Multidisciplinary System Design Optimization (MSDO)

Design Space Exploration

Lecture 5

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Today's Topics

- Design of Experiments Overview
- Full Factorial Design
- Parameter Study
- One at a Time
- Latin Hypercubes
- Orthogonal Arrays
- Effects
- DoE Paper Airplane Experiment

Design of Experiments

- A collection of statistical techniques providing a systematic way to sample the design space
- Useful when tackling a new problem for which you know very little about the design space.
- Study the effects of multiple input variables on one or more output parameters
- Often used before setting up a formal optimization problem
  - Identify key drivers among potential design variables
  - Identify appropriate design variable ranges
  - Identify achievable objective function values
- Often, DOE is used in the context of **robust design**. Today we will just talk about it for design space exploration.

Design of Experiments

Design variables = **factors**

Values of design variables = **levels**

Noise factors = variables over which we have no control

e.g. manufacturing variation in blade thickness

Control factors = variables we can control

e.g. nominal blade thickness

Outputs = **observations** (= objective functions)

(Often an analysis code)

Matrix Experiments

- Each row of the matrix corresponds to one experiment.
- Each column of the matrix corresponds to one factor.
- Each experiment corresponds to a different combination of factor levels and provides one observation.

Expt No.	Factor A	Factor B	Observation
1	A1	B1	η_1
2	A1	B2	η_2
3	A2	B1	η_3
4	A2	B2	η_4

Here, we have two factors, each of which can take two levels.

Full-Factorial Experiment

- Specify levels for each factor
- Evaluate outputs at every combination of values
  - complete but expensive!

n factors

l levels

→ l^n observations

2 factors, 3 levels each:

$l^n = 3^2 = 9$ expts

4 factors, 3 levels each:

$l^n = 3^4 = 81$ expts

Expt No.	Factor	
	A	**B**
1	A1	B1
2	A1	B2
3	A1	B3
4	A2	B1
5	A2	B2
6	A2	B3
7	A3	B1
8	A3	B2
9	A3	B3

Fractional Factorial Experiments

- Due to the combinatorial explosion, we cannot usually perform a full factorial experiment
- So instead we consider just *some* of the possible combinations
- Questions:
  - How many experiments do I need?
  - Which combination of levels should I choose?
- Need to balance experimental cost with design space coverage

Fractional Factorial Design

Initially, it may be useful to look at a large number of factors superficially rather than a small number of factors in detail:

f_1	$l_{11}, l_{12}, l_{13}, l_{14}, \ldots$
f_2	$l_{21}, l_{22},, l_{23}, l_{24}, \ldots$
f_3	$l_{31}, l_{32},, l_{33}, l_{34}, \ldots$

many levels

vs.

f_1	l_{11}, l_{12}
f_2	l_{21}, l_{22}
$\vdots$	
$\vdots$	
f_n	l_{n1}, l_{n2}

many factors

DoE Techniques Overview

TECHNIQUE	COMMENT	EXPENSE (l=# levels, n=# factors)
Full factorial design	Evaluates all possible designs.	l^n - grows exponentially with number of factors
Orthogonal arrays	Don't always seem to work - interactions?	Moderate – depends on which array
One at a time	Order of factors?	$1+n(l-1)$ - cheap
Latin hypercubes	Not reproducible, poor coverage if divisions are large.	l - cheap
Parameter study	Captures no interactions.	$1+n(l-1)$ - cheap

Parameter Study

- Specify levels for each factor
- Change one factor at a time, all others at base level
- Consider each factor at every level

n factors

l levels

$1+n(l-1)$ evaluations

4 factors, 3 levels each:

$1+n(l-1) =$

$1+4(3-1) = 9$ expts

Expt No.	Factor			
	A	**B**	**C**	**D**
1	A1	B1	C1	D1
2	A2	B1	C1	D1
3	A3	B1	C1	D1
4	A1	B2	C1	D1
5	A1	B3	C1	D1
6	A1	B1	C2	D1
7	A1	B1	C3	D1
8	A1	B1	C1	D2
9	A1	B1	C1	D3

Baseline : A1, B1, C1, D1

Parameter Study

- Select the best result for each factor

Expt No.	Factor				Observation
	A	B	C	D	
1	A1	B1	C1	D1	η_1
2	A2	B1	C1	D1	η_2
3	A3	B1	C1	D1	η_3
4	A1	B2	C1	D1	η_4
5	A1	B3	C1	D1	η_5
6	A1	B1	C2	D1	η_6
7	A1	B1	C3	D1	η_7
8	A1	B1	C1	D2	η_8
9	A1	B1	C1	D3	η_9

1. Compare η_1, η_2, η_3 $\Rightarrow$ A*
2. Compare η_1, η_4, η_5 $\Rightarrow$ B*
3. Compare η_1, η_6, η_7 $\Rightarrow$ C*
4. Compare η_1, η_8, η_9 $\Rightarrow$ D*

"Best design" is A*,B*,C*,D*

- Does not capture interaction between variables

One At a Time

- Change first factor, all others at base value
- If output is improved, keep new level for that factor
- Move on to next factor and repeat

n factors → 1+*n*(*l*-1) evaluations ← *l* levels

4 factors, 3 levels each:

$1+n(l-1) =$

1+4(3-1) = 9 expts

Expt No.	Factor			
	A	**B**	**C**	**D**
1	A1	B1	C1	D1
2	A2	B1	C1	D1
3	A3	B1	C1	D1
4	A*	B2	C1	D1
5	A*	B3	C1	D1
6	A*	B*	C2	D1
7	A*	B*	C3	D1
8	A*	B*	C*	D2
9	A*	B*	C*	D3

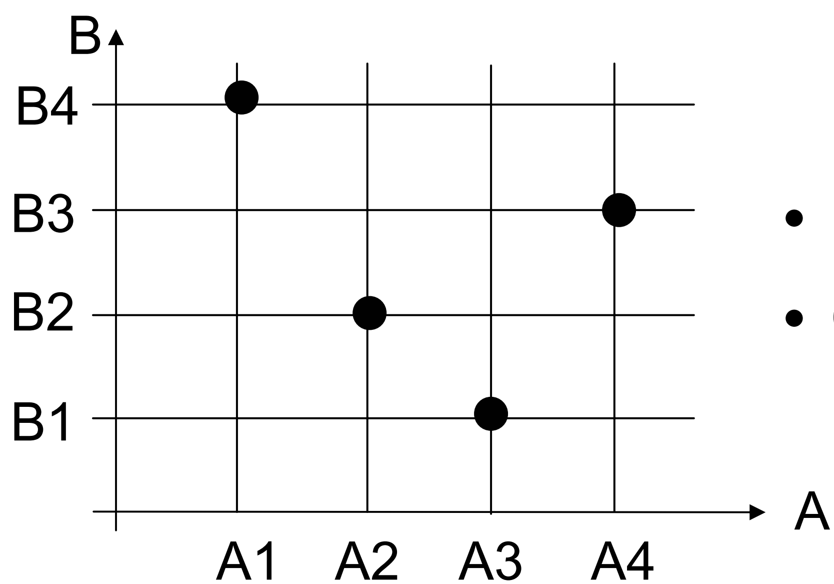
- Result depends on order of factors

Parameter Study vs. One at a Time

- Parameter study:
  - Chances are you will not actually evaluate the "best design" as part of your original experiment
  - "Best design" is chosen by extrapolating each factor's behavior, but interactions are not considered
- One at a Time:
  - The "best design" is a member of your matrix experiment
  - Some interactions are captured, even though the result depends on the order of the factors

Latin Hypercubes

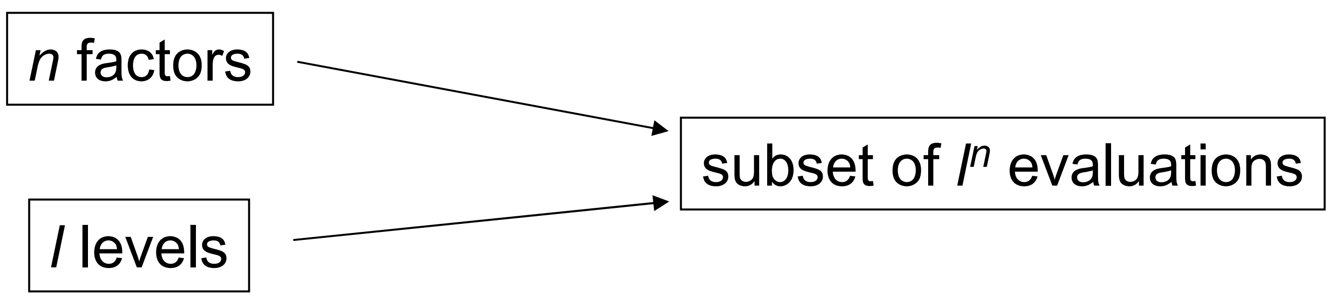
- Divide design space uniformly into *l* divisions for each factor
- Combine levels randomly
  - specify *l* points
  - use each level of a factor only once
- e.g. two factors, four levels each:



- Results not repeatable
- Can have poor coverage

Orthogonal Arrays

- Specify levels for each factor
- Use arrays to choose a subset of the full-factorial experiment
- Subset selected to maintain orthogonality between factors



- Does not capture all interactions, but is efficient
- Experiment is balanced

Orthogonal Arrays

Expt No.	Factor		
	A	**B**	**C**
1	A1	B1	C1
2	A1	B2	C2
3	A2	B1	C2
4	A2	B2	C1

$L_4(2^3)$

4 expts

2 levels

3 factors

Expt No.	Factor			
	A	**B**	**C**	**D**
1	A1	B1	C1	D1
2	A1	B2	C2	D2
3	A1	B3	C3	D3
4	A2	B1	C2	D3
5	A2	B2	C3	D1
6	A2	B3	C1	D2
7	A3	B1	C3	D2
8	A3	B2	C1	D3
9	A3	B3	C2	D1

$L_9(3^4)$

9 expts

3 levels

4 factors

Orthogonality

Notice that for any pair of columns, all combinations of factor levels occur and they occur an equal number of times.

This is the **balancing property**.

In general, the balancing property is sufficient for orthogonality.

There is a formal statistical definition of orthogonality, but we will not go into it here.

$L_9(3^4)$

Expt No.	Factor			
	A	B	C	D
1	A1	B1	C1	D1
2	A1	B2	C2	D2
3	A1	B3	C3	D3
4	A2	B1	C2	D3
5	A2	B2	C3	D1
6	A2	B3	C1	D2
7	A3	B1	C3	D2
8	A3	B2	C1	D3
9	A3	B3	C2	D1

All of the combinations (1-1, 1-2, 1-3, 2-1, 2-2, 2-3, 3-1, 3-2, 3-3) occur once for each pair of columns.

Effects

Once the experiments have been performed, the results can be used to calculate effects.

The *effect* of a factor is the change in the response as the level of the factor is changed.

- **Main effects**: averaged individual measures of effects of factors
- **Interaction effects**: the effect of a factor depends on the level of another factor

Often, the effect is determined for a change from a minus level (-) to a plus level (+) (2-level experiments).

Effects

Consider the following experiment:

– We are studying the effect of three factors on the price of an aircraft

– The factors are the number of seats, range and aircraft manufacturer

– Each factor can take two levels:

Factor 1: Seats	100<S1<150	150<S2<200
Factor 2: Range (nm)	2000<R1<2800	2800<R2<3500
Factor 3: Manufacturer	M1=Boeing	M2=Airbus

Main Effects

16.888
ESD.77

$L_8(2^3)$
(full factorial design)

Expt No.	Seats (S)	Range (R)	Mfr (M)	Price (observation)
1	S1	R1	M1	P_1
2	S1	R1	M2	P_2
3	S1	R2	M1	P_3
4	S1	R2	M2	P_4
5	S2	R1	M1	P_5
6	S2	R1	M2	P_6
7	S2	R2	M1	P_7
8	S2	R2	M2	P_8

The **main effect** of a factor is the effect of that factor on the output averaged across the levels of other factors.

Main Effects

Question: what is the main effect of manufacturer? *i.e.* from our experiments, can we predict how the price is affected by whether Boeing or Airbus makes the aircraft?

Expt No.	Seats (S)	Range (R)	Mfr (M)	Price (observation)
1	S1	R1	M1	P_1
2	S1	R1	M2	P_2
3	S1	R2	M1	P_3
4	S1	R2	M2	P_4
5	S2	R1	M1	P_5
6	S2	R1	M2	P_6
7	S2	R2	M1	P_7
8	S2	R2	M2	P_8

expts 1 and 2 differ only in the manufacturer

$$\frac{(P_2-P_1)+(P_4-P_3)+(P_6-P_5)+(P_8-P_7)}{4} = \text{main effect of manufacturer}$$

Main Effects – Another Interpretation

16.888 ESD.77

overall mean response:

$$m = \frac{P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 + P_8}{8}$$

avg over all expts when M=M1 :

$$m_{M1} = \frac{P_1 + P_3 + P_5 + P_7}{4}$$

effect of mfr level M1 $= m_{M1} - m$

effect of mfr level M2 $= m_{M2} - m$

Effect of factor level can be defined for multiple levels

main effect of mfr $= m_{M2} - m_{M1}$

Main effect of factor is defined as difference between two levels

NOTE: The main effect should be interpreted individually **only** if the variable does not appear to interact with other variables

Main Effect Example

Expt No.	Aircraft	Seats (S)	Range (R)	Mfr (M)	Price ($M)
1	**717**	S1	R1	M1	24.0
2	**A318-100**	S1	R1	M2	29.3
3	**737-700**	S1	R2	M1	33.0
4	**A319-100**	S1	R2	M2	35.0
5	**737-900**	S2	R1	M1	43.7
6	**A321-200**	S2	R1	M2	48.0
7	**737-800**	S2	R2	M1	39.1
8	**A320-200**	S2	R2	M2	38.0

$100 < S1 < 150$ $150 < S2 < 200$

$2000 < R1 < 2800$ $2800 < R2 < 3500$

M1=Boeing M2=Airbus

Sources:
Seats/Range data: Boeing Quick Looks
Price data: Aircraft Value News
Airline Monitor, May 2001 issue

Main Effect Example

overall mean price = 1/8*(24.0+29.3+33.0+35.0+43.7+48.0+39.1+38.0)
= 36.26

mean of experiments with M1 = 1/4*(24.0+33.0+43.7+39.1)
= 34.95

mean of experiments with M2 = 1/4*(29.3+35.0+48.0+38.0)
= 37.58

Main effect of Boeing (M1) = 34.95 – 36.26 = -1.3

Main effect of Airbus (M2) = 37.58 – 36.26 = 1.3

Main effect of manufacturer = 37.58 – 34.95 = 2.6

Interpretation?

Interaction Effects

We can also measure interaction effects between factors.

Answers the question: does the effect of a factor depend on the level of another factor?

e.g. Does the effect of manufacturer depend on whether we consider shorter range or longer range aircraft?

The interaction between manufacturer and range is defined as half the difference between the average manufacturer effect with range 2 and the average manufacturer effect with range 1.

$$\text{mfr} \times \text{range interaction} = \frac{\text{avg mfr effect with range 2} - \text{avg mfr effect with range 1}}{2}$$

Interaction Effects

16.888 ESD.77

range R1 : expts 1,2,5,6

range R2 : expts 3,4,7,8

Expt No.	Seats (S)	Range (R)	Mfr (M)	Price ($M)
1	S1	R1	M1	24.0
2	S1	R1	M2	29.3
3	S1	R2	M1	33.0
4	S1	R2	M2	35.0
5	S2	R1	M1	43.7
6	S2	R1	M2	48.0
7	S2	R2	M1	39.1
8	S2	R2	M2	38.0

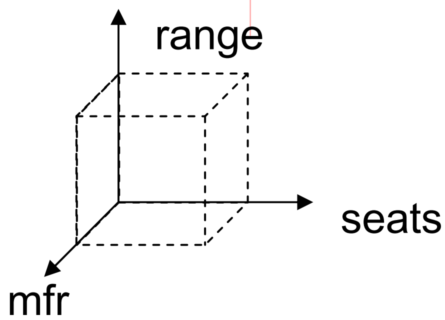
$$\text{avg mfr effect with range 1} = \frac{(P_2-P_1)+(P_6-P_5)}{2} = \frac{(29.3-24.0)+(48.0-43.7)}{2} = 4.8$$

$$\text{avg mfr effect with range 2} = \frac{(P_4-P_3)+(P_8-P_7)}{2} = \frac{(35.0-33.0)+(38.0-39.1)}{2} = 0.45$$

$$\text{mfr} \times \text{range interaction} = \frac{0.45-4.8}{2} = -2.2$$

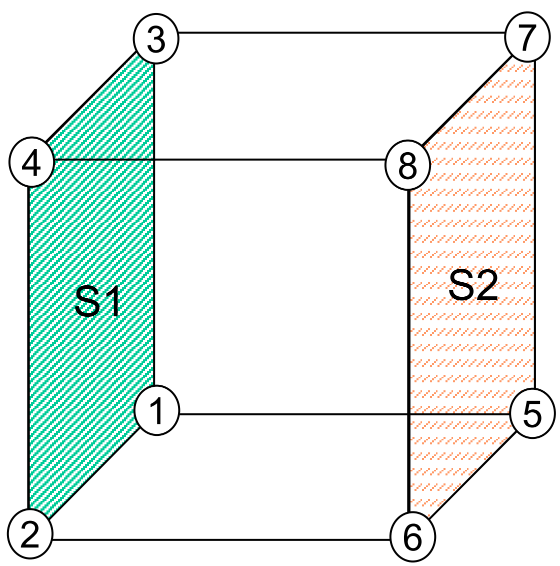
Interpretation?

Interpretation of Effects

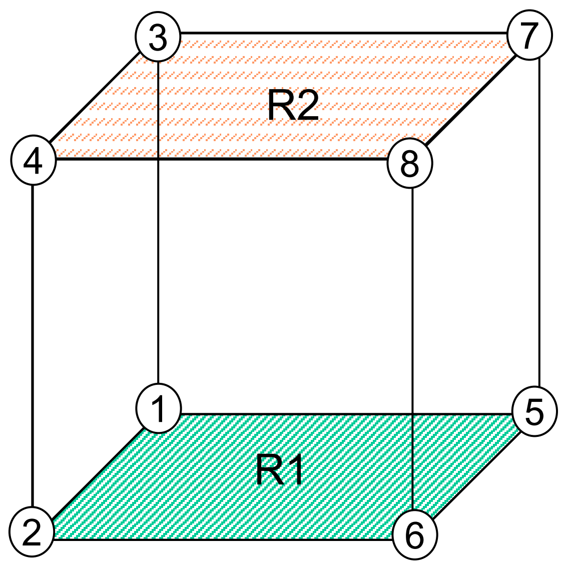


Main effects are the difference between two averages

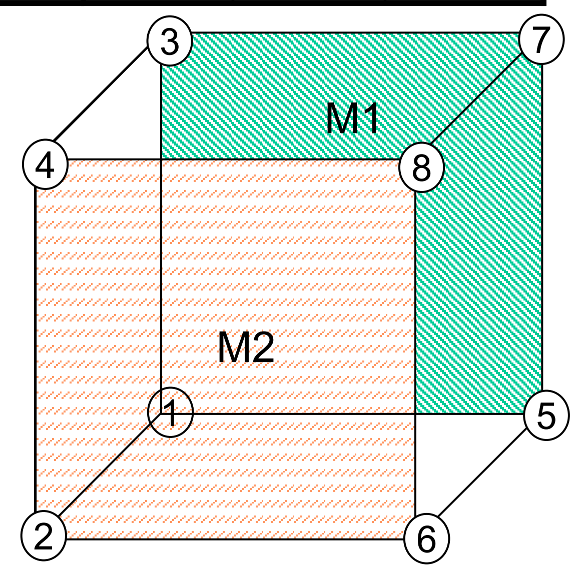
Expt No.	Seats (S)	Range (R)	Mfr (M)
1	S1	R1	M1
2	S1	R1	M2
3	S1	R2	M1
4	S1	R2	M2
5	S2	R1	M1
6	S2	R1	M2
7	S2	R2	M1
8	S2	R2	M2



seats



range

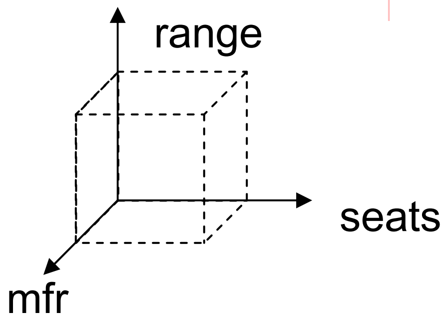


manufacturer

$$\text{main effect of mfr} = \frac{(P_8+P_6+P_4+P_2)-(P_7+P_5+P_3+P_1)}{4}$$

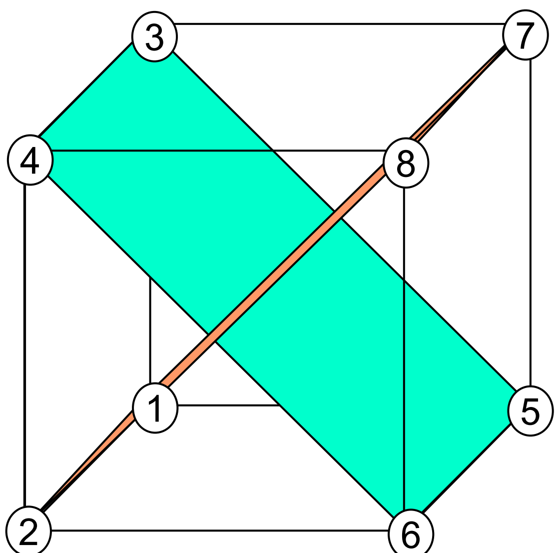
Adapted from Fig 10.2 Box, Hunter & Hunter

Interpretation of Effects

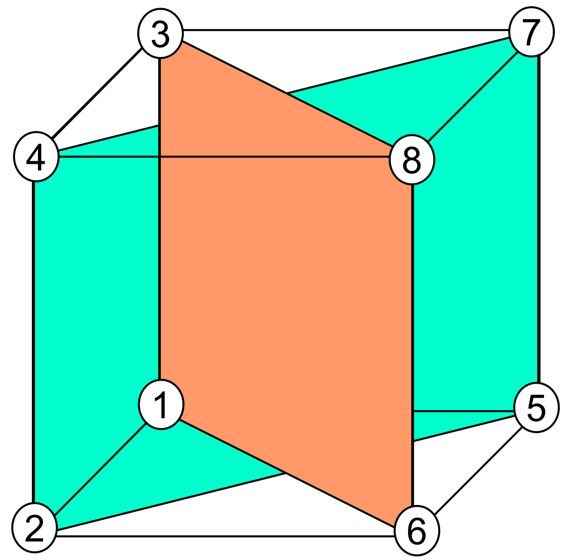


Interaction effects are also the difference between two averages, but the planes are no longer parallel

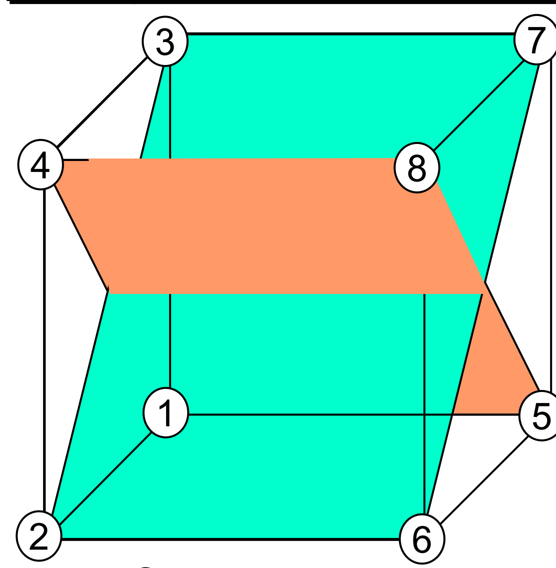
Expt No.	Seats (S)	Range (R)	Mfr (M)
1	S1	R1	M1
2	S1	R1	M2
3	S1	R2	M1
4	S1	R2	M2
5	S2	R1	M1
6	S2	R1	M2
7	S2	R2	M1
8	S2	R2	M2



seats × range



mfr × seats



mfr × range

mfr × range interaction $= \dfrac{(P_8+P_5+P_4+P_1)-(P_7+P_6+P_3+P_2)}{4}$

Adapted from Fig 10.2 Box, Hunter & Hunter

Design Experiment

Objective: Maximize Airplane Glide Distance

Design Variables:

- Weight Distribution
- Stabilizer Orientation
- Nose Length
- Wing Angle

Three levels for each design variable.

Experiment courtesy of Prof. Eppinger

Design Experiment

Full factorial design : 3^4=81 experiments

We will use an $L_9(3^4)$ orthogonal array:

Expt No.	Weight A	Stabilizer B	Nose C	Wing D
1	A1	B1	C1	D1
2	A1	B2	C2	D2
3	A1	B3	C3	D3
4	A2	B1	C2	D3
5	A2	B2	C3	D1
6	A2	B3	C1	D2
7	A3	B1	C3	D2
8	A3	B2	C1	D3
9	A3	B3	C2	D1

Design Experiment

Things to think about ...

Given just 9 out of a possible 81 experiments, can we predict the optimal airplane?

Do some design variables seem to have a larger effect on the objective than others (sensitivity)?

Are there other factors affecting the results (noise)?

References

- Phadke, : *Quality Engineering Using Robust Design*, Prentice Hall, 1995
- Box, G.; Hunter, W. and Hunter, J.: *Statistics for Experimenters*, John Wiley & Sons, 1978.