (6), and the covariance matrix relationships are given by (9), (10), and (14-18). The special case of a WSSUS channel[4,5] occurs when $\tilde{K}_Y(x,t;y,\tau)$ can be written

$$\tilde{K}_Y(x,t;y,\tau) = \tilde{K}_D(x,t-\tau)\,\delta(x-y). \tag{19}$$

That is, $\tilde{Y}(x,t)$ is spatially white and temporally stationary. From (18), this condition is satisfied if $\underline{C}(x,t)$ is only a function of x and if

$$\tilde{\underline{K}}_x(x,t;y,\tau) = \tilde{\underline{K}}(x,t-\tau)\,\delta(x-y). \tag{20}$$

For (20) to hold, inspection of (14) indicates that $\tilde{\underline{F}}(x,t)$, $\tilde{\underline{G}}(x,t)$, and $\tilde{\underline{Q}}(x,y,t)$ are constant with respect to t, and furthermore

$$\tilde{\underline{Q}}(x,y,t) = \tilde{\underline{Q}}(x)\,\delta(x-y). \tag{21}$$

Thus the WSSUS state-variable model is

$$\begin{aligned}\frac{\partial\tilde{\underline{X}}(x,t)}{\partial t} &= \tilde{\underline{F}}(x)\,\tilde{\underline{X}}(x,t) + \tilde{\underline{G}}(x)\,\tilde{\underline{U}}(x,t) \\ \tilde{Y}(x,t) &= \tilde{\underline{C}}(x)\,\tilde{\underline{X}}(x,t)\end{aligned} \tag{22}$$

$$\begin{aligned}E[\tilde{\underline{U}}(x,t)\tilde{\underline{U}}^\dagger(y,\tau)] &= \tilde{\underline{Q}}(x)\,\delta(x-y)\,\delta(t-\tau) \\ E[\tilde{\underline{U}}(x,t)\tilde{\underline{U}}^T(y,\tau)] &= \underline{0}.\end{aligned} \tag{23}$$

The covariance matrices for the WSSUS model follow directly from the more general case above. From (19), (20), and (22),

$$\tilde{K}_D(x,\tau) = \tilde{\underline{C}}(x)\,\tilde{\underline{K}}(x,\tau)\,\tilde{\underline{C}}^\dagger(x). \tag{24}$$

From (8) and (10),

$$\underline{\tilde{K}}(x,\tau) = \begin{cases} \underline{\tilde{\theta}}(x,\tau)\,\underline{\tilde{K}}_o(x) & \tau > 0 \\ \underline{\tilde{K}}_o(x)\,\underline{\tilde{\theta}}^{\dagger}(x,-\tau) & \tau < 0 \end{cases} \tag{25}$$

where $\underline{\tilde{\theta}}(x,\tau)$ is the solution to

$$\frac{\partial\,\underline{\tilde{\theta}}(x,t)}{\partial t} = \underline{\tilde{F}}(x)\,\underline{\tilde{\theta}}(x,t)$$
$$\underline{\tilde{\theta}}(x,0) = \underline{I} \tag{26}$$

The matrix $\underline{\tilde{K}}_o(x)$ is the steady-state solution of (14)

$$\underline{0} = \underline{\tilde{F}}(x)\,\underline{\tilde{K}}_o(x) + \underline{\tilde{K}}_o(x)\,\underline{\tilde{F}}^{\dagger}(x) + \underline{\tilde{G}}(x)\,\underline{\tilde{Q}}(x)\,\underline{\tilde{G}}^{\dagger}(x). \tag{27}$$

The scattering function for the WSSUS channel is defined as

$$\tilde{S}(x,f) = a\int_{-\infty}^{\infty} \tilde{K}_D(x,\tau)\,e^{-j2\pi f\tau}\,d\tau, \tag{28}$$

where the constant a normalizes $\tilde{S}(x,f)$ to unit volume. $\tilde{S}(x,f)$ is positive and real for all x and f, since $\underline{\tilde{Q}}(x)$ is Hermitian with a non-negative definite real part.

Example: Consider a first-order WSSUS model.

Then

$$\underline{\tilde{F}}(x) = -\tilde{k}(x) = -k_r(x) - jk_i(x)$$
$$\underline{\tilde{G}}(x) = \underline{\tilde{C}}(x) = 1 \tag{29}$$
$$\underline{\tilde{Q}}(x) = Q(x)$$

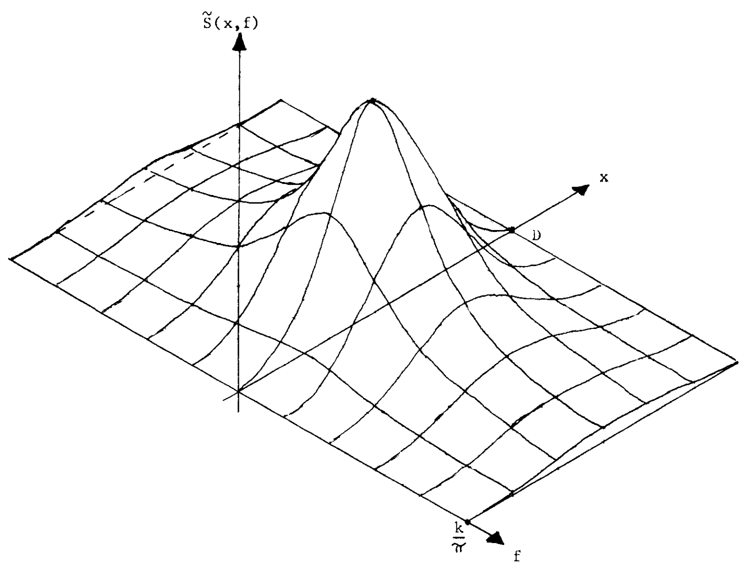
with

$$Q(x) \geqslant 0$$
$$k_r(x) > 0 \tag{30}$$

From (26) and (27)

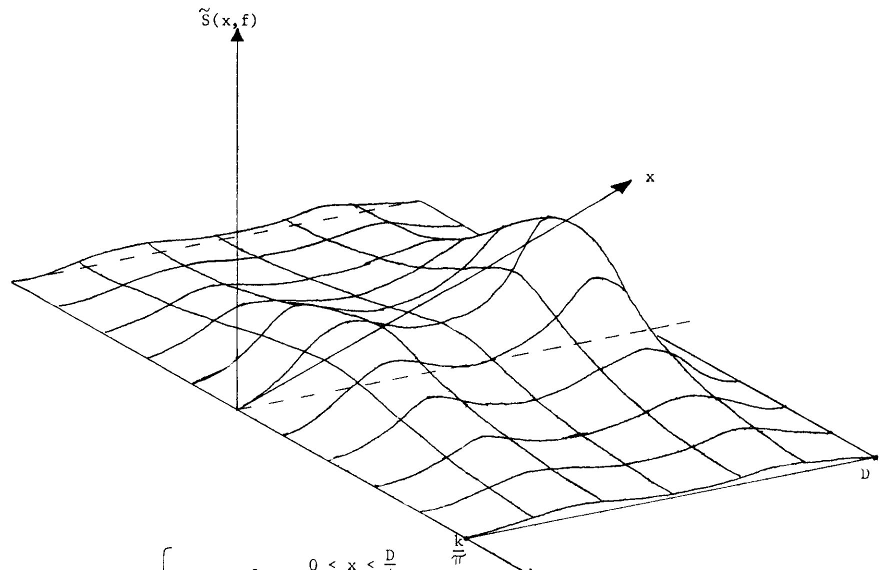
$$\tilde{\theta}(x,\tau) = \exp[-k_r(x)|\tau| - jk_i(x)\tau] \tag{31}$$

$$\tilde{K}_o(x) = \frac{Q(x)}{2k_r(x)} \tag{32}$$



(a) $\tilde{Q}(x) = \begin{cases} 1 - \cos\frac{2\pi x}{D} & 0 \le x \le D \\ 0 & \text{elsewhere} \end{cases}$

$\tilde{k}(x) = k(1 - \frac{1}{2}\sin\frac{\pi x}{D})$



(b) $\tilde{Q}(x) = \begin{cases} 1 - \cos\frac{2\pi x}{D} & 0 \le x \le \frac{D}{4} \\ & \frac{3D}{4} \le x \le D \\ 2 + \cos\frac{\pi x}{D} & \frac{D}{4} \le x \le \frac{3D}{4} \\ 0 & \text{elsewhere} \end{cases}$

$\tilde{k}(x) = k(1 - \frac{x}{2D}) - j\,\frac{3kx}{5\pi D}$

Fig. IX-1. Examples of scattering functions associated with a first-order model.

Thus

$$K_D(x,\tau) = \frac{Q(x)}{2k_r(x)} \exp[-k_r(x)|\tau|-jk_i(x)\tau] \tag{33}$$

$$\tilde{S}(x,f) = \frac{aQ(x)}{(2\pi f+k_i(x))^2 + k_r^2(x)} \tag{34}$$

where

$$a^{-1} = \int_{-\infty}^{\infty} \frac{Q(x)}{2k_r(x)} dx \tag{35}$$

The scattering function in (34), considered as a function of frequency at any value of x, is a one-pole spectrum centered at $f=-k_i(x)/2\pi$ with a peak value $aQ(x)/k_r^2(x)$ and 3 dB points $\pm k_r(x)/2\pi$ about the center frequency. Except for the constraints of (30), $Q(x)$ and $\tilde{K}(x)$ are arbitrary. This permits considerable flexibility in the choice of $\tilde{S}(x,f)$, even for this first order model. For instance, if $k_i(x)$ is proportional to x, then $\tilde{S}(x,f)$ is sheared in the x-f plane. Also, $Q(x)$ can be chosen so that $\tilde{S}(x,f)$ is multimodal in the x direction. Figure IX-1 shows several plots of possible $\tilde{S}(x,f)$.

The example above indicates that the class of scattering functions which can be described by the model of (22) are those for which $\tilde{S}(x,f)$ is a rational function in f. The poles and zeros of this particular function may depend on x in an arbitrary manner, except for conditions such as those of (30). Thus higher order distributed-state models permit more degrees of freedom in the specification of the scattering function. For example, a $\tilde{S}(x,f)$ which exhibits multimodal behavior in f can be obtained from a second or higher order state model.

2. A Realizable Optimum Detector

We shall now consider the simple binary detection of a signal transmitted over a Gaussian doubly spread channel and received in white Gaussian noise. In complex notation the two hypotheses are

$$\begin{aligned} H_1&: \tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t) \\ H_2&: \tilde{r}(t) = \tilde{w}(t), \end{aligned} \tag{36}$$

with $\tilde{s}(t)$ given by (4), and the noise covariance by (3).

The optimum detector for a wide class of criteria compares the likelihood ratio with a threshold. One way to realize this detector is to compare the statistic[1]

$$\ell = \frac{1}{2N_o} \int_{T_i}^{T_f} \left\{ -|\hat{\tilde{s}}(t)|^2 + 2\,\mathrm{Re}\,[\hat{\tilde{s}}(t)\tilde{r}^*(t)] - \tilde{\xi}_P(t) \right\} dt \tag{37}$$

with a threshold. The waveform $\hat{\tilde{s}}(t)$ is the minimum-mean-square-error realizable estimate of $\tilde{s}(t)$ in (2), and $\tilde{\xi}_P(t)$ is the filtering error

$$\tilde{\xi}_P(t) = E[|\tilde{s}(t) - \hat{\tilde{s}}(t)|^2]. \tag{38}$$

Thus if the realizable MMSE estimator for $\tilde{s}(t)$ can be found when the doubly spread channel model is used, the optimum detector of (37) can be realized.

In order to obtain the MMSE estimate of $\tilde{s}(t)$, the MMSE estimate of $\underline{\tilde{X}}(x,t)$ will be derived first. This estimate is the result of the linear operation

$$\hat{\underline{\tilde{X}}}(x,t) = \int_{T_i}^{t} \underline{\tilde{h}}_o(x,t,\sigma)\,\underline{\tilde{r}}(\sigma)\,d\sigma \qquad t > T_i, \tag{39}$$

where $\underline{h}_o(x,t,\tau)$ is the $n \times 1$ matrix impulse response that minimizes the error

$$\underline{\tilde{\xi}}(x,y,t) = E\{[\underline{\tilde{X}}(x,t) - \hat{\underline{\tilde{X}}}(x,t)][\underline{\tilde{X}}(y,t) - \hat{\underline{\tilde{X}}}(y,t)]^\dagger\} \tag{40}$$

for all x and y. The MMSE estimate of $\tilde{s}(t)$ is then

$$\hat{\tilde{s}}(t) = \int_{-\infty}^{\infty} \tilde{f}(t-\sigma)\,\underline{\tilde{C}}(\sigma,t)\,\hat{\underline{\tilde{X}}}(\sigma,t)\,d\sigma, \tag{41}$$

with

$$\tilde{\xi}_p(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(t-\sigma)\,\underline{\tilde{C}}(\sigma,t)\,\underline{\tilde{\xi}}(\sigma,a,t)\,\underline{\tilde{C}}^\dagger(a,t)\,\tilde{f}^*(t-a)\,d\sigma da. \tag{42}$$

The derivation of the realizable MMSE estimator for the distributed state vector $\underline{\tilde{X}}(x,t)$ parallels that of Van Trees[1,3] for the Kalman-Bucy filter. The result is a modification of the estimator obtained by Tzafestas.[4]

The starting point of the derivation is the generalized Wiener-Hopf equation[2] in complex notation:

$$\int_{-\infty}^{\infty} \underline{\tilde{K}}_x(x,t;\sigma,\tau)\,\underline{\tilde{C}}^\dagger(\tau,\sigma)\,\tilde{f}^*(\tau-\sigma)\,d\sigma = \int_{T_i}^{t} \underline{\tilde{h}}_o(x,t,\sigma)\,\tilde{K}_r(\sigma,\tau)\,d\sigma \qquad T_i < \tau < t. \tag{43}$$

The left-hand side of (43) is $E[\underline{\tilde{X}}(x,t)\tilde{r}^*(\tau)]$, and

$$\tilde{K}_r(\sigma, \tau) = E[\tilde{r}(\sigma)\tilde{r}^*(\tau)]. \tag{44}$$

Differentiating (43) with respect to t gives

$$\int_{-\infty}^{\infty} \frac{\partial \underline{\tilde{K}}_x(x, t; \sigma, \tau)}{\partial t} \underline{\tilde{C}}^{\dagger}(\sigma, \tau) \tilde{f}^*(\tau-\sigma)\, d\sigma = \underline{\tilde{h}}_o(x, t, t)\, \tilde{K}_r(t, \tau) + \int_{T_i}^{t} \frac{\partial \underline{\tilde{h}}_o(x, t, \sigma)}{\partial t} \tilde{K}_r(\sigma, \tau)\, d\sigma$$

$$T_i < \tau < t. \tag{45}$$

From (5) and (6),

$$E\left\{\frac{\partial \underline{\tilde{X}}(x, t)}{\partial t} \underline{\tilde{X}}^{\dagger}(\sigma, \tau)\right\} = \underline{\tilde{F}}(x, t)\, \underline{\tilde{K}}_x(x, t; \sigma, \tau) \qquad \tau < t. \tag{46}$$

Then the left-hand side of (45), with the relation of (43), becomes

$$\int_{-\infty}^{\infty} \underline{\tilde{F}}(x, t)\, \underline{\tilde{K}}_x(x, t; \sigma, \tau)\, \underline{\tilde{C}}^{\dagger}(\sigma, \tau)\, \tilde{f}^*(\tau-\sigma)\, d\sigma = \int_{T_i}^{t} \underline{\tilde{F}}(x, t)\, \underline{\tilde{h}}_o(x, t, \sigma)\, \underline{\tilde{K}}_r(\sigma, \tau)\, d\sigma$$

$$T_i < \tau < t. \tag{47}$$

From (5) and (43), with $T_i < \tau < t$, the first term on the right-hand side of (45) is

$$\begin{aligned} &\underline{\tilde{h}}_o(x, t, t)\, \tilde{K}_r(t, \tau) \\ &= \underline{\tilde{h}}_o(x, t, t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(t-\alpha)\, \underline{\tilde{C}}(\alpha, t)\, \underline{\tilde{K}}_x(\alpha, t; \sigma, \tau)\, \underline{\tilde{C}}^{\dagger}(\sigma, \tau)\, \tilde{f}^*(\tau-\alpha)\, d\sigma d\alpha \\ &= \int_{T_i}^{t} \int_{-\infty}^{\infty} \underline{\tilde{h}}_o(x, t, t)\, \tilde{f}(t-\alpha)\, \underline{\tilde{C}}(\alpha, t)\, \underline{\tilde{h}}_o(\alpha, t, \sigma)\, \tilde{K}_r(\sigma, \tau)\, d\alpha d\sigma. \end{aligned} \tag{48}$$

Substitution of (47) and (48) in (45) yields

$$\frac{\partial \underline{\tilde{h}}_o(x, t, \sigma)}{\partial t} = \underline{\tilde{F}}(x, t)\, \underline{\tilde{h}}_o(x, t, \sigma) + \int_{-\infty}^{\infty} \underline{\tilde{h}}_o(x, t, t)\, \tilde{f}(t-\alpha)\, \underline{\tilde{C}}(\alpha, t)\, \underline{\tilde{h}}_o(\alpha, t, \sigma)\, d\alpha. \tag{49}$$

Differentiation of (39) with respect to t and substitution of (49) give the differential equation

$$\frac{\partial \hat{\tilde{\underline{X}}}(x,t)}{\partial t} = \tilde{\underline{F}}(x,t)\, \hat{\tilde{\underline{X}}}(x,t) + \tilde{\underline{h}}_o(x,t,t)[\tilde{r}(t)-\hat{\tilde{s}}(t)]$$

$$\hat{\tilde{s}}(t) = \int_{-\infty}^{\infty} \tilde{f}(t-\sigma)\, \tilde{\underline{C}}(\sigma,t)\, \hat{\tilde{\underline{X}}}(\sigma,t)\, d\sigma. \tag{50}$$

The MMSE estimate of $\tilde{\underline{X}}(x,t)$ is the solution of the distributed parameter differential equation (50). The initial conditions are

$$\hat{\tilde{\underline{X}}}(x,T_i) = E[\tilde{\underline{X}}(x,T_i)] = \underline{0}. \tag{51}$$

The homogeneous system in (50) is just that of the model for the generation of $\tilde{\underline{X}}(x,t)$. The driving term in (50) is a scalar, $\tilde{r}(t) - \hat{\tilde{s}}(t)$, multiplied by a vector gain $\tilde{\underline{h}}_o(x,t,t)$. The next section shows that $\tilde{\underline{h}}_o(x,t,t)$ does not depend on $\tilde{r}(t)$. Thus the distributed estimator can be diagrammed as shown in Fig. IX-2. For the special case of the WSSUS model, $\tilde{\underline{F}}(x,t)$ in (50) is replaced by $\tilde{\underline{F}}(x)$.

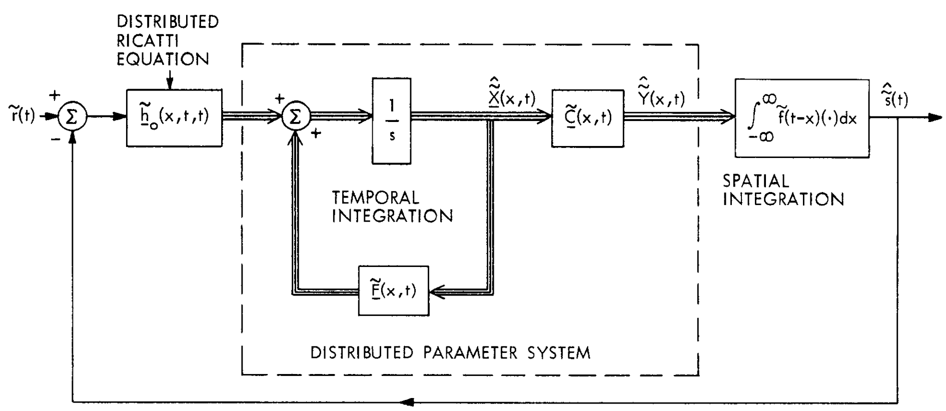


Fig. IX-2. Realizable MMSE estimator for distributed-parameter state-variable model.

3. An Equation for the Covariance Matrix

We shall relate the gain $\tilde{\underline{h}}_o(x,t,t)$ of (50) to the error covariance matrix $\tilde{\underline{\xi}}(x,y,t)$ of (40). Then a differential equation for $\tilde{\underline{\xi}}(x,y,t)$ is obtained. The derivation follows Van Trees,[3] with appropriate modifications to account for the complex distributed-state model.[1,2]

The optimum filter $\tilde{\underline{h}}_o(x,t,\tau)$ satisfies the integral equation

$$\int_{-\infty}^{\infty} \tilde{\underline{K}}_x(x,t;\sigma,t)\, \tilde{\underline{C}}^{\dagger}(\sigma,t)\, \tilde{f}^*(t-\sigma)\, d\sigma = N_o \tilde{\underline{h}}_o(x,t,t)$$

$$+ \int_{T_i}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\underline{h}}_o(x,t,\sigma)\, \tilde{f}(\sigma-a)\, \tilde{\underline{C}}(a)\, \tilde{\underline{K}}_x(a,\sigma;\beta,t)\, \tilde{\underline{C}}^{\dagger}(\beta)\, \tilde{f}^*(t-\beta)\, da\, d\beta\, d\sigma. \tag{52}$$

(This is (43) for $\tau = t$.[3]) From (40), (39), and (43),

$$\tilde{\underline{\xi}}(x,\beta,t) = E\left\{\left[\tilde{\underline{X}}(x,t) - \int_{T_i}^{t} \tilde{\underline{h}}_o(x,t,\tau)\, \tilde{r}(\tau)\, d\tau\right]\left[\tilde{\underline{X}}(\beta,t) - \int_{T_i}^{t} \tilde{\underline{h}}_o(\beta,t,\tau)\, \tilde{r}(\tau)\, d\tau\right]^{\dagger}\right\}$$

$$= \tilde{\underline{K}}_x(x,t;\beta,t) - \int_{T_i}^{t} \int_{-\infty}^{\infty} \tilde{\underline{h}}_o(x,t,\sigma)\, \tilde{f}(\sigma-a)\, \tilde{\underline{C}}(a)\, \tilde{\underline{K}}_x(a,\sigma;\beta,t)\, da\, d\sigma. \tag{53}$$

Postmultiplying (53) by $\tilde{\underline{C}}^{\dagger}(\beta)\tilde{\underline{f}}^*(t-\beta)$, integrating over β, and combining the result with (52) gives

$$\tilde{\underline{h}}_o(x,t,t) = \frac{1}{N_o} \int_{-\infty}^{\infty} \tilde{\underline{\xi}}(x,\sigma,t)\, \tilde{\underline{C}}^{\dagger}(\sigma,t)\, \tilde{f}^*(t-\sigma)\, d\sigma. \tag{54}$$

This specifies the gain $\tilde{\underline{h}}_o(x,t,t)$ in terms of the error covariance matrix.

The first step in obtaining a differential equation for $\tilde{\underline{\xi}}(x,y,t)$ is to recognize from (5) and (50) that the error

$$\tilde{\underline{X}}_{\epsilon}(x,t) = \tilde{\underline{X}}(x,t) - \hat{\tilde{\underline{X}}}(x,t) \tag{55}$$

satisfies the differential equation

$$\frac{\partial \hat{\tilde{\underline{X}}}(x,t)}{\partial t} = \tilde{\underline{F}}(x,t)\, \tilde{\underline{X}}_{\epsilon}(x,t) - \tilde{\underline{h}}_o(x,t,t) \int_{-\infty}^{\infty} \tilde{f}(t-\sigma)\, \tilde{\underline{C}}(\sigma,t)\, \tilde{\underline{X}}_{\epsilon}(\tau,t)\, d\tau$$

$$+ \tilde{\underline{G}}(x,t)\, \tilde{\underline{U}}(x,t) - \tilde{\underline{h}}_o(x,t,t)\, \tilde{w}(t). \tag{56}$$

Now

$$\frac{\partial \tilde{\underline{\xi}}(x,y,t)}{\partial t} = E\left[\frac{\partial \tilde{\underline{X}}_{\epsilon}(x,t)}{\partial t}\, \tilde{\underline{X}}_{\epsilon}^{\dagger}(y,t)\right] + E\left[\tilde{\underline{X}}_{\epsilon}(x,t)\, \frac{\partial \tilde{\underline{X}}_{\epsilon}^{\dagger}(y,t)}{\partial t}\right]. \tag{57}$$

From (56) and (13), the first term in (57) is

$$E\left[\frac{\partial \underline{\tilde{X}}_\epsilon(x,t)}{\partial t} \underline{\tilde{X}}_\epsilon^\dagger(y,t)\right] = \underline{\tilde{F}}(x,t)\, \underline{\tilde{\xi}}(x,y,t) - \underline{\tilde{h}}(x,t,t) \int_{-\infty}^{\infty} \tilde{f}(t-\sigma)\, \underline{\tilde{C}}(\sigma,t)\, \underline{\tilde{\xi}}(\sigma,y,t)\, d\sigma$$

$$+\frac{1}{2} \underline{\tilde{G}}(x,t)\, \underline{\tilde{Q}}(x,y,t)\, \underline{\tilde{G}}^\dagger(y,t) + \underline{\tilde{h}}_o(x,t,t)\, \frac{N_o}{2}\, \underline{\tilde{h}}_o^\dagger(y,t,t) \tag{58}$$

Evaluating the other term in (57) in like manner gives the distributed differential equation for the error covariance

$$\frac{\partial \underline{\tilde{\xi}}(x,y,t)}{\partial t} = \underline{\tilde{F}}(x,t)\, \underline{\tilde{\xi}}(x,y,t) + \underline{\tilde{\xi}}(x,y,t)\, \underline{\tilde{F}}^\dagger(y,t) + \underline{\tilde{G}}(x,t)\, \underline{\tilde{Q}}(x,y,t)\, \underline{\tilde{G}}^\dagger(y,t)$$

$$-\frac{1}{N_o} \int_{-\infty}^{\infty} \underline{\tilde{\xi}}(x,\sigma,t)\, \underline{\tilde{C}}^\dagger(\sigma,t)\, \tilde{f}^\dagger(t-\sigma)\, d\sigma \int_{-\infty}^{\infty} \tilde{f}(t-\sigma)\, \underline{\tilde{C}}(\alpha,t)\, \underline{\tilde{\xi}}(\alpha,y,t)\, d\alpha. \tag{59}$$

The initial condition for (59) is

$$\begin{aligned} \underline{\xi}(x,y,T_i) &= \underline{\tilde{K}}_x(x,T_i;y,T_i) \\ &= \underline{\tilde{P}}_i(x,y). \end{aligned} \tag{60}$$

From (54) and (59), it is evident that $\underline{\tilde{h}}_o(x,t,t)$ does not depend on $\tilde{r}(t)$. Furthermore, $\underline{\tilde{h}}_o(x,t,t)$ can be computed in principle from (54) and (59), since the right-hand side of (59) at any time t' depends only on $\underline{\tilde{\xi}}(x,y,t')$ and the known matrices of the model. Given $\underline{\tilde{h}}_o(x,y,t)$, the filtering error $\tilde{\xi}_p(t)$ of (42) and its integral over the observation interval can be found. The latter quantity is useful in evaluating bounds on and asymptotic expressions for the detection error probabilities.[2]

For the special case of the WSSUS channel model, (54), (59), and (60) reduce to

$$\underline{\tilde{h}}_o(x,t,t) = \frac{1}{N_o} \int_{-\infty}^{\infty} \underline{\tilde{\xi}}(x,\sigma,t)\, \underline{\tilde{C}}^\dagger(\sigma)\, \tilde{f}^*(t-\sigma)\, d\sigma \tag{61}$$

$$\frac{\partial \underline{\tilde{\xi}}(x,y,t)}{\partial t} = \underline{\tilde{F}}(x)\, \underline{\tilde{\xi}}(x,y,t) + \underline{\tilde{\xi}}(x,y,t)\, \underline{\tilde{F}}^\dagger(y) + \underline{\tilde{G}}(x)\, \underline{\tilde{Q}}(x)\, \underline{\tilde{G}}^\dagger(x)\, \delta(x-y)$$

$$-\frac{1}{N_o} \int_{-\infty}^{\infty} \underline{\tilde{\xi}}(x,\sigma,t)\, \underline{\tilde{C}}^\dagger(\sigma)\, \tilde{f}^*(t-\sigma)\, d\sigma \int_{-\infty}^{\infty} \tilde{f}(t-\alpha)\, \underline{\tilde{C}}(\alpha)\, \underline{\tilde{\xi}}(\alpha,y,t)\, d\alpha \tag{62}$$

$$\underline{\tilde{\xi}}(x,y,T_i) = \underline{\tilde{K}}_o(x)\, \delta(x-y), \tag{63}$$

where $\underline{\tilde{K}}_o(x)$ is specified by (27). If a solution of the form

$$\tilde{\underline{\xi}}(x,y,t) = \tilde{\underline{\xi}}_i(x,t)\,\delta(x-y) + \tilde{\underline{p}}(x,y,t) \tag{64}$$

is assumed and substituted in (62), application of the initial condition (63) gives

$$\tilde{\underline{\xi}}_i(x,t) = \tilde{\underline{K}}_o(x) \tag{65}$$

$$\tilde{\underline{h}}_o(x,y,t) = \frac{1}{N_o}\,\tilde{\underline{K}}_o(x)\,\tilde{\underline{C}}^{\dagger}(x)\,\tilde{f}^*(t-x) + \frac{1}{N_o}\int_{-\infty}^{\infty} \tilde{\underline{p}}(x,\sigma,t)\,\tilde{\underline{C}}^{\dagger}(\sigma)\,\tilde{f}^*(t-\sigma)\,d\sigma \tag{66}$$

$$\begin{aligned}\frac{\partial\tilde{\underline{p}}(x,y,t)}{\partial t} &= \tilde{\underline{F}}(x)\,\tilde{\underline{p}}(x,y,t) + \tilde{\underline{p}}(x,y,t)\,\tilde{\underline{F}}^{\dagger}(y)\\ &\quad - \frac{1}{N_o}\left[\tilde{\underline{K}}_o(x)\,\tilde{\underline{C}}^{\dagger}(x)\,\tilde{f}^*(t-x) + \int_{-\infty}^{\infty}\tilde{\underline{p}}(x,\sigma,t)\,\tilde{\underline{C}}^{\dagger}(\sigma)\,\tilde{f}^*(t-\sigma)\,d\sigma\right]\\ &\quad \cdot\left[\tilde{f}(t-y)\,\tilde{\underline{C}}(y)\,\tilde{\underline{K}}_o(y) + \int_{-\infty}^{\infty}\tilde{f}(t-\sigma)\,\tilde{\underline{C}}(\sigma)\,\tilde{\underline{p}}(\sigma,y,t)\,d\sigma\right]\end{aligned} \tag{67}$$

$$\tilde{\underline{p}}(x,y,T_i) = \underline{0}. \tag{68}$$

4. Comparison of the Distributed Model with a Tapped Delay Line Model for the WSSUS Channel

We shall now relate the distributed-parameter state-variable model to a tapped delay line model for the WSSUS channel. Various tapped delay line models have been suggested for this channel.[5] One of them is derived by assuming that f(t) is strictly bandlimited to W Hz. Then, from the sampling theorem,[2]

$$\tilde{f}(t-x) = \sum_{i=-\infty}^{\infty} \tilde{f}\left(t - \frac{i}{W_s}\right)\frac{1}{W_s}\,\frac{\sin \pi W_s\left(x - \frac{i}{W_s}\right)}{\pi\left(x - \frac{i}{W_s}\right)}, \tag{69}$$

where $W_s > W$. From (4)

$$\tilde{s}(t) = \sum_{i=-\infty}^{\infty} \tilde{f}\left(t - \frac{i}{W_s}\right)\tilde{y}\left(\frac{i}{W_s}, t\right)\frac{1}{W_s}, \tag{70}$$

where the tap gain processes $\tilde{y}\left(\frac{i}{W_s}, t\right)$ are defined as

$$\tilde{y}\left(\frac{i}{W_s}, t\right) = \int_{-\infty}^{\infty} \tilde{Y}(x,t) \frac{\sin \pi W_s \left(x - \frac{i}{W_s}\right)}{\pi \left(x - \frac{i}{W_s}\right)} dx. \tag{71}$$

From (19), the cross-covariance functions for the $\tilde{y}\left(\frac{i}{W_s}, t\right)$ are

$$E\left[\tilde{y}\left(\frac{i}{W_s}, t\right) \tilde{y}^*\left(\frac{j}{W_s}, \tau\right)\right]$$

$$= \int_{-\infty}^{\infty} \tilde{K}_D(x, t-\tau) \frac{\sin \pi W_s \left(x - \frac{i}{W_s}\right) \sin \pi W_s \left(x - \frac{j}{W_s}\right)}{\pi \left(x - \frac{i}{W_s}\right) \pi \left(x - \frac{j}{W_s}\right)} dx. \tag{72}$$

For large values of W_s, (72) can be written[5]

$$E\left[\tilde{y}\left(\frac{i}{W_s}, t\right) \tilde{y}^*\left(\frac{j}{W_s}, \tau\right)\right] = \begin{cases} W_s \tilde{K}_D\left(\frac{i}{W_s}, t-\tau\right) + 0\left(\frac{1}{\sqrt{W_s}}\right) & i = j \\ 0\left(\frac{1}{\sqrt{W_s}}\right) & i \neq j \end{cases} \tag{73}$$

where $0\left(\frac{1}{\sqrt{W_s}}\right)$ consists of terms that disappear faster than $1/\sqrt{W_s}$.

Equation 70 is a tapped delay line representation of the channel with an infinite number of taps spaced $1/W_s$ sec apart. A realizable approximation to this model is a tapped delay line with a finite number of taps. If $\tilde{K}_D(x, \tau)$ is essentially zero for $D < x < 0$, then (70) will be approximately

$$\tilde{s}(t) \cong \tilde{s}_a(t) = \sum_{i=0}^{L} \tilde{f}\left(t - \frac{i}{W_s}\right) \tilde{y}\left(\frac{i}{W_s}, t\right) \frac{1}{W_s}, \tag{74}$$

where $L = DW_s$.

Van Trees[2] has pointed out that the approximate model of (74) can be described in terms of a lumped-parameter state-variable representation. This is accomplished by letting the $\tilde{y}\left(\frac{i}{W_s}, t\right)$ be components of the vector $\underline{\tilde{y}}(t)$ which is the output of of the system

$$\frac{d\underline{\tilde{x}}(t)}{dt} = \underline{\tilde{F}}_s\underline{\tilde{x}}(t) + \underline{\tilde{G}}_s\underline{\tilde{u}}(t)$$

$$\underline{\tilde{y}}(t) = \underline{\tilde{C}}_s\underline{\tilde{x}}(t) \tag{75}$$

$$E[\underline{\tilde{u}}(t)\underline{\tilde{u}}^{\dagger}(\tau)] = \underline{\tilde{Q}}_s\delta(t-\tau),$$

where $\underline{\tilde{y}}(t)$ is $(L+1) \times 1$, $\underline{x}(t)$ is $N \times 1$, $\underline{\tilde{u}}(t)$ is $P \times 1$, with $N \geq L+1$ and $P \geq L+1$. It is, however, not evident how to pick $\underline{\tilde{C}}_s$, $\underline{\tilde{F}}_s$, $\underline{\tilde{G}}_s$ and $\underline{\tilde{Q}}_s$ to obtain the covariance matrix specified in (72) or under what conditions on $\tilde{K}_D(x,\tau)$ such a choice is possible.

The parameters of the state-variable model of (75) can be found if a further approximation is made. Equation 73 indicates that the tap gains become uncorrelated as $1/W_s$ approaches zero. This suggests modifying the model of (75) so that the $\tilde{y}\left(\frac{i}{W_s}, t\right)$ are uncorrelated for finite W_s. This can be accomplished with the system

$$\frac{d}{dt}\underline{\tilde{x}}_i(t) = \underline{\tilde{F}}_i\underline{\tilde{x}}_i(t) + \underline{\tilde{G}}_i\underline{\tilde{u}}_i(t)$$

$$\tilde{y}\left(\frac{i}{W_s}, t\right) = \underline{\tilde{C}}_i\underline{\tilde{x}}_i(t) \qquad i = 0, \dots, L+1 \tag{76}$$

$$E\left[\underline{\tilde{u}}_i(t)\underline{\tilde{u}}_j^{\dagger}(\tau)\right] = \begin{cases} \underline{\tilde{Q}}_i\delta(t-\tau) & i = j \\ 0 & i \neq j \end{cases}$$

with

$$E\left[\tilde{y}\left(\frac{i}{W_s}, t\right)\tilde{y}^*\left(\frac{i}{W_s}, \tau\right)\right] = W_s\tilde{K}_D\left(\frac{i}{W_s}, t-\tau\right). \tag{77}$$

The vectors $\underline{\tilde{x}}_i(t)$ and $\underline{\tilde{u}}_i(t)$ have dimension N_i and P_i, respectively. Adjoining the $\underline{\tilde{x}}_i(t)$ and $\tilde{y}_i(t)$ gives the composite vectors $\underline{\tilde{x}}(t)$ and $\underline{\tilde{y}}(t)$ in (75), along with $\underline{\tilde{F}}_s$, $\underline{\tilde{C}}_s$, $\underline{\tilde{G}}_s$, and $\underline{\tilde{Q}}_s$. In order to specify these matrices, (77) indicates that the scattering function associated with $\tilde{K}_D(x,\tau)$ should be a rational function of f at $x = i/W_s$.

The lumped-parameter state-variable model of (76-77) is an approximation to the tapped delay line WSSUS model (70-72) under the assumptions of a finite number of taps and uncorrelated tap gains. As the tap gain spacing goes to zero, this model converges to the distributed-parameter state-variable model of the WSSUS channel, provided that each subsystem in (76) is of the same dimension, $N_i = n$. For, if $x = i/W_s$ and $W_s \to \infty$ (hence $L \to \infty$), then $\tilde{y}\left(\frac{i}{W_s}, t\right) \to \tilde{Y}(x,t)$, $\underline{\tilde{x}}_i(t) \to \underline{\tilde{X}}(x,t)$, and the sum in (70) becomes the integral of (4). The differential equations for $\underline{\tilde{x}}_i(t)$ are replaced by the distributed equation (22). The estimator and error covariance differential equations[2] associated with (76-77) converge to those for the distributed model, (50) and (62-63). The spatial impulse associated with $\underline{\tilde{Q}}(x)$ in (62) comes from the limit of (77).

5. Conclusion

We have presented a distributed-parameter state-variable model for a doubly spread Gaussian channel, and discussed the special case of the WSSUS channel.

We have outlined the derivation of the realizable MMSE estimator for the distributed-state vector of the channel when a known narrow-band signal is transmitted over the channel and received in additive white Gaussian noise. The optimum detector for this channel can then be specified in terms of this state estimate. A distributed differential equation is given for the error covariance matrix associated with the optimum estimate. Solution of this equation provides the filtering error, which in turn permits calculation of detection error probability bounds.

An approximate, tapped delay line, lumped-parameter state-variable model for the WSSUS channel has been reviewed, which converges to the distributed model as the delay line tap spacing goes to zero.

Computational methods for solving the distributed-parameter error covariance equation are being investigated. For example, integration of Eq. 62 at discrete values of x and y is equivalent, under some circumstances, to solving the variance equation associated with the tapped delay line model. More efficient techniques may be applicable to the distributed model, however. Another advantage of the distributed formulation is that it provides a way of handling the impulsive quantities arising from the spatially white character of the channel model, as in Eqs. 64-68.

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B. AN APPLICATION OF STATE-VARIABLE ESTIMATION TECHNIQUES TO THE DETECTION OF A SPATIALLY DISTRIBUTED SIGNAL

1. Introduction

The problem that we consider is a spatial version of the Gauss-in-Gauss detection problem. We have taken a distributed state-variable approach to this problem, rather than the more usual eigenfunction approach. Our motivation is that state-variable models have provided useful insight and receiver realizations for the nondistributed case. The purpose of this report is to indicate the current status of this study.

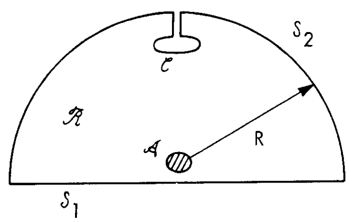

Fig. IX-3. Model geometry.

An example of the spatial Gauss-in-Gauss detection problem arises in the optical cloud channel when the quantum effects associated with physical detectors are neglected. The geometry associated with this example is shown in Fig. IX-3. The model comprises

(i) a free-space region $\mathcal{R}$ that supports propagation from the cloud surface $\mathcal{C}$ to the receiving aperture $\mathcal{A}$;

(ii) surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, which with $\mathcal{C}$ completely enclose the region $\mathcal{R}$. The cloud is illuminated by a source located somewhere above the cloud and exterior to the region $\mathcal{R}$. The resulting field on the bottom surface of the cloud will then excite a field throughout the region $\mathcal{R}$. This field, which we call m, is described below.

2. Message Field State Equation

Let $\{m(t,\underline{r}),\ t \in \mathcal{T},\ \underline{r} \in \mathcal{R}\}$ be a distributed scalar noise field with $\mathcal{T} = (0,T]$ as its temporal domain, and the region $\mathcal{R}$ as its spatial domain. Let $\mathcal{S}$ be the surface enclosing $\mathcal{R}$. We assume that m satisfies the wave equation

$$\frac{\partial^2}{\partial t^2} m(t,\underline{r}) = \Delta m(t,\underline{r}), \qquad t \in \mathcal{T},\ \underline{r} \in \mathcal{R} \tag{1}$$

in which Δ is the Laplacian. By defining the two states, $m_1 = m$ and $m_2 = \partial m/\partial t$, the wave equation (1) for m can be expressed as

$$\frac{\partial}{\partial t} \underline{m}(t, \underline{r}) = \underline{F}\, \underline{m}(t, \underline{r}), \qquad t \in \mathscr{T}, \underline{r} \in \mathscr{R} \tag{2}$$

in which

$$\underline{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$$

and $\underline{F}$ is a differential operator defined by

$$\underline{F} = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}.$$

The state satisfies appropriate initial and boundary conditions on all boundary surfaces, and, in particular, the state is driven by a boundary condition at the cloud surface. These conditions will be described.

(i) Message Field Initial Conditions

Let the initial state $\underline{m}$ be

$$\underline{m}(0, \underline{r}) = \underline{m}_0(\underline{r}), \qquad \underline{r} \in \mathscr{R}, \tag{3}$$

where $\underline{m}_0$ is a Gaussian noise field with mean $\overline{\underline{m}}_0(\underline{r})$ and covariance

$$E[(\underline{m}_0(\underline{r}) - \overline{\underline{m}}_0(\underline{r}))(\underline{m}_0(\underline{r}') - \overline{\underline{m}}_0(\underline{r}'))^T] = \underline{M}(\underline{r}, \underline{r}'), \qquad \underline{r}, \underline{r}' \in \mathscr{R}.$$

(ii) Message Field Boundary Conditions

The boundary conditions will be described with reference to Fig. IX-3. The process on the cloud's surface is characterized by the state $\{\underline{x}(t, \underline{r}), t \in \mathscr{T}, \underline{r} \in \mathscr{R}\}$, which is a Gaussian-Markov process defined by

$$\frac{\partial}{\partial t} \underline{x}(t, \underline{r}) = \underline{A}(t, \underline{r})\, \underline{x}(t, \underline{r}) + \underline{B}(t, \underline{r})\, u(t, \underline{r}), \qquad t \in \mathscr{T}, \underline{r} \in \mathscr{R}, \tag{4}$$

where $\{u(t, \underline{r}), t \in \mathscr{T}, \underline{r} \in \mathscr{C}\}$ is a white Gaussian process with covariance

$$E[u(t, \underline{r}) u(t', \underline{r}')] = U\delta(t-t')\, \delta(\underline{r} - \underline{r}'), \qquad t, t' \in \mathscr{T}, \underline{r}, \underline{r}' \in \mathscr{R}.$$

The initial condition for (4) is

$$\underline{x}(0, \underline{r}) = \underline{x}_0(\underline{r}), \qquad \underline{r} \in \mathscr{R}, \tag{5}$$

where $\underline{x}_0$ is Gaussian with mean $\overline{\underline{x}}_0(\underline{r})$ and covariance

$$E[(\underline{x}_0(\underline{r})-\bar{\underline{x}}_0(\underline{r}))(\underline{x}_0(\underline{r}')-\bar{\underline{x}}_0(\underline{r}'))] = \underline{X}(\underline{r},\underline{r}'), \qquad \underline{r}, \underline{r}' \in \mathscr{R}.$$

We now impose the following boundary conditions:

(i) $$m_1(t,\underline{r}) - a_c \frac{\partial}{\partial\eta} m_1(t,\underline{r}) = \underline{C}(t,\underline{r})\, \underline{x}(t,\underline{r}). \qquad t \in \mathscr{T}, \underline{r} \in \mathscr{C} \tag{6}$$

This describes the coupling of the signal-induced noise field on the cloud's surface to the field $\underline{m}$ in the cloud-to-sensor channel. The constant a_c is non-negative ($a_c \geq 0$), and $\partial/\partial\eta$ denotes the inward-directed normal derivative at the boundary.

(ii) $$m_1(t,\underline{r}) - a_1 \frac{\partial}{\partial\eta} m_1(t,\underline{r}) = 0 \qquad t \in \mathscr{T}, \underline{r} \in \mathcal{S}_1. \tag{7}$$

$\mathcal{S}_1$ is the surface in the proximity of the sensor: for example, the earth's surface. We assume that this surface does not interact with the field seen by the aperture,

(iii) $$\lim_{|\underline{r}|\to\infty} |\underline{r}| \left[m_1(t,\underline{r}) - a_2 \frac{\partial}{\partial\eta} m_1(t,\underline{r}) \right] = 0, \qquad t \in \mathscr{T}, \underline{r} \in \mathcal{S}_2, |\underline{r}| = \mathscr{R}. \tag{8}$$

$\mathcal{S}_2$ is a spherically shaped surface. Equation 8 is the Sommerfeld radiation condition.

This model provides a reasonable first-order description of the optical field at the "bottom" surface of a cloud. At optical frequencies a cloud is dispersive in time, frequency, and angle. A recent theoretical study has established that, for an optically thick cloud (optical thickness ≥ 5), the field below the cloud is reasonably modeled as a zero-mean Gaussian process in time and space.[1] In terms of a complex envelope representation, the real and imaginary parts are independent jointly Gaussian random processes with identical covariance functions. The field intensity also has spatial properties. For instance, if the incident intensity has a symmetrical Gaussian spatial distribution, the spatial distribution, the spatial dependence of the mean-square value of the field below the cloud has a symmetrical Gaussian shape, with a variance depending on the incident intensity variance and the cloud parameters (physical thickness, optical thickness, and particle scattering pattern). Furthermore, the spatial correlation distance of the field at a receiving place below the cloud is very small (of the order of wavelengths). According to the study, the Doppler spread of a typical cloud is of the order of megacycles, and the corresponding range spread is in the microsecond range. Thus the cloud channel is typically overspread.

In the model used here, it is assumed that the signal-bandwidth range-delay is small, so that the range spread is not important. For example, the signal may be a pulse that is long relative to the range spread. The Gaussian-Markov process $\underline{x}(t,\underline{r})$ is used to model, in a first-order way, the Doppler spread and spatial correlation properties of the field at the bottom of the cloud. The modulation matrix $\underline{C}(t, \underline{r})$ can be used to model the spatial intensity distribution at the cloud "bottom," as well as the amplitude variations of the transmitted signal and any deterministic, possibly position-dependent, delay. For simplicity, only a single polarization component of the field is assumed.

The techniques to be used here could be applied to a more realistic cloud model. It would be possible to account for the range spread and to approximate more accurately the Doppler spread and spatial correlation properties by incorporating a more complicated propagation model for the cloud, for example, a doubly spread channel model. This is deferred for the present to concentrate on the form of the estimation problem.

3. Detection Problem

We wish to consider making observations of the field in $\mathscr{R}$ and then deciding whether or not m(t, r) is present, that is, whether or not the cloud is illuminated. The observations are taken with a sensor having an aperture $\mathscr{A}$, and there is a white Gaussian background noise $n(t, \underline{r})$ representing scattered light. The observations are defined by

$$e(t, \underline{r}) = \begin{cases} m_1(t, \underline{r}) + n(t, \underline{r}) & t \in \mathscr{T}, \underline{r} \in \mathscr{A}: H_1 \\ n(t, \underline{r}) & t \in \mathscr{T}, \underline{r} \in \mathscr{A}: H_0 \end{cases} \tag{9}$$

where $n(t, \underline{r})$ is white Gaussian noise with covariance

$$E[n(t, \underline{r})n(t', \underline{r}')] = N_0 \delta(t-t') \delta(\underline{r}-\underline{r}'), \qquad t, t' \in \mathscr{T}, \underline{r}, \underline{r}' \in \mathscr{A}.$$

Because of the Gaussian model, the general detector structure for deciding between H_0 and H_1 is well-known.[1] One version of the detector incorporates the noncausal, minimum-mean-square-error estimator of $\{m_1(t, \underline{r}), t \in \mathscr{T}, \underline{r} \in \mathscr{A}\}$; this estimator is designed under the assumption that $\{e(t, \underline{r}), t \in \mathscr{T}, \underline{r} \in \mathscr{A}\}$ is given and H_1 is true. This is the estimator-correlator structure shown in Fig. IX-4. The constant γ is a predetermined constant that depends on the particular criterion by which the detector's performance is judged.

The estimator required to generate the optimal estimate can be specified as the solution of an integral equation involving the covariance function of m_1 in the aperture $\mathscr{A}$. This covariance function can presumably be determined from the model that we have formulated. Even if the covariance function is known, however, the integral equation for the estimator may be difficult to solve. This is because m_1 will not have the covariance of a lumped-parameter process, because of the inherent coupling between time and space associated with the wave equation. An alternative procedure is to derive equations that determine estimates directly. We shall explore this latter state-variable approach.

The rest of the discussion is concerned with this estimator, the form of which was determined by using the technique of minimizing a quadratic functional containing

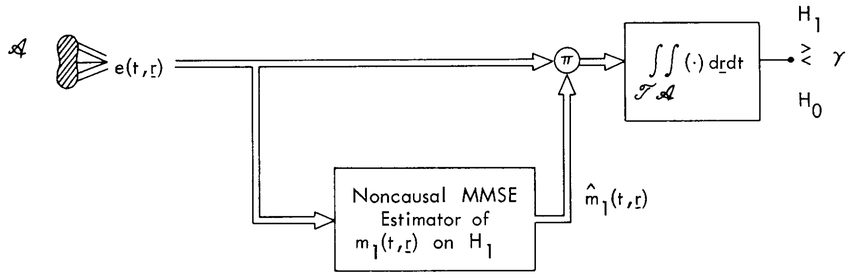

Fig. IX-4. Estimator-Correlator receiver.

Lagrange multipliers to account for constraints. These are the constraints, for example, of the propagation model and boundary conditions. This technique has been used previously by Bryson and Frazier[3] and Baggeroer.[4]

4. Estimation Problem

The relevant estimation problem is that of estimating $\{m_1(t,\underline{r}), t \in \mathscr{T}, \underline{r} \in \mathscr{A}\}$, given $\{e(t,\underline{r}), t \in \mathscr{T}, \underline{r} \in \mathscr{A}\}$, where $e(t,\underline{r}) = m_1(t,\underline{r}) + n(t,\underline{r})$. It is convenient for us to rewrite $e(t,\underline{r})$ as

$$e(t,\underline{r}) = \underline{H}(\underline{r})\, \underline{m}(t,\underline{r}) + n(t,\underline{r}), \qquad t \in \mathscr{T}, \underline{r} \in \mathscr{A}, \tag{10}$$

where

$$\underline{H}(\underline{r}) = [1(\underline{r}) \quad 0]$$

and

$$1(\underline{r}) = \begin{cases} 1, & \underline{r} \in \mathscr{A} \\ 0, & \underline{r} \notin \mathscr{A} \end{cases}$$

The first step in obtaining equations for the desired estimate is to reduce the estimation problem to a problem of minimizing a quadratic functional. The desired equations are then obtained by carrying out the minimization, using Lagrange multipliers to account for constraints on the minimizing solution.

The quadratic functional to be minimized is

$$\begin{aligned} J = &\frac{1}{2} \int_{\mathscr{R}} \int_{\mathscr{R}} [\underline{m}_0(r)-\overline{\underline{m}}_0(r)]^T\, \underline{M}^{-1}(\underline{r},\underline{r}')[\underline{m}_0(\underline{r}')-\overline{\underline{m}}_0(\underline{r}')]\, d\underline{r}d\underline{r}' \\ &+ \frac{1}{2U} \int_{\mathscr{T}} \int_{\mathscr{C}} u^2(t,\underline{r})\, dtd\underline{r} + \frac{1}{2} \int_{\mathscr{C}} \int_{\mathscr{C}} [\underline{x}_0(\underline{r})-\overline{\underline{x}}_0(\underline{r})]^T\, \underline{X}^{-1}(\underline{r},\underline{r}')[\underline{x}_0(\underline{r}')-\overline{\underline{x}}_0(\underline{r}')]\, d\underline{r}d\underline{r}' \\ &+ \frac{1}{2N_0} \int_{\mathscr{T}} \int_{\mathscr{A}} [e(t,\underline{r})-\underline{H}(\underline{r})\underline{m}(t,\underline{r})]^2\, dtd\underline{r}. \end{aligned} \tag{11}$$

In Eq. 11, the two terms $\underline{M}^{-1}(\underline{r},\underline{r}')$ and $\underline{X}^{-1}(\underline{r},\underline{r}')$ are inverse kernels satisfying

$$\int_{\mathscr{R}} \underline{M}(\underline{r},\underline{r}'')\, \underline{M}^{-1}(\underline{r}'',\underline{r}')\, d\underline{r}'' = \underline{I}\delta(\underline{r}-\underline{r}'), \qquad \underline{r}, \underline{r}' \in \mathscr{R} \tag{12}$$

and

$$\int_{\mathscr{C}} \underline{X}(\underline{r},\underline{r}'')\,\underline{X}^{-1}(\underline{r}'',\underline{r}')\,d\underline{r}'' = \underline{I}\delta(\underline{r}-\underline{r}'), \qquad \underline{r},\underline{r}' \in \mathscr{C}. \tag{13}$$

The minimization of J is subject to the following several constraints:

1. $$\frac{\partial}{\partial t}\underline{m}(t,\underline{r}) = \underline{F}\underline{m}(t,\underline{r}), \qquad t\in\mathscr{T},\ \underline{r}\in\mathscr{R} \tag{14}$$

2. $$\underline{m}(0,\underline{r}) = \underline{m}_0(\underline{r}), \qquad \underline{r}\in\mathscr{R} \tag{15}$$

3. $$\frac{\partial}{\partial t}\underline{x}(t,\underline{r}) = \underline{A}(t,\underline{r})\,\underline{x}(t,\underline{r}) + \underline{B}(t,\underline{r})\,u(t,\underline{r}), \qquad t\in\mathscr{T},\ \underline{r}\in\mathscr{C} \tag{16}$$

4. $$\underline{x}(0,\underline{r}) = \underline{x}_0(\underline{r}), \qquad \underline{r}\in\mathscr{C} \tag{17}$$

5. $$m_1(t,\underline{r}) - \alpha_c\frac{\partial}{\partial\eta}m_1(t,\underline{r}) = \underline{C}(t,\underline{r})\,\underline{x}(t,\underline{r}), \qquad t\in\mathscr{T},\ \underline{r}\in\mathscr{C} \tag{18}$$

6. $$m_1(t,\underline{r}) - \alpha_1\frac{\partial}{\partial\eta}m_1(t,\underline{r}) = 0, \qquad t\in\mathscr{T},\ \underline{r}\in\mathcal{S}_1 \tag{19}$$

7. $$\lim_{|\underline{r}|\to\infty} |\underline{r}|\left[m_1(t,\underline{r})-\alpha_2\frac{\partial}{\partial\eta}m_1(t,\underline{r})\right] = 0, \qquad t\in\mathscr{T},\ \underline{r}\in\mathcal{S}_2,\ |\underline{r}| = \mathscr{R}. \tag{20}$$

We now incorporate these seven constraints into J by using Lagrange multipliers:

$$\begin{aligned}
J = {} & \frac{1}{2}\int_{\mathscr{R}}\int_{\mathscr{R}} [\underline{m}_0(\underline{r})-\overline{\underline{m}}_0(\underline{r})]^T\,\underline{M}^{-1}(\underline{r},\underline{r}')[\underline{m}_0(\underline{r}')-\overline{\underline{m}}_0(\underline{r}')]\,d\underline{r}d\underline{r}' \\
& + \frac{1}{2U}\int_{\mathscr{T}}\int_{\mathscr{C}} u^2(t,\underline{r})\,dtd\underline{r} + \frac{1}{2}\int_{\mathscr{C}}\int_{\mathscr{C}} [\underline{x}_0(\underline{r})-\overline{\underline{x}}_0(\underline{r})]^T\,\underline{X}^{-1}(\underline{r},\underline{r}')[\underline{x}_0(\underline{r}')-\overline{\underline{x}}_0(\underline{r}')]\,d\underline{r}d\underline{r}' \\
& + \frac{1}{2N_0}\int_{\mathscr{T}}\int_{\mathscr{A}} [e(t,\underline{r})-m_1(t,\underline{r})]^2\,dtd\underline{r} + \int_{\mathscr{T}}\int_{\mathscr{R}} \underline{p}^T(t,\underline{r})\left[\frac{\partial}{\partial t}\underline{m}(t,\underline{r})-\underline{F}\underline{m}(t,\underline{r})\right]dtd\underline{r} \\
& + \int_{\mathscr{R}} \underline{p}_0^T(\underline{r})[\underline{m}(0,\underline{r})-\underline{m}_0(r)]\,d\underline{r} \\
& + \int_{\mathscr{T}}\int_{\mathscr{C}} \underline{q}^T(t,\underline{r})\left[\frac{\partial}{\partial t}\underline{x}(t,\underline{r})-\underline{A}(t,\underline{r})\underline{x}(t,\underline{r})-\underline{B}(t,\underline{r})u(t,\underline{r})\right]dtd\underline{r} \\
& + \int_{\mathscr{C}} \underline{q}_0^T(\underline{r})[\underline{x}(0,\underline{r})-\underline{x}_0(\underline{r})]\,d\underline{r} \\
& + \int_{\mathscr{T}}\int_{\mathscr{C}} \lambda_1(t,\underline{r})\left[m_1(t,\underline{r})-\alpha_c\frac{\partial}{\partial\eta}m_1(t,\underline{r})-\underline{C}(t,\underline{r})\underline{x}(t,\underline{r})\right]dtd\underline{r} \\
& + \int_{\mathscr{T}}\int_{\mathcal{S}_1} \lambda_2(t,\underline{r})\left[m_1(t,\underline{r})-\alpha_1\frac{\partial}{\partial\eta}m_1(t,\underline{r})\right]dtd\underline{r} \\
& + \lim_{R\to\infty}\int_{\mathscr{T}}\int_{\mathcal{S}_2} \lambda_3(t,\underline{r})\,|\underline{r}|\left[m_1(t,\underline{r})-\alpha_2\frac{\partial}{\partial\eta}m_1(t,\underline{r})\right]dtd\underline{r}.
\end{aligned} \tag{21}$$

Before carrying out the minimization of J, there are several terms to be examined.

We simply list the results that we need.

1. $$\int_{\mathscr{T}}\int_{\mathscr{R}} \underline{p}^T(t,\underline{r})\frac{\partial}{\partial t}\underline{m}(t,\underline{r})\,dtd\underline{r} = \int_{\mathscr{R}}[\underline{p}^T(T,\underline{r})\underline{m}(T,\underline{r})-\underline{p}^T(0,\underline{r})\underline{m}(0,\underline{r})]\,d\underline{r} - \int_{\mathscr{T}}\int_{\mathscr{R}}\left[\frac{\partial}{\partial t}\underline{p}^T(t,\underline{r})\right]\underline{m}(t,\underline{r})\,dtd\underline{r} \tag{22}$$

2. $$\int_{\mathscr{T}}\int_{\mathscr{R}} \underline{p}^T(t,\underline{r})\,\underline{F}\,\underline{m}(t,\underline{r})\,dtd\underline{r}$$

$$= \int_{\mathscr{T}}\int_{\mathscr{R}} p_1(t,\underline{r})\,m_2(t,\underline{r})\,dtd\underline{r} + \int_{\mathscr{T}}\int_{\mathscr{R}} p_2(t,\underline{r})\,\Delta m_1(t,\underline{r})\,dtd\underline{r}$$

$$= \int_{\mathscr{T}}\int_{\mathscr{R}} p_1(t,\underline{r})\,m_2(t,\underline{r})\,dtd\underline{r} + \int_{\mathscr{T}}\int_{\mathscr{R}} m_1(t,\underline{r})\,\Delta p_2(t,\underline{r})\,dtd\underline{r} + \int_{\mathscr{T}}\int_{\mathcal{S}} p_2(t,\underline{r})\frac{\partial}{\partial\eta}m_1(t,\underline{r})\,dtd\underline{r} - \int_{\mathscr{T}}\int_{\mathcal{S}} m_1(t,\underline{r})\frac{\partial}{\partial\eta}p_2(t,\underline{r})\,dtd\underline{r}$$

$$= \int_{\mathscr{T}}\int_{\mathscr{R}} \underline{m}^T(t,\underline{r})\,\underline{F}^T\underline{p}(t,\underline{r})\,dtd\underline{r} + \int_{\mathscr{T}}\int_{\mathcal{S}} p_2(t,\underline{r})\frac{\partial}{\partial\eta}m_1(t,\underline{r})\,dtd\underline{r} - \int_{\mathscr{T}}\int_{\mathcal{S}} m_1(t,\underline{r})\frac{\partial}{\partial\eta}p_2(t,\underline{r})\,dtd\underline{r} \tag{23}$$

The last result follows from Green's second identity, and in it $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathscr{C}$ is the surface enclosing $\mathscr{R}$.

3. $$\int_{\mathscr{T}}\int_{\mathscr{C}} \underline{q}^T(t,\underline{r})\frac{\partial}{\partial t}\underline{x}(t,\underline{r})\,dtd\underline{r} = \int_{\mathscr{C}}[\underline{q}^T(t,\underline{r})\underline{x}(t,\underline{r})-\underline{q}^T(0,\underline{r})\underline{x}(0,\underline{r})]\,d\underline{r} - \int_{\mathscr{T}}\int_{\mathscr{C}}\left[\frac{\partial}{\partial t}\underline{q}^T(t,\underline{r})\right]\underline{x}(t,\underline{r})\,dtd\underline{r} \tag{24}$$

We now incorporate these results into J, set the variation of J to zero at the optimizing values, and collect the terms common to each variation. The result is

$$0 = -\int_{\mathscr{R}} \delta \underline{m}_0^T(\underline{r}) \left[\underline{p}_0(\underline{r}) - \int_{\mathscr{R}} \underline{M}^{-1}(\underline{r},\underline{r}')[\hat{\underline{m}}_0(\underline{r}')-\overline{\underline{m}}_0(\underline{r}')]\, d\underline{r}' \right] d\underline{r} \tag{25}$$

$$+ \int_{\mathscr{T}} \int_{\mathscr{C}} \delta u(t,\underline{r})[U^{-1}u(t,\underline{r})-\underline{B}^T(t,\underline{r})\underline{q}(t,\underline{r})]\, dt d\underline{r}$$

$$- \int_{\mathscr{C}} \delta \underline{x}_0^T(\underline{r}) \left[\underline{q}_0(\underline{r}) - \int_{\mathscr{C}} \underline{X}^{-1}(\underline{r},\underline{r}')[\hat{\underline{x}}_0(\underline{r}')-\overline{\underline{x}}_0(\underline{r}')]\, d\underline{r}' \right] d\underline{r}$$

$$- \int_{\mathscr{T}} \int_{\mathscr{R}} \delta \underline{m}^T(t,\underline{r}) \left\{ \frac{\partial}{\partial t} \underline{p}(t,\underline{r}) + \underline{F}^T \underline{p}(t,\underline{r}) + \underline{H}^T(\underline{r}) N_0^{-1}[e(t,\underline{r})-\underline{H}(\underline{r})\hat{\underline{m}}(t,\underline{r})] \right\} dt d\underline{r}$$

$$+ \int_{\mathscr{R}} \delta \underline{m}^T(\underline{T},\underline{r})\, \underline{p}(T,\underline{r})\, d\underline{r} - \int_{\mathscr{R}} \delta \underline{m}(0,\underline{r})[\underline{p}(0,\underline{r})-\underline{p}_0(\underline{r})]\, d\underline{r}$$

$$- \int_{\mathscr{T}} \int_{\mathscr{S}_1} \delta \frac{\partial}{\partial \eta} m_1(t,\underline{r})[p_2(t,\underline{r})+a_1\lambda_2(t,\underline{r})]\, dt\, d\underline{r}$$

$$- \lim_{R\to\infty} \int_{\mathscr{T}} \int_{\mathscr{S}_2} \delta \frac{\partial}{\partial \eta} m_1(t,\underline{r})[p_2(t,\underline{r})+a_2|\underline{r}|\lambda_3(t,\underline{r})]\, dt d\underline{r}$$

$$- \int_{\mathscr{T}} \int_{\mathscr{C}} \delta \frac{\partial}{\partial \eta} m_1(t,\underline{r})[p_2(t,\underline{r})+a_c\lambda_1(t,\underline{r})]\, dt d\underline{r}$$

$$+ \int_{\mathscr{T}} \int_{\mathscr{S}_1} \delta m_1(t,\underline{r}) \left[\frac{\partial}{\partial \eta} p_2(t,\underline{r}) + \lambda_2(t,\underline{r}) \right] dt d\underline{r}$$

$$+ \lim_{R\to\infty} \int_{\mathscr{T}} \int_{\mathscr{S}_2} \delta m_1(t,\underline{r}) \left[\frac{\partial}{\partial \eta} p_2(t,\underline{r}) + |\underline{r}|\lambda_3(t,\underline{r}) \right] dt d\underline{r}$$

$$+ \int_{\mathscr{T}} \int_{\mathscr{C}} \delta m_1(t,\underline{r}) \left[\frac{\partial}{\partial \eta} p_2(t,\underline{r}) + \lambda_1(t,\underline{r}) \right] dt d\underline{r} + \int_{\mathscr{C}} \delta \underline{x}^T(T,\underline{r})\, \underline{q}(T,\underline{r})\, d\underline{r}$$

$$- \int_{\mathscr{C}} \delta \underline{x}^T(t,\underline{r})[\underline{q}(0,\underline{r})-\underline{q}_0(\underline{r})]\, d\underline{r} - \int_{\mathscr{T}} \int_{\mathscr{C}} \delta \underline{x}^T(t,\underline{r}) \left[\frac{\partial}{\partial t} \underline{q}(t,\underline{r}) + \underline{A}^T(t,\underline{r})\underline{q}(t,\underline{r}) + \underline{C}^T(t,\underline{r})\lambda_1(t,\underline{r}) \right] dt d\underline{r}.$$

Because of the arbitrariness of the variations, we get several conditions on the optimizing solution and the Lagrange multipliers. We also use the fact that noncausal, minimum-mean-square-error estimation commutes with linear operations.

a. Field Estimation Equations

The estimate of the state of the field $\underline{m}$ and the corresponding Lagrange multiplier $\underline{p}$ satisfy

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{\underline{m}}(t,\underline{r})\\ \underline{p}(t,\underline{r})\end{bmatrix} = \begin{bmatrix}\underline{F} & \underline{0}\\ \underline{H}^T(\underline{r})\,N_0^{-1}\underline{H}(\underline{r}) & -\underline{F}^T\end{bmatrix}\begin{bmatrix}\hat{\underline{m}}(t,\underline{r})\\ \underline{p}(t,\underline{r})\end{bmatrix} + \begin{bmatrix}\underline{0}\\ -\underline{H}^T(\underline{r})\,N_0^{-1}\end{bmatrix} e(t,\underline{r}) \qquad t\in\mathscr{T},\ \underline{r}\in\mathscr{A} \tag{26}$$

with the following ancillary conditions:

(i) Initial condition

$$\underline{p}(0,\underline{r}) = \int_{\mathscr{R}} \underline{M}^{-1}(\underline{r},\underline{r}')[\hat{\underline{m}}(0,\underline{r}')-\overline{\underline{m}}_0(\underline{r}')]\,d\underline{r}'. \tag{27}$$

The inverse kernal $\underline{M}^{-1}$ can be eliminated by multiplying by $\underline{M}$ and integrating. The The result is the initial condition

$$\hat{\underline{m}}(0,\underline{r}) = \overline{\underline{m}}_0(r) + \int_{\mathscr{R}} \underline{M}(\underline{r},\underline{r}')\,\underline{p}(0,\underline{r}')\,d\underline{r}', \qquad \underline{r}\in\mathscr{R}. \tag{28}$$

(ii) Final condition

$$\underline{p}(T,\underline{r}) = \underline{0}, \qquad \underline{r}\in\mathscr{R}. \tag{29}$$

(iii) Boundary conditions

$$\hat{m}_1(t,\underline{r}) - \alpha_1\frac{\partial}{\partial\eta}\hat{m}_1(t,\underline{r}) = 0, \qquad p_2(t,\underline{r}) - \alpha_1\frac{\partial}{\partial\eta}p_2(t,\underline{r}) = 0, \qquad t\in\mathscr{T},\ \underline{r}\in\mathcal{S}_1 \tag{30}$$

$$\lim_{|\underline{r}|\to\infty} |\underline{r}|\left[\hat{m}_1(t,\underline{r}) - \alpha_2\frac{\partial}{\partial\eta}\hat{m}_1(t,\underline{r})\right] = 0, \qquad \lim_{|\underline{r}|\to\infty}\left[p_2(t,\underline{r}) - \alpha_2\frac{\partial}{\partial\eta}p_2(t,\underline{r})\right] = 0,$$

$$t\in\mathscr{T},\ \underline{r}\in\mathcal{S}_2,\ |\underline{r}| = R \tag{31}$$

$$\hat{m}_1(t,\underline{r}) - \alpha_c\frac{\partial}{\partial\eta}\hat{m}_1(t,\underline{r}) = \underline{C}(t,\underline{r})\,\hat{\underline{x}}(t,\underline{r}), \qquad p_2(t,\underline{r}) - \alpha_c\frac{\partial}{\partial\eta}p_2(t,\underline{r}) = 0, \qquad t\in\mathscr{T},\ \underline{r}\in\mathscr{C} \tag{32}$$

Note that Eq. 26 is excited by e, which is the field in the aperture. There are three important points to made about this equation.

(i) It is a partial differential equation because F is a spatial differential operator.

(ii) The form of the equation is identical to that for a nondistributed system, except that here F is an operator.

(iii) m satisfies the same wave equation as the field $\hat{\underline{m}}$.

Furthermore, the estimate, $\hat{\underline{m}}$, is driven by a boundary condition at the cloud's surface, as was $\underline{m}$. Here the boundary condition involves the estimate of the state associated with the cloud process. The generation of this required estimate is described next.

b. Cloud Estimation Equation

The estimate of the state of the cloud process $\hat{\underline{x}}$ and the corresponding Lagrange multiplier $\underline{q}$ satisfy

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{\underline{x}}(t,r)\\ \underline{q}(t,\underline{r})\end{bmatrix}=\begin{bmatrix}\underline{A}(t,\underline{r}) & \underline{B}(t,\underline{r})\,U\underline{B}^T(t,\underline{r})\\ \underline{0} & -\underline{A}^T(t,\underline{r})\end{bmatrix}\begin{bmatrix}\hat{\underline{x}}(t,\underline{r})\\ \underline{q}(t,\underline{r})\end{bmatrix}+\begin{bmatrix}\underline{0}\\ \underline{C}^T(t,\underline{r})\end{bmatrix}\frac{\partial}{\partial\eta}p_2(t,\underline{r})\quad t\in\mathscr{T},\ \underline{r}\in\mathscr{C} \tag{33}$$

with the following ancillary conditions:

(i) Initial condition

$$\underline{q}(0,\underline{r})=\int_{\mathscr{C}}\underline{X}^{-1}(\underline{r},\underline{r}')[\underline{x}(0,\underline{r}')-\bar{\underline{x}}_0(\underline{r}')]\,d\underline{r}'. \tag{34}$$

As before, the inverse kernel can be eliminated to obtain the initial condition

$$\hat{\underline{x}}(0,\underline{r})=\bar{\underline{x}}_0(r)+\int_{\mathscr{C}}\underline{X}(\underline{r},\underline{r}')\,\underline{q}(0,\underline{r}')\,dr',\qquad \underline{r}\in C. \tag{35}$$

(ii) Final condition

$$\underline{q}(T,\underline{r})=\underline{0},\qquad r\in\mathscr{C}. \tag{36}$$

Equations 26-36 define $\hat{\underline{m}}$, $\underline{p}$, $\hat{\underline{x}}$, and $\underline{q}$ in terms of a two-point boundary-value problem. The equations are difficult, if not impossible, to solve analytically, and moreover, direct implementation of a processor to generate the solution is not possible because of the presence of both the initial and final value conditions. To circumvent this difficulty, we now convert the two-point boundary-value problem into an initial value problem whose solution can be generated causally. This is possible because of the linearity of the equations. For this purpose, we let $\tilde{\underline{m}}$, $\tilde{\underline{p}}$, $\tilde{\underline{x}}$, and $\tilde{\underline{q}}$ be solutions to (26) and (33), subject to the following initial conditions:

$$\tilde{\underline{m}}(0,\underline{r})=\bar{\underline{m}}_0(\underline{r}),\qquad \underline{r}\in\mathscr{R} \tag{37}$$

$$\tilde{\underline{p}}(0,r) = \underline{0}, \qquad \underline{r} \in \mathscr{R} \tag{38}$$

$$\tilde{\underline{x}}(0,r) = \bar{\underline{x}}_0(r), \qquad \underline{r} \in \mathscr{C} \tag{39}$$

$$\tilde{\underline{q}}(0,r) = \underline{0}, \qquad \underline{r} \in \mathscr{C}. \tag{40}$$

The boundary conditions for these functions will be defined as we proceed. We also define eight matrices $\underline{\Phi}_{mp}$, $\underline{\Phi}_{mq}$, $\underline{\Phi}_{pp}$, $\underline{\Phi}_{pq}$, $\underline{\Phi}_{xp}$, $\underline{\Phi}_{xq}$, $\underline{\Phi}_{qp}$, and $\underline{\Phi}_{qq}$ by the following homogeneous versions of (26) and (33):

$$\frac{\partial}{\partial t}\begin{bmatrix} \underline{\Phi}_{mp}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{pp}(t,\underline{r},\underline{r}') \end{bmatrix} = \begin{bmatrix} \underline{F} & \underline{0} \\ \underline{H}^T(\underline{r})\, N_0^{-1}\underline{H}(\underline{r}) & -\underline{F}^T \end{bmatrix}\begin{bmatrix} \underline{\Phi}_{mp}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{pp}(t,\underline{r},\underline{r}') \end{bmatrix}, \qquad t \in \mathscr{T},\ \underline{r},\ \underline{r}' \in \mathscr{R} \tag{41}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \underline{\Phi}_{mq}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{pq}(t,\underline{r},\underline{r}') \end{bmatrix} = \begin{bmatrix} \underline{F} & \underline{0} \\ \underline{H}^T(r)\, N_0^{-1}\underline{H}(\underline{r}) & -\underline{F}^T \end{bmatrix}\begin{bmatrix} \underline{\Phi}_{mq}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{pq}(t,\underline{r},\underline{r}') \end{bmatrix}, \qquad t \in \mathscr{T},\ \underline{r} \in \mathscr{R},\ \underline{r}' \in \mathscr{C} \tag{42}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} \underline{\Phi}_{xp}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{qp}(t,\underline{r},\underline{r}') \end{bmatrix} = \begin{bmatrix} \underline{A}(t,r) & \underline{B}(t,\underline{r})\, U\underline{B}^T(t,\underline{r}) \\ \underline{0} & -\underline{A}^T(t,\underline{r}) \end{bmatrix}\begin{bmatrix} \underline{\Phi}_{xp}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{qp}(t,\underline{r},\underline{r}') \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{C}^T(t,\underline{r}) \end{bmatrix} [0\ \ 1]\frac{\partial}{\partial \eta}\Phi_{pp}(t,\underline{r},\underline{r}'),$$
$$t \in \mathscr{T},\ \underline{r} \in \mathscr{C},\ \underline{r}' \in \mathscr{R} \tag{43}$$

and

$$\frac{\partial}{\partial t}\begin{bmatrix} \underline{\Phi}_{xq}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{qq}(t,\underline{r},\underline{r}') \end{bmatrix} = \begin{bmatrix} \underline{A}(t,\underline{r}) & \underline{B}(t,\underline{r})\, U\underline{B}^T(t,\underline{r}) \\ \underline{0} & -\underline{A}^T(t,\underline{r}) \end{bmatrix}\begin{bmatrix} \underline{\Phi}_{xq}(t,\underline{r},\underline{r}') \\ \underline{\Phi}_{qq}(t,\underline{r},\underline{r}') \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{C}^T(t,\underline{r}) \end{bmatrix} [0\ \ 1]\frac{\partial}{\partial \eta}\Phi_{pq}(t,\underline{r},\underline{r}'),$$
$$t \in \mathscr{T},\ \underline{r},\ \underline{r}' \in \mathscr{C} \tag{44}$$

The initial conditions for these eight matrices will be taken to be

$$\underline{\Phi}_{mp}(0,\underline{r},\underline{r}') = \underline{M}(\underline{r},\underline{r}') \qquad \underline{\Phi}_{mq}(0,\underline{r},\underline{r}') = \underline{0}$$

$$\underline{\Phi}_{pp}(0,\underline{r},\underline{r}') = \underline{I}\delta(\underline{r}-\underline{r}') \qquad \underline{\Phi}_{pq}(0,\underline{r},\underline{r}') = \underline{0}$$

$$\underline{\Phi}_{xp}(0,\underline{r},\underline{r}') = \underline{0} \qquad \underline{\Phi}_{xq}(0,\underline{r},\underline{r}') = \underline{X}(\underline{r},\underline{r}')$$

$$\underline{\Phi}_{qp}(0,\underline{r},\underline{r}') = \underline{0} \qquad \underline{\Phi}_{qq}(0,\underline{r},\underline{r}') = \underline{I}\delta(\underline{r}-\underline{r}').$$

We now express $\hat{\underline{m}}$, $\underline{p}$, $\hat{\underline{x}}$, and $\underline{q}$ in terms of these defined quantities. The

$$\underline{\hat{m}}(t,\underline{r}) = \underline{\tilde{m}}(t,\underline{r}) + \int_{\mathscr{R}} \underline{\Phi}_{mp}(t,\underline{r},\underline{r}')\,p(0,\underline{r}')\,d\underline{r}' + \int_{\mathscr{C}} \underline{\Phi}_{mq}(t,\underline{r},\underline{r}')\,q(0,\underline{r}')\,d\underline{r}', \qquad t \in \mathscr{T},\ \underline{r} \in \mathscr{R} \tag{45}$$

$$\underline{p}(t,\underline{r}) = \underline{\tilde{p}}(t,\underline{r}) + \int_{\mathscr{R}} \underline{\Phi}_{pp}(t,\underline{r},\underline{r}')\,\underline{p}(0,\underline{r}')\,d\underline{r}' + \int_{\mathscr{C}} \underline{\Phi}_{pq}(t,\underline{r},\underline{r}')\,\underline{q}(0,\underline{r}')\,d\underline{r}', \qquad t \in \mathscr{T},\ \underline{r} \in \mathscr{R} \tag{46}$$

$$\underline{\hat{x}}(t,\underline{r}) = \underline{\tilde{x}}(t,\underline{r}) + \int_{\mathscr{R}} \underline{\Phi}_{xp}(t,\underline{r},\underline{r}')\,\underline{p}(0,\underline{r}')\,d\underline{r}' + \int_{\mathscr{C}} \underline{\Phi}_{xq}(t,\underline{r},\underline{r}')\,q(0,\underline{r}')\,d\underline{r}', \qquad t \in \mathscr{T},\ \underline{r} \in \mathscr{C} \tag{47}$$

$$\underline{q}(t,\underline{r}) = \underline{\tilde{q}}(t,\underline{r}) + \int_{\mathscr{R}} \underline{\Phi}_{qp}(t,\underline{r},\underline{r}')\,\underline{p}(0,\underline{r}')\,d\underline{r}' + \int_{\mathscr{C}} \underline{\Phi}_{qp}(t,\underline{r},\underline{r}')\,\underline{q}(0,\underline{r}')\,d\underline{r}', \qquad t \in \mathscr{T},\ \underline{r} \in \mathscr{C}. \tag{48}$$

It is straightforward to verify by direct substitution that the definitions for $\underline{\tilde{m}}$, $\underline{\tilde{p}}$, $\underline{\tilde{x}}$, $\underline{\tilde{q}}$ and the eight matrices are consistent in the sense that (45) and (46) imply $\underline{\hat{m}}$, $\underline{p}$, $\underline{\hat{x}}$, and $\underline{q}$ continue to satisfy (26) and (33). Moreover, the assumed initial conditions are likewise consistent. In order to obtain a consistent set of boundary conditions, we simply substitute (45) and (46) in Eqs. 30-32. The resulting boundary conditions on $\mathscr{S}_1$ and $\mathscr{S}_2$ are all homogeneous. The conditions that we obtain on $\mathscr{C}$ are

(i) $\tilde{m}_1(t,r) - \alpha_c \frac{\partial}{\partial \eta} \tilde{m}_1(t,\underline{r}) = \underline{C}(t,\underline{x})\, \underline{\tilde{x}}(t,\underline{r})$

(ii) $\tilde{p}_2(t,\underline{r}) - \alpha_2 \frac{\partial}{\partial \eta} \tilde{p}_2(t,\underline{r}) = 0$

(iii) $[0 \quad 1]\, \underline{\Phi}_{pp}(t,\underline{r},\underline{r}') - \alpha_c \frac{\partial}{\partial \eta} \underline{\Phi}_{pp}(t,\underline{r},\underline{r}') = \underline{0} \qquad \underline{r}' \in \mathscr{R}$

(iv) $[0 \quad 1]\, \underline{\Phi}_{pq}(t,\underline{r},\underline{r}') - \alpha_c \frac{\partial}{\partial \eta} \underline{\Phi}_{pq}(t,\underline{r},\underline{r}') = \underline{0}, \qquad \underline{r}' \in \mathscr{C}$

(v) $[1 \quad 0]\, \underline{\Phi}_{mp}(t,\underline{r},\underline{r}') - \alpha_c \frac{\partial}{\partial \eta} \underline{\Phi}_{mp}(t,\underline{r},\underline{r}') = \underline{C}(t,\underline{x})\, \underline{\Phi}_{xp}(t,\underline{r},\underline{r}'), \qquad r' \in \mathscr{R}$

(vi) $[1 \quad 0]\, \underline{\Phi}_{mq}(t,\underline{r},\underline{r}') - \alpha_c \frac{\partial}{\partial \eta} \Phi_{mq}(t,\underline{r},\underline{r}') = \underline{C}(t,\underline{x})\, \underline{\Phi}_{xq}(t,\underline{r},\underline{r}'), \qquad r' \in \mathscr{C},$

for $t \in \mathscr{T}$, $\underline{r} \in \mathscr{R}$. Note that the last two conditions couple the solutions to (43) and (44).

5. Concluding Comments

The important aspect of Eqs. 45-48 is that $\underline{\hat{m}}$, $\underline{p}$, $\underline{\hat{x}}$, and $\underline{q}$ can be expressed in terms of a part (the tilde quantities) that can be generated causally as data arrive, and an additional part that can be computed after the final observation time. This additional part depends on precomputed quantities, the $\underline{\Phi}$'s, and the final values of the causal parts. The causal processing involves partial differential equations driven by the data received in the aperture. An important topic requiring further investigation is the procedure for generating the solutions to these equations; a method for simulating them on a reduced spatial scale would be desirable. For instance, the propagation implied by the wave equation for $\underset{\sim}{\tilde{m}}$ might be realized optically.

An additional topic requiring further investigation is the possibility of simplifying the processing required to generate the noncausal estimate needed in the detector.

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