Convergence of the FEM

Problem to solve is:

(1a) $\tau_{ij,j} + f_i^B = 0$ within the body

(1b) $\tau_{ij} n_j \big|_{S_f} = f_i^{S_f}$

(1c) $u_i \big|_{S_u} = 0$

Recast the problem into

$$(2) \quad \int_{Vol} \overline{\underline{\varepsilon}}^T \underline{\tau} \, dVol = \int_{Vol} \overline{\underline{u}}^T \underline{f}^B \, dVol + \int_{S_f} \underline{u}^{S_f T} \underline{f}^{S_f} \, dS_f$$

$$\overline{u}_i \big|_{S_u} = 0$$

Short hand notation for (2)

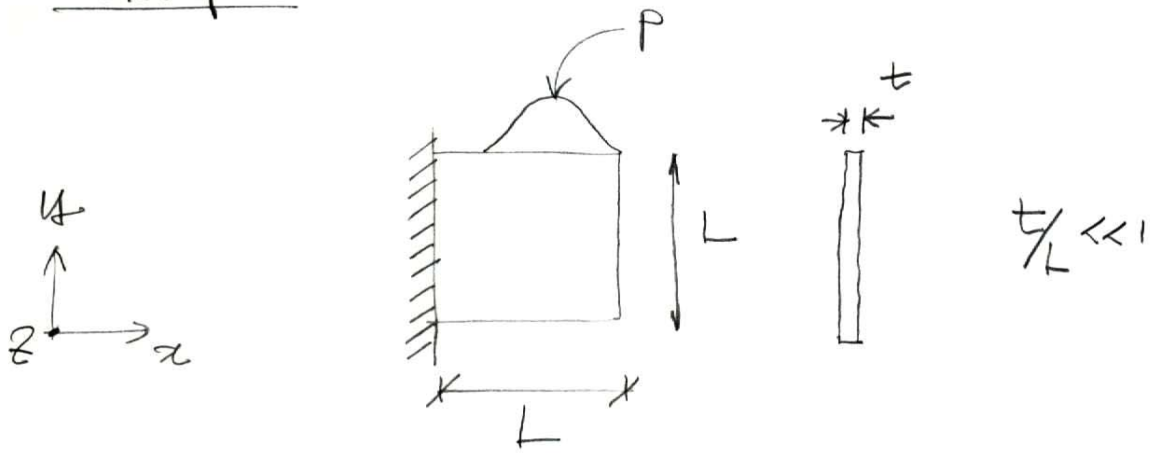
Find $\underline{u} \in V$ such that

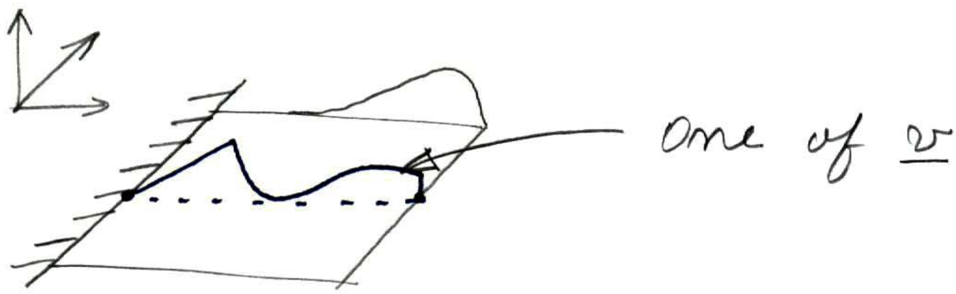
$$(3) \qquad a(u, v) = (f, v) \quad \forall \underline{v} \in V$$

V is the space of all continuous functions, zero on S_u

$\underline{u}$ = exact solution, $\underline{v}$ = virtual displacements

Example





Uniqueness of solution $\underline{u}$

Proof : Assume we have two solution $\underline{u}_1, \underline{u}_2$

Then we would have from (3)

$$a(\underline{u}_1, \underline{v}) = (\underline{f}, \underline{v}) \qquad (4.a)$$

$$a(\underline{u}_2, \underline{v}) = (\underline{f}, \underline{v}) \qquad (4.b)$$

Subtract (4b) from (4.a)

$$a(u_1, v) - a(u_2, v) = 0$$

$$a(u_1 - u_2, v) = 0 \qquad \forall \underline{v} \in V$$

but let $\underline{v} = \underline{u}_1 - \underline{u}_2$ then we have

$$a(u_1 - u_2, u_1 - u_2) = 0 \rightarrow \underline{u}_1 \equiv \underline{u}_2$$

Physically,

$a(w, w) =$ twice the strain energy in the body due to $\underline{w}$

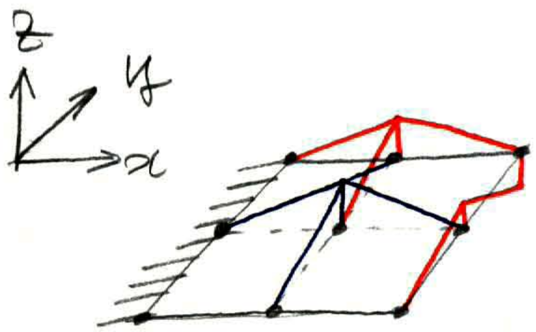
$$a(w, w) = \int_{Vol} \underline{\varepsilon}_w \underline{\tau}_w \, dVol \; ; \quad \varepsilon_w = \text{strain due to } w$$

FE Problem solution is

Find $\underline{u}_h \in V_h$ such that

$$(5) \qquad a(u_h, v_h) = (\underline{f}, v_h) \quad \forall \underline{v}_h \in V_h$$

V_h is the space of finite element functions: $V_h \subset V_h$.



$\therefore$ 6 linearly indep. ftn $\times \underbrace{2}_{u, v}$

Dimension 12

Let $\underline{e}_h$ = error = $\underline{u} - \underline{u}_h$

Properties

$$\boxed{\text{I.} \quad a(\underline{e}_h, \underline{v}_h) = 0 \qquad \forall\, \underline{v}_h \in V_h}$$

Proof

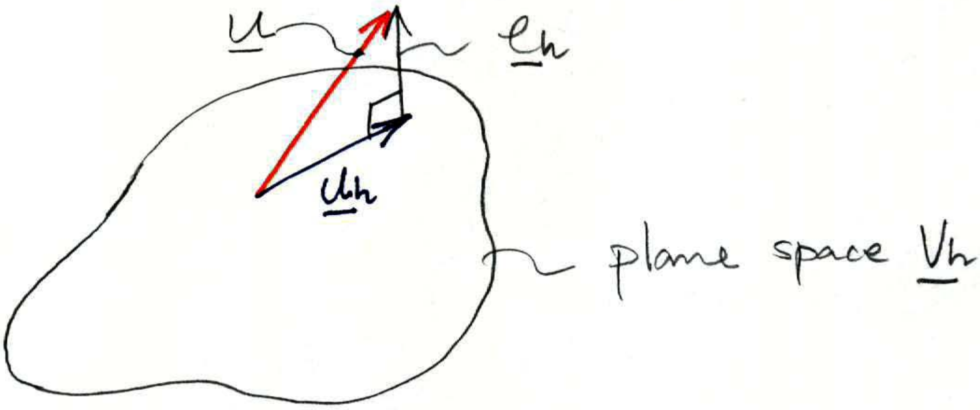
(6.a) $\quad a(u, v_h) = (f, v_h)$ from (3)

(6.b) $\quad a(u_h, v_h) = (f, v_h)$

Subtract :

$$a(\underline{u}, \underline{v}_h) - a(\underline{u}_h, \underline{v}_h) = 0$$

$$a(\underline{u} - \underline{u}_h, \underline{v}_h) = 0$$



$$\text{II.} \quad \boxed{a(\underline{u}, \underline{u}) \geq a(\underline{u}_h, \underline{u}_h)}$$

$$a(\underline{u}_h + \underline{e}_h, \underline{u}_h + \underline{e}_h) = a(\underline{u}_h, \underline{u}_h) + \overset{2}{\underbrace{a(\underline{e}_h, \underline{u}_h)}_{0}}$$
$$+ a(\cancel{\underline{u}_h}, \underline{e}_h) + a(\underline{e}_h, \underline{e}_h)$$

III. $\boxed{a(\underline{e}_h, \underline{e}_h) \leq a(\underline{u} - \underline{v}_h, \underline{u} - \underline{v}_h) \quad \forall \underline{v}_h \in V_h}$

Related w/ convergence

Proof : pick $\underline{w}_h \in \underline{V}_h$

$$a(\underline{e}_h + \underline{w}_h, \underline{e}_h + \underline{w}_h) = a(\underline{e}_h, \underline{e}_h) + \underbrace{a(\underline{w}_h, \underline{w}_h)}_{\text{always pos.}}$$

L.H.S is

$$a(\underline{u} - \underline{u}_h + \underline{w}_h, \underline{u} - \underline{u}_h + \underline{w}_h) \geq a(\underline{e}_h, \underline{e}_h)$$

pick $\underline{w}_h = \underline{u}_h - \underline{v}_h$

Pascal triangle

1

$x \quad y$ — $k=1$ 

$x^2 \quad xy \quad y^2$ — $k=2$ 

$x^3 \quad xy^2 \quad xy^2 \quad y^3$ — $k=3$

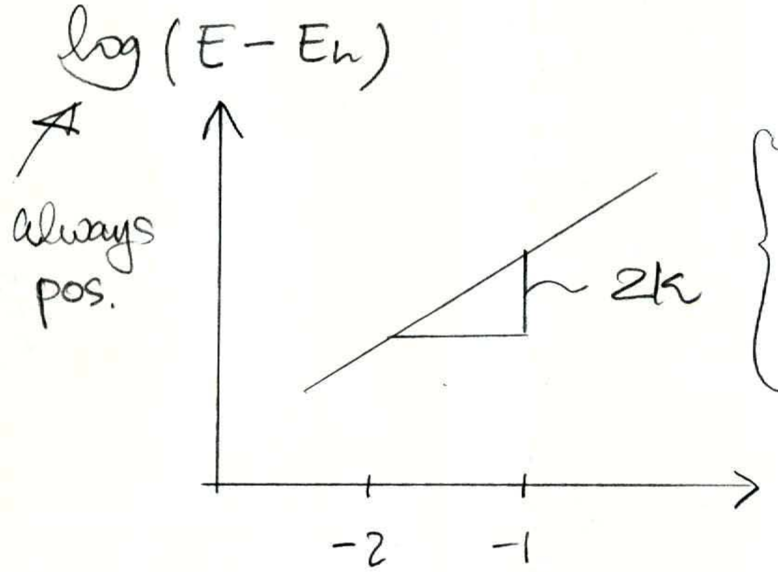
etc

- displ. error $\sim C\, h^{k+1}$
- strain / stress error $\sim C\, h^{k}$
- strain energy error $\sim C\, h^{2k}$

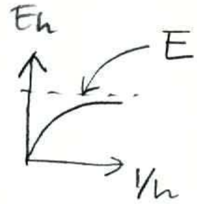
$$\underbrace{\tfrac{1}{2}a(u,u)}_{E} - \underbrace{\tfrac{1}{2}a(u_h,u_h)}_{E_h} \sim C\, h^{2k}$$

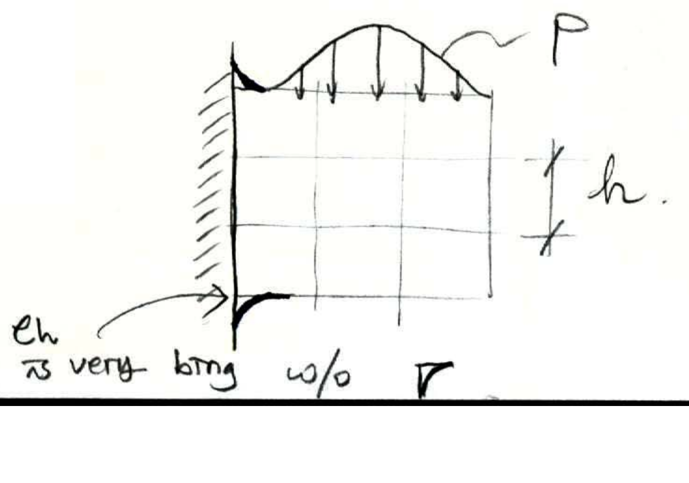
in practice

$$\log(E-E_h) = \log C + 2k \cdot \log h$$



In theory, we should get slope (2k)





Use graded mesh

e_h should be const. in X, Y, (Z)