FEM formulation cont'd

The mathematical model:

Differential formulation :

it should be smooth in this case

$$\begin{cases} EA \dfrac{d^2u}{dx^2} + f^B = 0 & \text{within the bar} \quad (1) \\[2ex] \underbrace{EA \dfrac{du}{dx}\Big|_{x=L} = R}_{(a)} \; ; \; \underbrace{u\Big|_{x=0} = 0}_{(b)} & (2) \end{cases}$$

Principle of virtual displacement (Work)

$$\left(EA \frac{d^2u}{dx^2} + f^B\right) v = 0$$

v = arbitrary continuous virtual displ.
with $v|_{x=0} = 0$

$$\rightarrow \int_0^L \left(EA \frac{d^2u}{dx^2} + f^B\right) v \, dx = 0$$

Using a transformation, Eq (2a) We obtain

$$\int \frac{dv}{dx} EA \frac{du}{dx} \, dx = \int v f^B dx + R v\Big|_{x=L} \qquad (3)$$

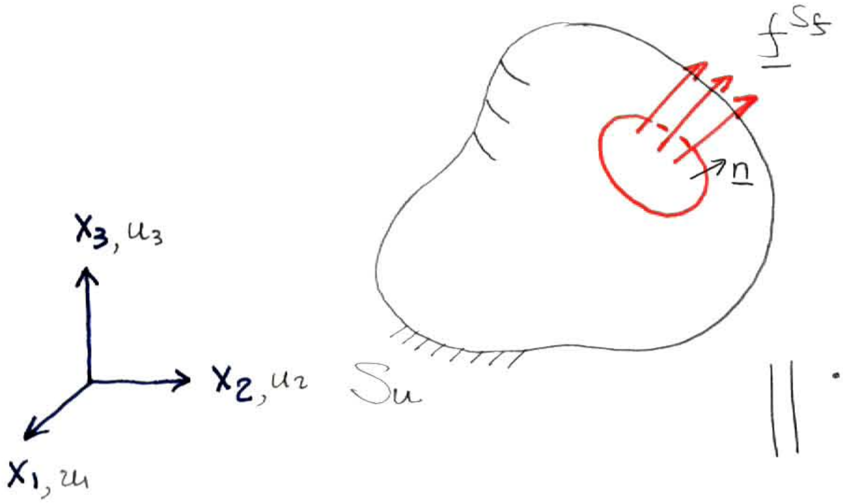
(v: infinite amount)

Variation formulation, generalized formulation.

in this case solution set is bigger than d.e.

General Case

- Differential formulation of the problem :



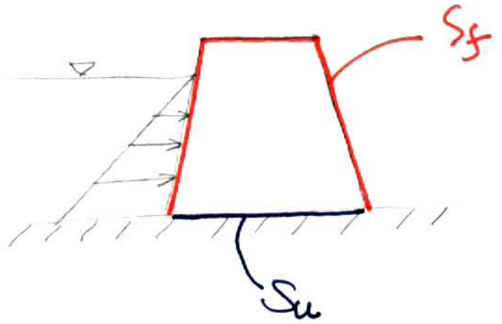
- 3D body supported on S_u

$S_u \cup S_f = S$

$S_u \cap S_f = \phi$

- Solve for the disp. strains, and stresses.

- Aside



$\underline{f}^B$ = body force / unit volumn

$\underline{f}^{S_f}$ = trantions applied per unit area.

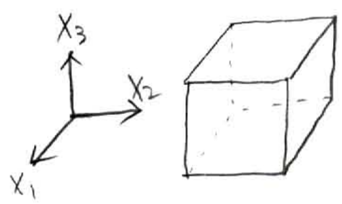
$\underline{n}$ = unit normal vector on S_f

For $i = 1, 2, 3$

$$\frac{\partial \tau_{ij}}{\partial x_j} + f_B = 0 \quad \text{(summing over)} \qquad \text{(A)}$$

- For $i=1$

$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + f_1^B = 0$$



Use (A) as in our simple example.

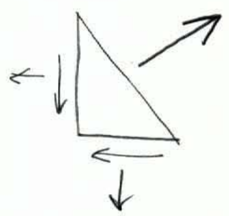
$$\int_V \left(\frac{\partial \tau_{ij}}{\partial x_j} + f_i^B \right) v_i \, dV = 0 \qquad \begin{cases} v_i(x_1, x_2, x_3) \\ \text{virtual displ.} \\ v_i\big|_{S_u} = 0 \end{cases} \qquad \text{(B)}$$

From (B), using a transformation (divergence theorem) and using (C) we obtain the general P.V.W.

$$\boxed{\tau_{ij}\, n_j = f_i^{S_f} \quad \text{on } S_f \qquad i = 1, 2, 3} \qquad \underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

⊙ Traction must be balanced with internal stresses.

(in 2D)



(C)

Aside

$$v \equiv \delta u \equiv \bar{u}$$

Only continuous is needed.

Now,

This satisfies eq (A)

(See 4.2)

$$\int_V \bar{\underline{\varepsilon}}^T \underline{\tau} \, dV = \int_V \bar{\underline{u}}^T \underline{f}^B \, dV + \int_{S_f} \bar{\underline{u}}^{S_f^T} \underline{f}^{S_f} \, dS_f \qquad \text{(D)}$$

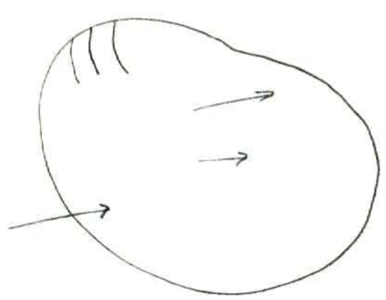
$$\underline{\tau} = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{bmatrix} = \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

$$X \equiv X_1$$
$$y \equiv X_2$$
$$z \equiv X_3$$

$$\overline{\underline{\varepsilon}} = \begin{bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{yy} \\ \bar{\varepsilon}_{zz} \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{yz} \\ \bar{\gamma}_{zx} \end{bmatrix} \qquad \overline{\underline{u}} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix} = \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix} \qquad \underline{f}^B = \begin{bmatrix} f_x^B \\ f_y^B \\ f_z^B \end{bmatrix}$$

$$\overline{\underline{u}}^{S_f} = \begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{bmatrix} \text{ on } S_f$$

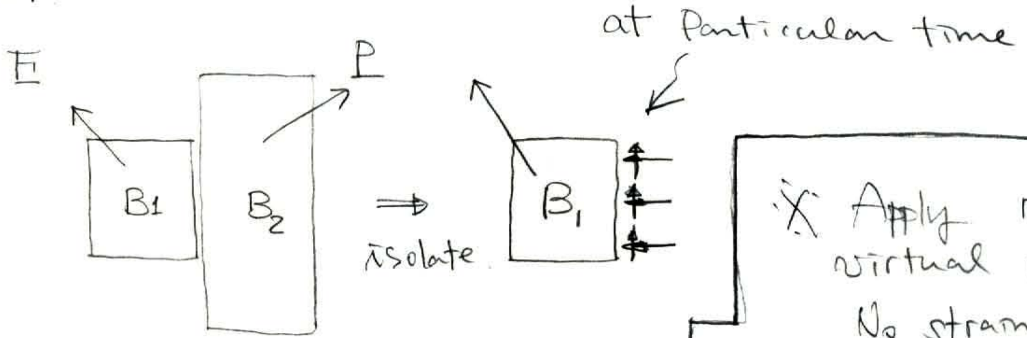
$$\underline{f}^B = \begin{bmatrix} \tilde{f}_x^B \\ \tilde{f}_y^B \\ \tilde{f}_z^B \end{bmatrix} \underbrace{- \rho \begin{bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{bmatrix}}_{\text{inertia force.}}$$



Assume :

$$S_f = S$$
$$S_u = \phi$$

Ex : contact



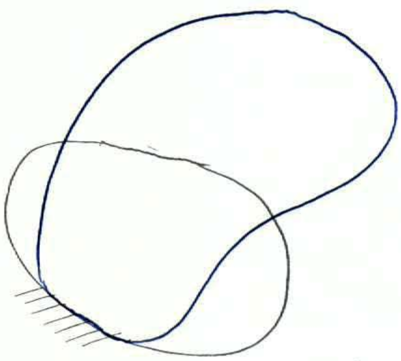
※ Apply rigid body virtual motion

No strain! R.H.S is also 0

① translation, ② Rotation.

→ Overall checking

Aside.



Still (C), & (A) are applicable.

We don't mention about how $\underline{\tau}$ is constructed here.

$$\int \underline{\bar{\varepsilon}}^T \underline{\tau} \, dV$$

$\blacksquare$