COMMUNICATION SCIENCES

AND

ENGINEERING

XVIII. PROCESSING AND TRANSMISSION OF INFORMATION*

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RESEARCH OBJECTIVES AND SUMMARY OF RESEARCH

1. Optical Communication

The fundamental limitations and efficient utilization of optical communication channels are the concern of these investigations. Our interests include the turbulent atmospheric channel, scattering channels, scatter channels with absorption, and quantum channels. During the coming year our activities will focus on improving the understanding of scattering channels, on practical, or at least feasible, methods of implementing near optimum detection and estimation systems, and on imaging radar systems.

The characteristics of scattering channels, such as clouds, fog, and haze, are being investigated experimentally and theoretically. The broad purpose of these investigations is to develop the quantitative description of these channels so that their utility for specific communication applications can be assessed. The studies of scatter channels with absorption such as the ocean have a similar purpose. The objective of one doctoral investigation is to relate the optical propagation characteristics of the cloud to its gross physical characteristics.[1] A second doctoral investigation is concerned with efficient means of measuring those parameters of scattering channels that are important in communication applications.[2] The objectives are to draw upon theoretical considerations to develop efficient measuring techniques and then to implement these techniques on the experimental scatter link that has been developed in cooperation with Lincoln Laboratory, M.I.T.

Our understanding of quantum communication theory has continued to increase during the past year. A general procedure for translating a classical description of a communication channel into a full quantum description was developed for linear channels.[3] In addition to simplifying many steps in the development of a quantum model, the procedure also avoids the need for an artificial representation for the system operators. In the same study some new and important properties of optimum receivers were developed.[3,4]

The problems of communicating through the clear turbulent atmosphere will continue to receive attention during the coming year. The feasibility of implementing a

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wavefront phase compensating receiver has been examined in an experimental thesis.[5] An investigation of variable-rate, or burstlike, communications systems has been completed.[6] This study emphasized the performance gains that can be achieved on an earth-to-space link. Both heterodyne and direct detection systems were considered. Detector output statistics and error bounds for direct-detection receivers have been considered and will receive further attention.[7,8] Two other atmospheric studies will be completed during the coming year. One of these is concerned with the performance and structure of receivers that track, or adapt to, the instantaneous channel state so as to make optimal use of the spatial diversity in the receiving aperture.[9] The other is addressed to the structure and performance of waveform estimators for Poisson processes.[10] This work, which will consider both optimal and suboptimal estimators, is directly applicable to analog communication systems that employ direct-detection receivers.

R. S. Kennedy, E. V. Hoversten

2. Coding for Noisy Channels

The goals of this work are: first, to develop fundamental limitations on data rate, reliability, and system complexity for data transmission over noisy channels; and second, to develop communication techniques that approach these limitations. One of the most promising such techniques is that of convolutional coding and sequential decoding. A new mathematical technique, called "branching random walks," is being developed to simplify and extend the analysis of convolutional codes and sequential decoding. In addition, two doctoral theses have recently been completed. The first dealt with the feasibility of concatenated convolutional codes,[11] and the second established basic limitations on mean-square error when transmitting analog samples over white Gaussian noise channels without intersample coding.[12]

R. G. Gallager

3. Simple Encoding Techniques for Analog Signals

Two simple techniques for encoding analog sources continue to be explored: quantization and feedback. A paper has been published during the year on optimum multivariate quantizers.[13] Doctoral research by Bernd Neumann is directed toward the use of noiseless feedback for communication over a forward channel with additive Gaussian noise which is not white, so that feedback increases channel capacity.

P. Elias

4. Digital Data Networks and Other Data-Processing Problems

Digital data networks are of great and still growing importance, both in local networks like the one connecting several hundred terminals to several computers at M.I.T. and in more widely dispersed systems. The possibility of buffering messages at intermediate nodes, of switching messages rather than lines, and of having networks in which nodes are more expensive than branches, present new areas for research which are now being defined with the help of several graduate students.

Information theory has relevance to other data-processing problems, and several are being explored. Terry A. Welch has been pursuing a doctoral investigation in information retrieval, relating the amount of information stored in a catalog to the relevance, recall, and retrieval effort involved in its use. Other work is under way using informational analysis in the study of switching systems, sorting algorithms, the storage and retrieval of simple information structures, and the generation of independent equiprobable random digits from less random input data.

P. Elias

5. Finite Memory-Learning Algorithms

The problem of testing among several hypotheses when the data must be summarized by a finite-valued statistic is being studied. The large sample size problem has been solved previously,[14] and promising results have been obtained for small sample sizes.[15] The differences between deterministic and stochastic algorithms are also being studied and substantial progress has been made in determining the properties of both types.[16-18]

M. E. Hellman

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A. ESTIMATION FOR POISSON PROCESSES

In this report we present some new results in the theory of minimum mean-square error estimation for Poisson processes. Proofs of the theorems, too involved to be presented here, will be given elsewhere.

It is well known that the minimum mean-square error (MMSE) estimate of a random variable z relative to a data set $\{y_\sigma, \sigma \in [t_o, t]\}$ is simply the conditional expectation $E\{z | y_\sigma, \sigma \in [t_o, t]\}$. Straightforward calculation of this quantity is usually a prohibitively difficult task. Using indirect methods, we have obtained a useful representation for the conditional mean when $\{y_\sigma, \sigma \in [t_o, t]\}$ is a doubly stochastic Poisson process; that is, when the data process is a counting process with a stochastic rate function such that $\{y_\sigma, \sigma \in [t_o, t]\}$ is conditionally Poisson.[1-3]

Theorem 1

Let $\{N_t, t \in T = [t_o, t_1]\}$ be a doubly stochastic vector Poisson process with conditionally independent components. Assume that the vector rate process $\{\lambda_t, t \in T\}$ of $\{N_t, t \in T\}$ is non-negative a. s. and integrable. Let z be a vector random variable with components in L_2, such that z is measurable with respect to B_{λ_t}, the Borel field generated by $\{\lambda_\sigma, \sigma \in [t_o, t]\}$. Define the process $\{\nu_t, t \in T\}$ by the stochastic differential equation $d\nu_t = dN_t - \hat{\lambda}_t dt$, where $\hat{\lambda}_t = E\{\lambda_t | B_{N_t}\}$. Finally, let $\hat{\Lambda}_t$ be a diagonal matrix with nonzero entries, the components of the vector $\hat{\lambda}_t$. Then

(i) $B_{N_t} \equiv B_{\nu_t}$, all $t \in T$;

(ii) the conditional mean $\{\hat{z}_t = E(z | B_{N_t}), t \in T\}$ can be represented by the stochastic integral equation,

$$\hat{z}_t = \hat{z}_{t_o} + \int_{t_o}^{t} E\{z(\lambda_\sigma - \hat{\lambda}_\sigma)' | B_{\nu_\sigma}\} \hat{\Lambda}_\sigma^{-1} d\nu_\sigma, \tag{1}$$

where prime indicates transpose.

This result is in the spirit of recent work of Frost[4] for Gaussian processes, and has wide applicability in problems of filtering, parameter estimation, prediction, smoothing, and detection. Stochastic integrals such as that in (1) are discussed in detail in references 5-8.

The significance of this theorem is that $\hat{z}_t$ is expressed as an explicit functional of the observations. Thus stochastic differential equations can be derived with ease for many estimators of interest. Consider the second-order vector Ito process $\{x_t, t \in T\}$, defined by $dx_t = f_t dt + g_t d\xi_t$, where $\{\xi_t, t \in T\}$ is an independent increment martingale, and $\{f_t, t \in T\}$ and $\{g_t, t \in T\}$ are processes depending on x_t and obeying mild regularity conditions. Assume that λ_t is a causal functional of x_t. Using (1), and taking the limit of $E\{x_{t+\Delta t} | B_{N_{t+\Delta t}}\} - E\{x_t | B_{N_t}\}$ as $\Delta t \to 0$, we get the <u>filtering equation</u>,

$$d\hat{x}_t = \hat{f}_t dt + E\{x_t(\lambda_t - \hat{\lambda}_t)' | B_{\nu_t}\} \hat{\Lambda}_t^{-1} d\nu_t. \tag{2}$$

As another filtering example, suppose we wish to estimate the scalar process $\{y_t = \exp(iv'x_t), t \in T\}$. By a similar argument, we find that the <u>conditional characteristic function</u> $\hat{y}_t$ satisfies the Ito equation

$$d\hat{y}_t = E\{\psi_t(v) \exp(iv'x_t) | B_{\nu_t}\} dt + E\{(\lambda_t - \hat{\lambda}_t)' \exp(iv'x_t) | B_{\nu_t}\} \hat{\Lambda}_t^{-1} d\nu_t, \tag{3}$$

where $\psi_t(v)$ is the characteristic form of the differential generator[9] of x_t,

$$\psi_t(v) = p \lim_{\Delta t \to 0} \Delta t^{-1} E\{\exp[iv'(x_{t+\Delta t} - x_t)] - 1 | B_{\xi_t} \cup B_{\nu_t}\}. \tag{4}$$

This "canonical filtering theorem" was obtained by Snyder[1] by different means. Note that $\hat{y}_t$ is a sufficient statistic for $\hat{x}_t$ relative to $\{N_\sigma, \sigma \in [t_o, t]\}$.

As an example of <u>noncausal MMSE filtering</u>, let us estimate x_t based on data up to time $\tau > t$. From our representation theorem we have

$$E\{x_t | B_{N_\tau}\} = E\{x_t | B_{N_t}\} + \int_t^\tau E\{x_t(\lambda_\sigma - \hat{\lambda}_\sigma)' | B_{\nu_\sigma}\} \hat{\Lambda}_\sigma^{-1} d\nu_\sigma. \tag{5}$$

Note that this is the sum of two orthogonal estimates because the stochastic integral is a martingale in its upper limit; the first term is the causal filtered estimate, and the second is an estimate "backward in time" of x_t given data up to time τ. This backward estimate is causal relative to a reversed time scale. By using (5) it is easy to verify the intuitive notion that the causal mean-square error must be greater than the noncausal mean-square error.[10]

We now consider a simple example of MMSE parameter estimation. Let the intensity of N_t be $a\lambda_t + \lambda_o$, with a a zero-one random variable, λ_t a non-negative process, and λ_o a non-negative constant. Let us estimate the parameter a. Since a is an Ito process (albeit somewhat degenerate), (2) applies to give

$$d\hat{a}_t = E\{a(a\lambda_t' - \hat{a}_t\hat{\lambda}_t')|B_{\nu_t}\}\hat{\Lambda}_t^{-1}d\nu_t, \tag{6}$$

where $\hat{\lambda}_t = E\{\lambda_t|B_{\nu_t}, a=1\}$, and $\hat{\Lambda}_t$ is a diagonal matrix with nonzero entries, the components of $\hat{a}_t\hat{\lambda}_t + \lambda_o$. This can be solved explicitly by making the change of variable $\ell_t = \ln(\hat{a}_t/1-\hat{a}_t)$. Omit the details, and the end result of this is

$$d\ell_t = \sum_{j=1}^{D}\left\{-\hat{\lambda}_t'e_j + \ln\left[1+\frac{\hat{\lambda}_t'e_j}{\lambda_o'e_j}\right]dN_t'e_j\right\},$$

or in integral form,

$$\ell_t = \ell_{t_o} + \sum_{j=1}^{D}\left\{-\int_{t_o}^{t}\hat{\lambda}_\sigma'e_j\,d\sigma + \int_{t_o}^{t}\ln\left[1+\frac{\hat{\lambda}_\sigma'e_j}{\lambda_o'e_j}\right]dN_\sigma'e_j\right\}. \tag{7}$$

The observation vector N_t is assumed to be D-dimensional, and $\{e_j\}$ is the set of unit vectors in R^D.

This simple example of MMSE parameter estimation is significant because ℓ_t is the log-likelihood ratio for deciding between the hypotheses

H_1: rate process of N_t is $\lambda_t + \lambda_o$;

H_o: rate process of N_t is λ_o;

based on the observations $\{N_\sigma, \sigma \in [t_o, t]\}$. The fact that $\hat{a}_t = E\{a|B_{N_t}\} = \Pr\{a=1|B_{N_t}\}$ makes this apparent. ℓ_{t_o} is the a priori log-likelihood ratio $\ln[\Pr(a=1)/\Pr(a=0)]$. It should be noted that $\hat{\lambda}_t = E\{\lambda_t|B_{\nu_t}, a=1\}$ has an interpretation as an MMSE causal estimate only when $a = 1$.

Equation 7 is the extension of the well-known Reiffen-Sherman detector[11] to stochastic intensity functions and vector observations. In scalar form it is identical to the result obtained by Synder[1] under somewhat more restrictive assumptions. The scalar version of (7) was also obtained by Evans[12] using different methods.

Other applications of the representation theorem, Eq. 1, will be considered in detail elsewhere.

We now state an interesting result that provides a link between doubly stochastic Poisson process characterization and MMSE estimation. The conditional joint probability density function of the number of events N_t in $(t_o, t]$, and the event times $\{\tau_j\}$, of a doubly stochastic scalar Poisson process with rate function λ_t, is given by[13,14]

$$p(\{\tau_j\}, N_t \mid B_{\lambda_t}) = \exp\left[-\int_{t_o}^{t} \lambda_\sigma \, d\sigma\right] \prod_{i=1}^{N_t} \lambda_{\tau_i}. \tag{8}$$

This can be written in the form

$$\exp\left\{-\int_{t_o}^{t} \lambda_\sigma \, d\sigma + \int_{t_o}^{t} \ln \lambda_\sigma \, dN_\sigma\right\},$$

where the second integral is a stochastic integral.[8] By using techniques of the stochastic calculus and properties of conditional expectation, this can be averaged over all possible sample paths of $\{\lambda_\sigma, \sigma \in [t_o, t]\}$ to yield the following theorem.

Theorem 2

$$p(\{\tau_j\}, N_t) = \exp\left[-\int_{t_o}^{t} \hat{\lambda}_\sigma \, d\sigma\right] \prod_{i=1}^{N_t} \hat{\lambda}_{\tau_i^-}, \tag{9}$$

where $\hat{\lambda}_t = E\{\lambda_t \mid B_{N_t}\}$, and $\hat{\lambda}_{t-} = \lim_{\Delta \downarrow 0} \hat{\lambda}_{t-\Delta}$.

This result, which is identical in form to Eq. 8, has far-reaching implications. It provides a closed-form representation for doubly stochastic Poisson processes, which can serve as the basis for the design of detectors and demodulators for random signals. The detection equation, (7), can be derived immediately from the vector version of (9), as can likelihood ratios for other hypotheses. The form in which (9) is written is somewhat misleading, since $\hat{\lambda}_t$ depends implicitly on $\{\tau_j\}$ and N_t. Nevertheless, that $p(\{\tau_j\}, N_t)$ can be written so simply and succinctly is propitious.

We have presented two theorems that provide powerful tools for the design of systems using doubly stochastic Poisson processes. Proofs and other applications will be given elsewhere.

J. R. Clark

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B. PHOTOCOUNT STATISTICS: A REPRESENTATION THEOREM

1. Introduction

Calculation of the photocount distributions arising from the incidence of a random optical field upon an ideal photodetector can be difficult if attempted by direct methods. In this report we show how judicious application of a representation theorem using Poisson functions succeeds, in some cases, in reformulating the calculation into the determination of a linear transformation that maps a given photocount distribution into the actual photodetector output distribution.

2. Poisson Expansions

Consider functions $x(\cdot)$ which may be represented as follows:

$$x(u) = \sum_{n=0}^{\infty} x_n p_n(u); \qquad u \in [0, \infty) \tag{1}$$

where $p_n(\cdot)$ represents the n^{th} Poisson function

$$p_n(u) = \frac{u^n}{n!} e^{-u}; \qquad u \in [0,1); \qquad n = 0, 1, \ldots . \tag{2}$$

The Poisson functions are the probabilities associated with a Poisson random variable n of mean $\bar{n} = u$.

In order to find the expansion coefficients $\{x_n\}$ associated with $x(\cdot)$ we may note that the Taylor series (if it exists) for $e^u x(u)$ is

$$e^u x(u) = \sum_{n=0}^{\infty} x_n \frac{u^n}{n!}. \tag{3}$$

Using the Taylor inversion formula, we find

$$x_n = \frac{d^n}{du^n} [e^u x(u)] \Big|_{u=0} \tag{4}$$

which, by Leibnitz' rule for product differentiation, is

$$x_n = \sum_{k=0}^{n} \binom{n}{k} \frac{d^n}{du^n} x(u) \Big|_{u=0} . \tag{5}$$

In the rest of the report Poisson expansions are used to calculate photocount statistics.

3. Reception in the Presence of Fading

The random process resulting from the incidence of a random field on an ideal photodetector has been characterized as a doubly stochastic Poisson process.[1] If the randomness of the received field arises solely from a constant multiplicative fading in intensity, then the resulting conditional counting statistics are given by

$$\Pr \{\# \text{ counts} = m | a\} = p_m(au), \tag{6}$$

where u is the nominal average number of counts, and a, the intensity fading, is a random variable with density $q(\cdot)$. We wish to examine the unconditional counting statistics

$$\Pr \{\# \text{ counts} = m\} \triangleq g_m(u) = \int_0^{\infty} p_m(au)\, q(a)\, da. \tag{7}$$

Suppose $g_m(u)$ has a Poisson expansion for non-negative m:

$$g_m(u) = \sum_{n=0}^{\infty} r_{mn} p_n(u). \tag{8}$$

Formula (5) may be applied to find r_{mn}. After considerable algebraic manipulation we find that the expansion coefficients are expressible as a simple function of the moments of the fading variable:

$$r_{mn} = \begin{cases} \binom{n}{m} \overline{a^m (1-a)^{n-m}}; & n \geq m \\ 0; & n < m \end{cases} \tag{9}$$

That is,

$$g_m(u) = \sum_{n=m}^{\infty} \binom{n}{m} \overline{a^m (1-a)^{n-m}}\, p_n(u) \tag{10}$$

One may interpret this result as a linear transformation $R: \underline{p}(u) \rightarrow \underline{g}(u)$, where

$$\underline{p}(u) = \begin{Bmatrix} p_o(u) \\ p_1(u) \\ \vdots \end{Bmatrix} \qquad \underline{g}(u) = \begin{Bmatrix} g_o(u) \\ g_1(u) \\ \vdots \end{Bmatrix} \tag{11}$$

R is then represented by the infinite matrix $R = \{r_{mn}\}$, and the result is written

$$\underline{p}(u) = R\, \underline{g}(u). \tag{12}$$

An immediate generalization of (12) is obtained by noting that since the elements of R depend only upon moments of the fading variable a, and not on the "signal intensity" u, then u may be treated as a random variable and (12) may be averaged over u to yield

$$\overline{\underline{p}(u)}^{u} = \overline{R\, \underline{g}(u)}^{u} \tag{13}$$

This may be regarded as an expansion in terms of the counting statistics for random u without fading; viewed in this light the interesting facet is that the linear transformation remains unchanged when random source intensity is introduced. One may prefer to interpret (13) as a rule for performing expansions over basis sets other than Poisson. If one can obtain the basis set by averaging the Poisson functions, then the transformation matrix (9) yields the correct expansion.

4. Reception in the Presence of Fading and Additive Noise

Consider the case in which a single-mode received signal consists of a faded replica plus a white Gaussian noise sample. The intensity in that received signal is proportional to $|\sqrt{au}+n|^2$, where a and u are as before and n is a circular complex Gaussian random variable with zero mean, and $\overline{|n|^2} = \lambda$. The probability of m counts is

$$g_m(u) = \overline{\frac{|\sqrt{au}+n|^{2m}}{m!} e^{-|\sqrt{au}+n|^2}}^{a,n} \tag{14}$$

The average over n yields[2]

$$g_m(u) = \overline{\frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^m e^{-\frac{au}{1+\lambda}} L_m\left[-\frac{au}{\lambda(1+\lambda)}\right]}^{a}, \tag{15}$$

where $L_m(\cdot)$ is the m^{th} Laguerre polynomial

$$L_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-x)^k}{k!}. \tag{16}$$

Using the series (16) in (15), we obtain

$$g_m(u) = \frac{\lambda^m}{(1+\lambda)^{m+1}} \sum_{j=0}^{m} \binom{m}{j} \left[\frac{u}{\lambda(1+\lambda)}\right]^j \int_0^\infty \frac{a^j}{j!} e^{-\frac{au}{1+\lambda}} q(a)\, da. \tag{17}$$

It is easy to show that in the limit $\lambda \to 0$ (17) becomes identical to (7), which was derived for the noiseless case. Again assume that $g_m(u)$ has a Poisson expansion of the form (8) and apply the coefficient rule (5) to it. Then we find that

$$r_{mn}(\lambda) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{1+\lambda}\right)^{m+k+1} \overline{(-a)^k} \sum_{j=0}^{\min(m,k)} \binom{m}{j}\binom{n}{j} \lambda^{m-j}(-1)^j.$$

Equation 18 is difficult to simplify, but the asymptotic behavior for small λ can be found via a Taylor series in λ for $r_{mn}(\lambda)$:

$$r_{mn}(\lambda) \approx \begin{cases} \binom{m}{n} \overline{a^n}\, \lambda^{m-n}; & m \geq n \quad (19a) \\ \overline{(1-a)^n} + \lambda\left[\overline{n\,a(1-a)^{n-1}} - \overline{(1-a)^n}\right]; & m = 0,\ n \geq 1 \quad (19b) \\ \binom{n}{m} \overline{a^m(1-a)^{n-m}} - \lambda\binom{n}{m}\left[\frac{m^2}{n-m+1}\, \overline{a^{m-1}(1-a)^{n-m-1}} \right. & \\ \left. + (2m+1)\, \overline{a^m(1-a)^{n-m}} + (n-m)\, \overline{a^{m+1}(1-a)^{n-m-1}}\right]; & n > m > 0 \quad (19c) \end{cases}$$

In the limit $\lambda = 0$ (19) agrees with (9).

5. Comparison with Previous Results

Solimeno, et al.[3] have used a different method to solve these same problems. For the noiseless case they express the linear operator R as

$$R = \int p(\chi)\, e^{-2\chi(I+\bigcirc)}\, d\chi, \tag{20}$$

where $\bigcirc$ stands for the linear differential operator $u\,\frac{d}{du}$, I is the identity operator, and $p(\cdot)$ is the probability density of the logarithm of the fading amplitude $\chi = \ln\sqrt{a}$. $(I+\bigcirc)$ can be represented as an infinite matrix over the Poisson basis and then R calculated by a matrix exponentiation and average over χ. The last manipulations are not carried out explicitly by Solimeno, but they may be done by using a Cauchy integral theorem for functions of an operator Γ in Hilbert space[4]:

$$f(\Gamma) = \frac{1}{2\pi j} \oint_C f(s)(sI-\Gamma)^{-1}\, ds, \tag{21}$$

where C is an appropriate contour in the complex s-plane. The result is

$$r_{mn} = \begin{cases} \binom{n}{m} \overline{e^{2\chi m}\, (1-e^{2\chi})^{n-m}}^{\,\chi}; & n \geq m \\ 0; & n < m \end{cases} \tag{22}$$

which agrees with (9) under the substitution $\chi = \ln\sqrt{a}$. Equivalence of the noisy result (18) and Solimeno's comparable formula has not yet been demonstrated.

6. Applications

It has been argued, with justification, that the output statistics of an ideal photodetector excited by a signal plus independent background noise can be modelled by a Poisson process. Clark and Hoversten[5] have investigated the validity of this assumption from the point of view of cumulant comparison of counting statistics. The techniques outlined in this report might be used to yield an alternative measure of "Poisson-ness" in terms of an L_1 or L_2 norm of the transformation (R-I). Such investigations are, at present, under way.

R. S. Orr

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