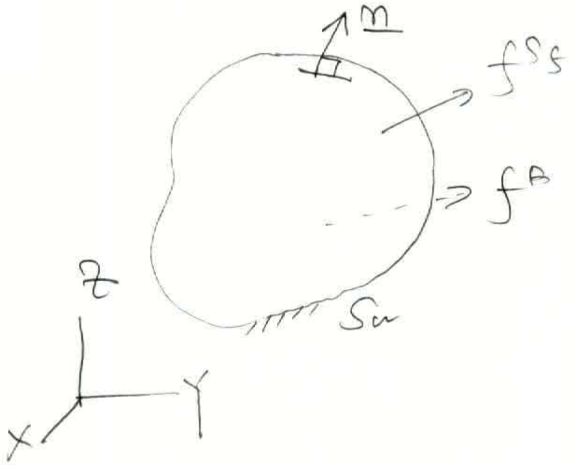
Finite element formulation – an example.



(1) $\begin{cases} \tau_{ij,j} + f_i^B = 0 & \text{in } V \\ \tau_{ij} n_j = f_i^{S_f} & \text{in } S_f \\ u_i|_{S_u} = 0 & \text{on } S_u \end{cases}$

Recall we set for the moment

$$S = S_f, \quad S_u = \phi$$

(2) $\underline{K}\,\underline{U} = \underline{R}$

$$\underline{K} = \sum_m \underline{K}^{(m)}, \quad K^{(m)} = \int \underline{B}^{(m)T} \underline{C}^{(m)} \underline{B}^{(m)} \, dV^{(m)}$$

$$\underline{R} = \underline{R}_B + \underline{R}_S, \quad \underline{R}_B = \sum \underline{R}_B^{(m)}; \quad \underline{R}_S = \sum \underline{R}_S^{(m)}$$

$$\underline{R}_B^{(m)} = \int H^{(m)T} f^{B(m)} \, dV^{(m)}$$

$$\underline{R}_S^{(m)} = \int H^{S(m)T} f^{S(m)} \, dS^{(m)}$$

$S^{(m)}$ = all external surface of element (m)

The key assumption.

$$\underline{u}^{(m)} = \underline{H}^{(m)} \underline{u} \quad (A)$$

$$\Downarrow$$

$$\underline{\varepsilon}^{(m)} = \underline{B}^{(m)} \underline{u} \quad (B)$$

In dynamic analysis

$$f^{B(m)} = \tilde{f}^{B(m)} - \rho^{(m)}\, \ddot{u}^{(m)} \quad ; \quad \tilde{f}^{B(m)} = \text{all body forces excluding inertia force}$$

Using (A)

$$\ddot{u}^{(m)} = H^{(m)}\, \ddot{U} \qquad \text{(C)}$$

Eq (2) now becomes

$$\underline{K}\,\underline{U} = \underline{R} - \left[\sum_m \int H^{(m)T} \rho^{(m)} H^{(m)}\, dV^{(m)}\right] \underline{\ddot{U}}$$

where $\underline{R}$ is constructing using $\underline{\tilde{f}}^{B(m)}$ & $\underline{f}^{S(m)}$

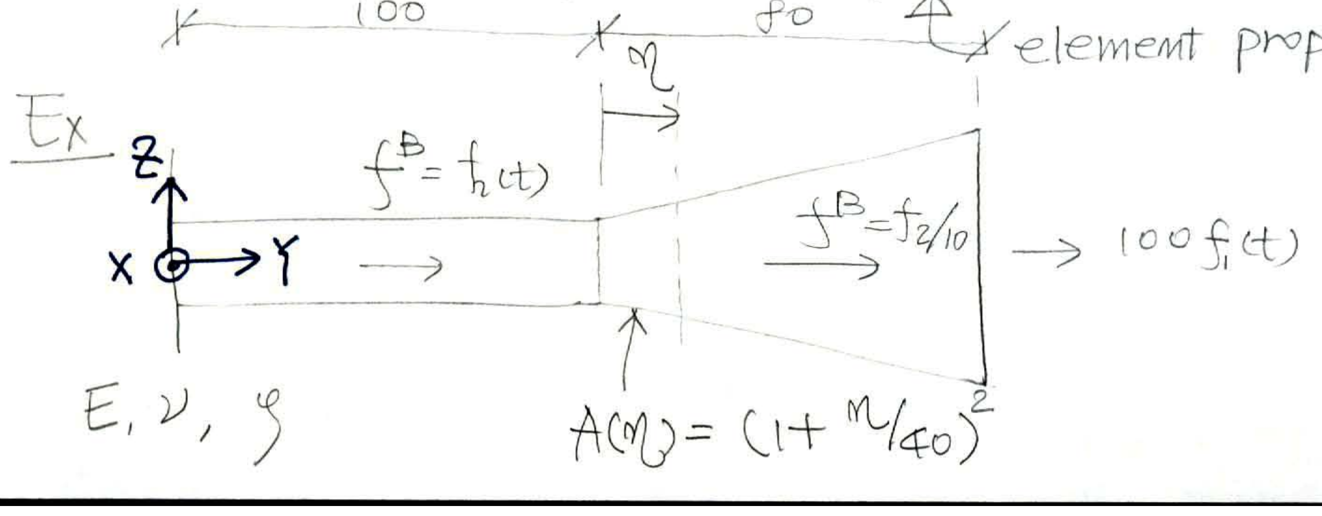
$$\underline{M}\,\underline{\ddot{U}} + \underline{K}\,\underline{U} = \underline{R} \quad ; \quad \underline{M} = \sum_m \underline{M}^{(m)} \; ; \; M^{(m)} = \int \underline{H}^{(m)T} \rho^{(m)} H^{(m)}\, dV^{(m)}$$

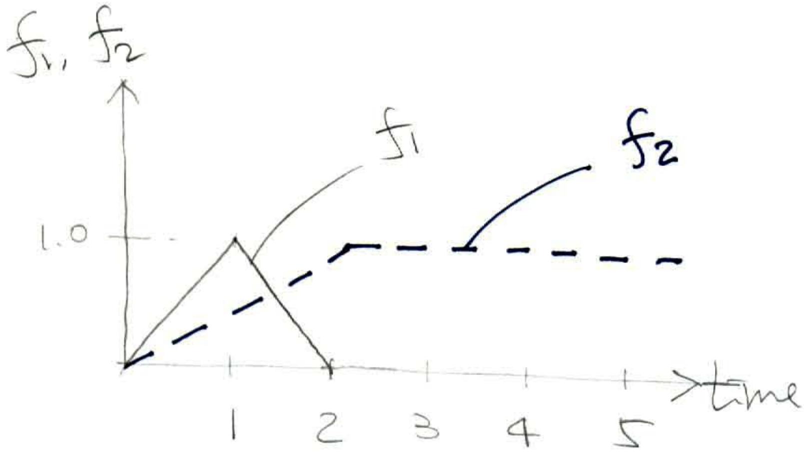
Similarly, we could attempt to include damping forces

$$f^{B(m)} = \tilde{f}^{B(m)} - \rho^{(m)}\ddot{u}^{(m)} - \kappa^{(m)}\dot{u}^{(m)} \; ; \; \dot{u}^{(m)} = H^{(m)}\dot{U}$$

element property.

Ex





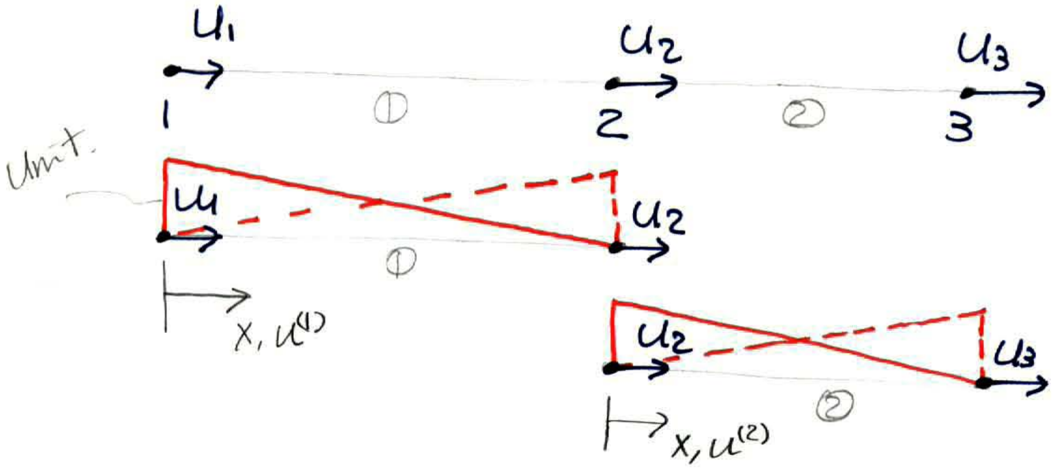
Math Model

Eq(1):

$$\frac{\partial}{\partial y}\left(EA\frac{\partial v}{\partial y}\right) = -f^B A$$

$$EA\frac{\partial v}{\partial y}\Big|_{y=180} = 100 f_1$$

$$v\big|_{y=0} = 0$$



$$u^{(1)}(x) = \underbrace{\left[\, 1-\frac{x}{100} \;\middle|\; \frac{x}{100} \;\middle|\; 0 \,\right]}_{\underline{H}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

U_3 doesn't affect

$$\ddot{u}^{(1)}(x,t) = \left[\, 1-\frac{x}{100} \;\middle|\; \frac{x}{100} \;\middle|\; 0 \,\right] \begin{bmatrix} \ddot{U}_1 \\ \ddot{U}_2 \\ \ddot{U}_3 \end{bmatrix}$$

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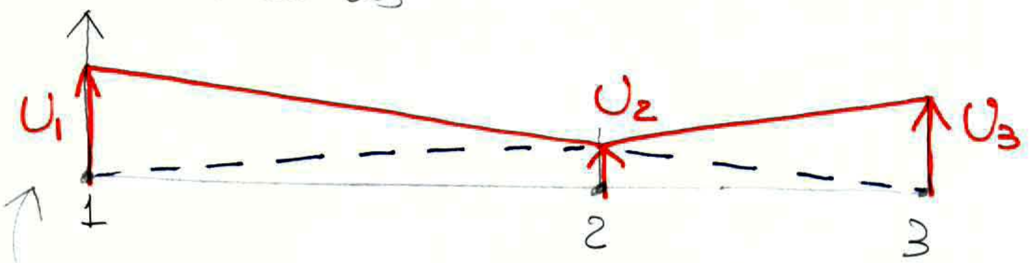
$$\underline{u}^{(2)} = \left[0 \;\vdots\; 1-\frac{x}{80} \;\vdots\; \frac{x}{80} \right] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Note: The displ in the structure are assumed as



In this problem, we set as 0

$$\ddot{u}^{(2)} = H^{(2)} \, \underline{\ddot{u}}$$

$$\underline{\varepsilon}^{(1)} = \varepsilon_{xx}^{(1)} = \frac{\partial u^{(1)}}{\partial x} = \left[-\frac{1}{100} \;\vdots\; \frac{1}{100} \;\vdots\; 0 \right] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\underline{\varepsilon}^{(2)} = \varepsilon_{xx}^{(2)} = \frac{\partial u^{(2)}}{\partial x} = \left[0 \;\vdots\; -\frac{1}{80} \;\vdots\; \frac{1}{80} \right] \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\underline{K} = \underline{K}^{(1)} + \underline{K}^{(2)} = \int_0^{100} (1) E \begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \\ 0 \end{bmatrix} \left[-\frac{1}{100} \;\vdots\; \frac{1}{100} \;\vdots\; 0 \right] dx$$

$$+ \int_0^{80} E \left(1+\frac{x}{40}\right)^2 \begin{bmatrix} 0 \\ -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left[0 \;\vdots\; -\frac{1}{80} \;\vdots\; \frac{1}{80} \right] dx$$

$$\underline{K} = \frac{E \cdot 1}{100} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13E}{240} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$1 < \; < 9$

make sense

$$\frac{13}{3}\frac{E}{80} \leftarrow \frac{EA^*}{L} \;;\; \underline{A^* = \frac{13}{3}}$$

$$\underline{K} = E \begin{bmatrix} \frac{1}{100} & -\frac{1}{100} & \\ -\frac{1}{100} & \frac{1}{100}+\frac{13}{240} & -\frac{13}{240} \\ 0 & -\frac{13}{240} & \frac{13}{240} \end{bmatrix} ;$$

Column Sum will be 0

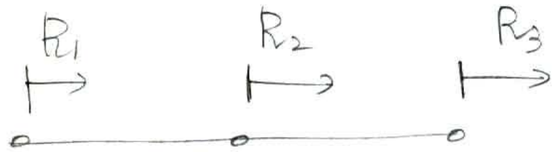
Assume $U_1 = 0,\ U_2 = 1,\ U_3 = 0$

$-\frac{1}{100}E$ $\quad\left(\frac{1}{100}+\frac{13}{240}\right)E$ $\quad -\frac{13}{240}E.$

$$\underline{R}_B = (1)\int_0^{100} \underline{H}^{(1)T} \underline{f}^{B(1)}(t)\, dx + \int_0^{80} \underline{H}^{(2)T} \underline{f}^{B(2)}(t) \underbrace{\left(1+\frac{x}{80}\right)^2 dx}_{dV}$$

$$\underline{R}_S = \begin{bmatrix} 0 \\ 0 \\ 100 f_1 \end{bmatrix} \equiv R^{S(2)} = \int_{S^{(2)}\big|_{x=80}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{100 f_1(t)}{9}\, dS$$

$$\underline{K} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \underline{R}$$



Singular, We cannot solve.

Rigid body Motion.

$\underline{\text{impose}}$

$U_1 = 0$

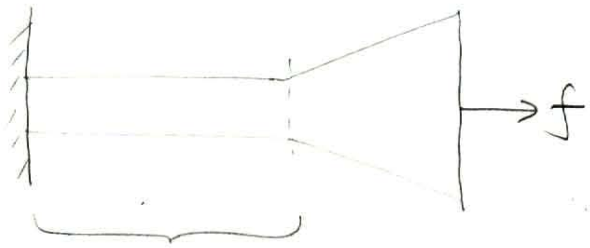
$$E \begin{bmatrix} \frac{1}{100} + \frac{13}{240} & -\frac{13}{240} \\ -\frac{13}{240} & \frac{13}{240} \end{bmatrix} \begin{pmatrix} U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} R_2 \\ R_3 \end{pmatrix}$$

Now calculate U_2, U_3, $(u_1 = 0)$ and obtain

$$\tau_{xx}^{(1)} = E \cdot \underline{B}^{(1)} \underline{U}$$

$$\tau_{xx}^{(2)} = E \cdot \underline{B}^{(2)} \underline{U}$$

Real stress is quadratic, so the FEM sol is Not exact.



In this case, we will get exact solution.