Solution of Dynamic equations

$$\underline{M}\,\ddot{\underline{u}} + \underline{C}\,\dot{\underline{u}} + \underline{K}\,\underline{u} = R(t) \qquad (1)$$

Aside for ex.

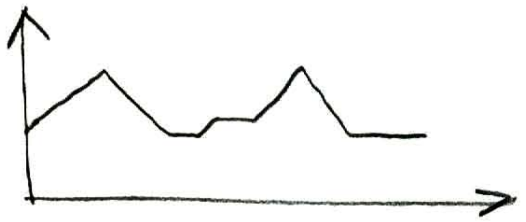
$$\underline{C} = \alpha\underline{M} + \beta\underline{K}$$

(2) $\;{}^{0}\dot{\underline{u}} = \dot{\underline{u}}(t=0)$

(3) $\;{}^{0}\underline{u} = \underline{u}(t=0)$

Note : $\;{}^{0}\ddot{\underline{u}} = \ddot{\underline{u}}(t=0)$ is given by Eq (1) !

loading $R_i(t)$



To solve Eq(1) in Finite element Analysis, we use.

- Direct integration
  - explicit method. e.g central difference.
  - implicit method. eg. Newmark method
- Mode superposition

Transform eq. (1)

$\underline{M} \Rightarrow$ can be a consistent mass matrix

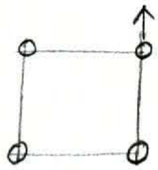
" " " lumped " "

lumping masses: calculate total mass of element and allocate reasonable fraction to its nodes

e.g.

2D

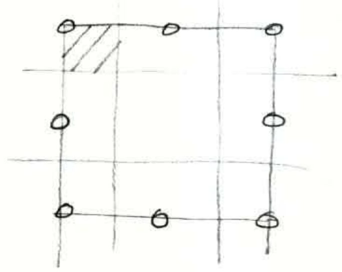
plane stress



$$m = \int_V \rho \, dV$$

$$* \quad \underline{M}_e = \frac{m}{4} \begin{bmatrix} 1 & & & \text{zeros} \\ & 1 & & \\ & & 1 & \\ \text{zeros} & & & 1 \end{bmatrix}$$

<u>e.g</u>



No unique way!

$$* \quad \underline{M}_c \Big|_{ee} = \int_V \underbrace{\underline{H}^T \rho \, \underline{H}}_{} \, dV$$

<u>another way</u>. → use only diagonal term

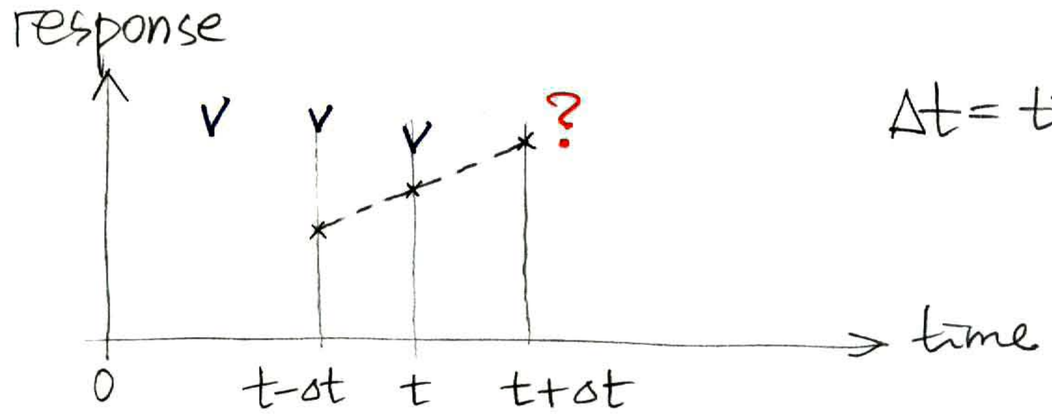
Wave propagation → lowerorder el.
lumped.

<u>lower order</u> ← linear displ.

Central difference method (explicit method)

(Collatz)

Solve eq (1) directly by time stepping



Δt = time step.

Assume "solution" is known at t, $t-\Delta t$ ...., calculate solution at time $t+\Delta t$

CDM :

Assume ${}^{t}\underline{U}$, ${}^{t-\Delta t}\underline{U}$, ... known

Caculate ${}^{t+\Delta t}\underline{U}$

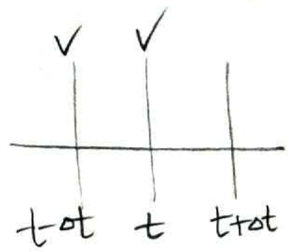
In explicit method, use eq (1) at time "t" to calculate ${}^{t+\Delta t}\underline{U}$ → conditionally stable

In implicit " , " " at time "$t+\Delta t$" to calculate ${}^{t+\Delta t}\underline{U}$ → always stable

$$(1)^{*} \quad \underline{M}\,{}^{t}\ddot{\underline{u}} + \underline{C}\,{}^{t}\dot{\underline{u}} + \underline{K}\,{}^{t}\underline{u} = {}^{t}\underline{R}$$

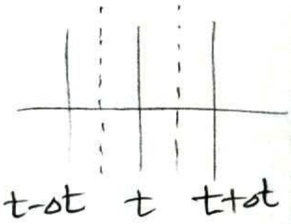
For ${}^{t}\ddot{\underline{u}}$, ${}^{t}\dot{\underline{u}}$ we use :

$$(2^{*}) \quad {}^{t}\dot{\underline{u}} = \frac{1}{2\Delta t}\left({}^{t+\Delta t}\overset{?}{\underline{u}} - {}^{t-\Delta t}\underline{u}\right)$$



$${}^{t}\ddot{u} = \left({}^{t+\Delta t/2}\dot{u} - {}^{t-\Delta t/2}\dot{u}\right)/\Delta t$$

$$(3^{*}) \quad = \frac{1}{\Delta t^2}\left({}^{t+\Delta t}\overset{?}{\underline{u}} - 2\,{}^{t}\underline{u} + {}^{t-\Delta t}\underline{u}\right)$$



$${}^{t+\frac{\Delta t}{2}}\dot{u} = \frac{1}{\Delta t}\left({}^{t+\Delta t}\underline{u} - {}^{t}\underline{u}\right)$$

$${}^{t-\frac{\Delta t}{2}}\dot{u} = \frac{1}{\Delta t}\left({}^{t}\underline{u} - {}^{t-\Delta t}\underline{u}\right)$$

Substitute in eq (1)

$$(A) \quad \underbrace{\left(\frac{1}{(\Delta t)^2}\underline{M} + \frac{1}{2\Delta t}\underline{C}\right)}_{\underline{K} \text{ is not here!}}{}^{t+\Delta t}\underline{u} = {}^{t}\underline{R} - \left(\underline{K} - \frac{2}{(\Delta t)^2}\underline{M}\right){}^{t}\underline{u} - \left(\frac{1}{(\Delta t)^2}\underline{M} - \frac{1}{2\Delta t}\underline{C}\right){}^{t-\Delta t}\underline{u}$$

By an analysis, we find that the CDM is conditionally stable, meaning $\Delta t \le \Delta t_{crit} = \frac{T_n}{\pi}$ ☆

T_n = smallest period in f.e. model.

We know that the f.e. model contains n frequencies

$$0 \le \omega_1 \le \omega_2 \le \ldots \le \omega_n$$

with n degrees of freedom in $\underline{U}$

$$T_i = 2\pi / \omega_i$$

In practice, we use $\underline{M}$ = lumped mass matrix

$\underline{C} = \underline{0}$

Then (A) is

$$\left(\frac{1}{(\Delta t)^2}\underline{M}\right)^{t+\Delta t}\underline{U} = \underbrace{{}^{t}\underline{R} - \overbrace{\underline{K}\,{}^{t}\underline{U}}^{{}^{t}\underline{F}} + \frac{2}{(\Delta t)^2}\underline{M}\,{}^{t}\underline{U} - \frac{1}{(\Delta t)^2}\underline{M}\,{}^{t-\Delta t}\underline{U}}_{{}^{t}\hat{\underline{R}}}$$

$$\underline{K}\,{}^{t}\underline{U} = \left(\sum_m \underline{K}^{(m)}\right){}^{t}\underline{U} = \sum_m \left(\underline{K}^{(m)}\,{}^{t}\underline{U}\right)$$

$$= \sum_m {}^{t}\underline{F}^{(m)} \; ; \quad {}^{t}\underline{F}^{(m)} = \underline{K}^{(m)}\,{}^{t}\underline{U}$$

$$= \int_{V^{(m)}} \underline{B}^{(m)T}\,\underline{C}\,\underline{B}^{(m)}\,dV^{(m)}\,\underline{U}$$

$${}^{t}\underline{F}^{(m)} = \int_{V^{(m)}} \underline{B}^{(m)T}\,\underline{\tau}^{(m)}\,dV^{(m)}$$

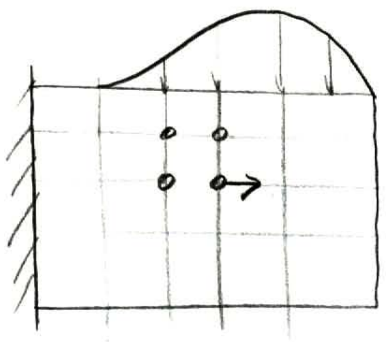
vector

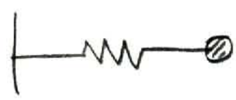
$$\therefore \underline{K}\,{}^{t}\underline{U} = \begin{bmatrix} \times & \\ \times & \\ \times & \circ \\ \times & \circ \\ & \circ \\ & \circ \\ & \\ \end{bmatrix} = {}^{t}\underline{F} = \sum_m {}^{t}\underline{F}^{(m)}$$

We have

$$ {}^{t+\Delta t}u_i = \frac{{}^{t}\hat{R}_i \cdot (\Delta t)^2}{m_{ii}} $$

<u>Ex</u>





$\omega = \sqrt{\frac{K}{m}}$.

if set that $\equiv 0$. $u_i\ \phi^{\infty}$, at ϕ_0. <u>my note</u>*