XII. DIGITAL SIGNAL PROCESSING

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A. POLE-ZERO MODELING USING CEPSTRAL PREDICTION

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1. Introduction

Recently, there has been considerable interest in the use of linear prediction for all-pole modeling of signals. When the signal to be analyzed is best approximated as the impulse response of a filter having both poles and zeros, it is straightforward to use the technique of linear prediction to determine the pole locations.[1] It is considerably more difficult, however, to determine the location of the zeros. In this report, we propose an approach for determining both poles and zeros by applying linear prediction analysis to the complex cepstrum of the signal rather than to the signal itself. We shall review briefly some of the properties of the complex cepstrum[2] and then discuss the use of the complex cepstrum to obtain poles and zeros using linear prediction.

2. Complex Cepstrum

With $s(n)$ denoting a sequence and $\hat{s}(n)$ its complex cepstrum and with $S(z)$ and $\hat{S}(z)$ denoting their z-transforms, the complex cepstrum is defined so that

$$\hat{S}(z) = \log S(z). \tag{1}$$

If $S(z)$ is a rational function of z, then $-z\frac{d\hat{S}(z)}{dz}$, the z-transform of $n\hat{s}(n)$, is also a rational function of z. In particular, with $S(z) = N(z)/D(z)$, $-z\frac{d\hat{S}(z)}{dz}$ is of the form

$$-z\frac{d\hat{S}(z)}{dz} = -z\frac{D(z)\,N'(z) - N(z)\,D'(z)}{N(z)\,D(z)}. \tag{2}$$

Thus the poles of $n\hat{s}(n)$ correspond to the poles and zeros of $s(n)$.

If $s(n)$ and $\hat{s}(n)$ are both causal, i.e., are both zero for $n < 0$, then $n\hat{s}(n)$ can be obtained directly from $s(n)$ by the recursion relation

$$n\hat{s}(n) = \frac{1}{s(0)}\left[ns(n) - \sum_{k=0}^{n-1} k\hat{s}(k)\, s(n-k)\right] \tag{3}$$

$$\hat{s}(0) = \log s(0).$$

The requirement that $\hat{s}(n)$ be causal is equivalent to requiring that all poles and zeros of $s(n)$ be inside the contour of integration associated with $\hat{S}(z)$. If it is assumed that $\hat{s}(n)$ is stable, then the requirement that $\hat{s}(n)$ be causal is equivalent to requiring that $s(n)$ be minimum phase, since all poles and zeros of $s(n)$ must then be inside the unit circle.

For minimum-phase sequences, the complex cepstrum can be computed directly from the recursion relation (3). Alternatively, it can be computed by first computing the cepstrum $c(n)$ for which the Fourier transform is the log magnitude of the Fourier transform of $s(n)$. The complex cepstrum $\hat{s}(n)$ is then obtained from $c(n)$ by means of the relation

$$\begin{aligned} \hat{s}(n) &= 2c(n) \qquad n > 0 \\ &= 0 \qquad\qquad n < 0 \end{aligned} \tag{4}$$

and

$$\hat{s}(0) = c(0).$$

If $s(n)$ is not minimum phase, but $s(n)$ is nevertheless computed according to (4), the complex cepstrum that is obtained will correspond to the minimum-phase counterpart of $s(n)$; i.e., it will be the complex cepstrum of a sequence $s_1(n)$ that has the same spectral magnitude as $s(n)$. If a zero of $S(z)$ occurs outside the unit circle, say at $z = z_1$, then there will be a corresponding zero of $S_1(z)$ at $z = 1/z_1$.

3. Determination of Poles and Zeros Using Linear Prediction on the Complex Cepstrum

Since, according to Eq. 2, the poles of $n\hat{s}(n)$ correspond to the poles and zeros of $s(n)$, if linear prediction is applied to $n\hat{s}(n)$ the roots of the resulting predictor polynomial will be the poles and zeros of $s(n)$. We note that since the z-transform of $n\hat{s}(n)$ also has zeros, the linear prediction must be applied to a portion of $n\hat{s}(n)$ that is not influenced by the zeros. As is usual in the linear prediction problem, this requires that if there are m zeros, linear prediction is applied for $n > m$.

By analyzing the sequence $n\hat{s}(n)$, we can, in principle, obtain a polynomial, which we denote $\hat{P}(z)$, the roots of which are the poles and zeros of $s(n)$. The analysis does not, however, indicate which of these roots are poles and which are zeros.

To separate the roots of the predictor polynomial into the poles and zeros of s(n), we can apply a linear prediction analysis to s(n) separately to obtain a predictor polynomial P(z) the roots of which are the poles of s(n). By dividing out these roots from $P_1(z)$, we can identify the remaining roots as the zeros of s(n).

For minimum-phase data, the complex cepstrum can be computed by using either the recursion relation of Eq. 3 or from the cepstrum obtained through computation of the Fourier transform and application of Eq. 4. For nonminimum-phase data there are several possibilities. If the recursion relation (3) is applied to the data, the resulting complex cepstrum is assumed to be causal and consequently will be unstable, since some of the roots of N(z) will lie outside the unit circle in the z plane. Nevertheless, since only a relatively short data length is required, linear prediction analysis can still be applied to this sequence and in principle the roots of the predictor polynomial obtained will still correspond to the poles and zeros of s(n). Since s(n) is nonminimum-phase, some of these roots, corresponding to zeros, will lie outside the unit circle. Under the assumption that the original sequence was causal and stable, these roots can then be identified immediately as zeros. The separation of the remaining roots into poles and zeros can then be accomplished by the same procedure as that discussed previously. While this procedure for nonminimum-phase data will work in principle, it would seem to be limited in practice to cases in which zeros do not lie at large distances outside the unit circle, since, in those cases, noise in the original data would tend to be amplified. An alternative procedure for nonminimum-phase data is to compute the complex cepstrum of the minimum-phase counterpart according to Eq. 4. Linear prediction analysis applied to this sequence yields a predictor polynomial whose roots correspond to the poles of s(n) and either the zeros or their reciprocals. Again, the poles and zeros can be separated as before. In some instances we may not be concerned with the phase of the zeros, i.e., whether they are inside the unit circle or should be reflected outside the unit circle. In speech analysis, for example, linear prediction is often applied as a first step in obtaining the spectral magnitude, in which case the phase of the zeros is not important. Similarly, in using the predictor coefficients for speech synthesis it is possible that the phase of the zeros need not be determined. If the phase of the zeros must be determined, this can be done by examining the error obtained with each of the two possible phases assigned to each zero. The one yielding the minimum error will then correspond to the correct phase assignment.

References

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