$$\underline{M}\,\ddot{\underline{u}} + \underline{C}\,\dot{\underline{u}} + \underline{K}\,\underline{u} = R(t) \qquad (1)$$

${}^{0}\underline{u}$ ${}^{0}\dot{\underline{u}}$ given initial conditions

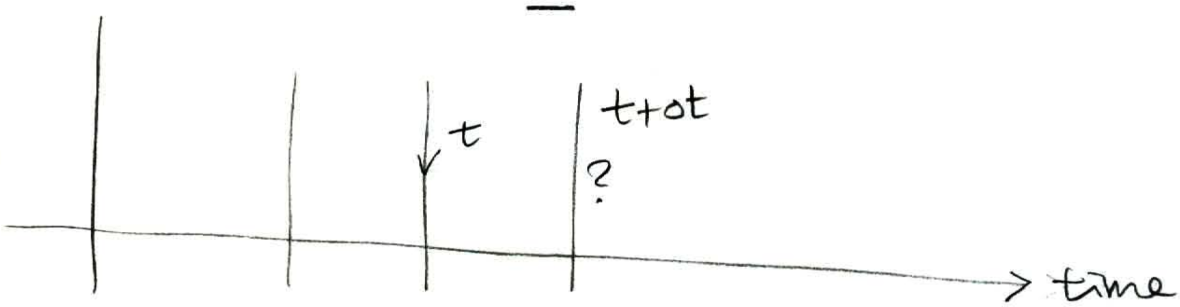
For solution:

1) Explicit method, we use central difference scheme.

Properties of explicit methods

use $$\underline{M}\,{}^{t}\ddot{\underline{u}} + \underline{C}\,{}^{t}\dot{\underline{u}} + \underline{K}\,{}^{t}\underline{U} = {}^{t}\underline{R} \qquad (a)$$

to obtain ${}^{t+\Delta t}\underline{u}$



Hence, we have conditional stability.

For CDM

$$\Delta t \leq \Delta t_{cr} = \frac{T_n}{\pi} \qquad T_n = \frac{2\pi}{\omega_n}$$

We neglect damping (i.e. $\underline{C} = 0$),

We use $\underline{M}$ = lumped mass matrix.

$${}^{t+\Delta t}u_i = {}^{t}\hat{R}_i \left(\frac{(\Delta t)^2}{m_{ii}} \right)$$

more on this later!

(2) Implicit methods

Use (1) at time $t+\Delta t$, to obtain the solution at time "$t+\Delta t$".

We use

$$\underline{M}\,{}^{t+\Delta t}\ddot{\underline{u}} + \underline{C}\,{}^{t+\Delta t}\dot{\underline{u}} + \underline{K}\,{}^{t+\Delta t}\underline{u} = {}^{t+\Delta t}\underline{R} \qquad (b)$$

unknowns

Newmark method

$$(c)\qquad {}^{t+\Delta t}\dot{\underline{u}} = {}^{t}\dot{\underline{u}} + \left[(1-\delta)\,{}^{t}\ddot{\underline{u}} + \delta\,{}^{t+\Delta t}\ddot{\underline{u}}\right]\Delta t$$

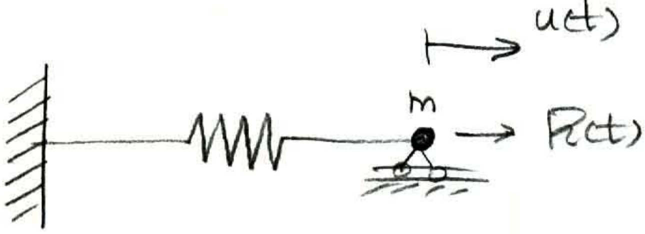
$${}^{t+\Delta t}\underline{u} = {}^{t}\underline{u} + {}^{t}\dot{\underline{u}}\,\Delta t + \left[\left(\tfrac{1}{2}-\alpha\right){}^{t}\ddot{\underline{u}} + \alpha\,{}^{t+\Delta t}\ddot{\underline{u}}\right](\Delta t)^2 \qquad (d)$$

Trapezoidal rule : $\delta = \frac{1}{2}$, $\alpha = \frac{1}{4}$ unconditional stability.

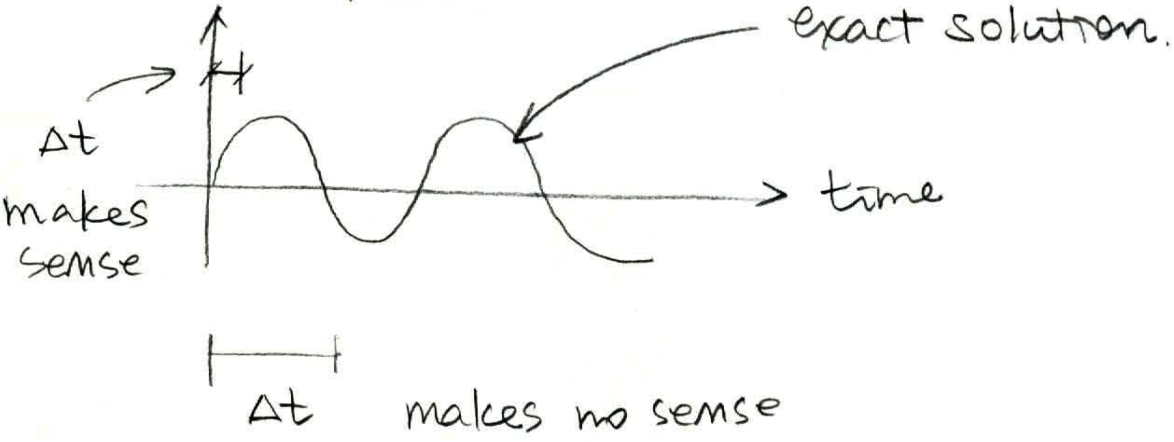
This means that for any Δt, the solution will not blow up.

Δt needs to be selected based on accuracy consideration.

e.g



Assume



Implicit method / trapezoidal rule

$$\hat{\underline{K}}\ {}^{t+\Delta t}\underline{u} = {}^{t+\Delta t}\hat{\underline{R}} \quad \leftarrow \text{like static} ☆$$

$$\hat{\underline{K}} = f_n\left(\underline{K}, \underbrace{\frac{1}{\Delta t^2}\underline{M}, \frac{1}{\Delta t}\underline{C}}_{\substack{\Delta t \downarrow \ \text{effect} \uparrow \\ \Delta t \uparrow \ \text{effect} \downarrow}}\right)$$

Mode superposition, transform eq (1) into a form that is "easier to integrate."

Mode superposition

(A) $\quad \underset{n\times 1}{\underline{u}(t)} = \underset{n\times n}{\underline{P}} \; \underline{X}(t) \;;\; \underline{P} =$ indep. of time.

(B) $\underbrace{\underline{P}^T \underline{M}\, \underline{P}}\, \underline{\ddot{X}} + \underbrace{\underline{P}^T \underline{C}\, \underline{P}}\, \underline{\dot{X}} + \underbrace{\underline{P}^T \underline{K}\, \underline{P}}_{\text{Symm } \forall P}\, \underline{X} = \underline{R}$

We want a "P" that diagonalizes $\underline{P}^T \underline{K}\, \underline{P}$

$\underline{P}^T \underline{M}\, \underline{P}$.

Consider the free-vibration eqns

(C) $\quad \underline{M}\,\underline{\ddot{u}} + \underline{K}\,\underline{u} = \underline{0}$

Assume solution

$$\underline{u} = \underline{\phi} \sin\omega(t-t_0)$$

then substitute into (C)

$$-\underline{M}\phi\,\omega^2 \sin\omega(t-t_0) + K\phi \sin\omega(t-t_0) = 0$$

$$\underline{K}\,\underline{\phi} = \omega^2 \underline{M}\,\underline{\phi} \qquad (D)$$

For a solution to exist (D) has to be satisfied

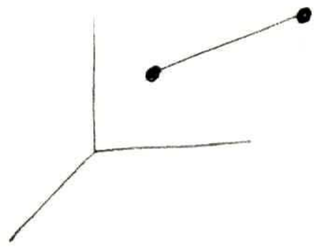
There are n solutions to (D)

$$\underline{K}\,\underline{\phi}_i = \omega_i^2\,\underline{M}\,\underline{\phi}_i \qquad i=1,2,\ldots n$$

(ϕ_i, ω_i^2) eigenpair

$$\begin{array}{cccc} 0 \le \omega_1^2 \le & \omega_2^2 & \cdots & \le \omega_n^2 \\ \phi_1 & \phi_2 & & \phi_n \end{array}$$

e.g. truss



$6 - 1 = 5$ //

We can prove

δ_{ij} = Kronecker delta

(i) $\underline{\phi}_i^T \underline{K}\,\underline{\phi}_j = \omega_i^2 \delta_{ij}$

(ii) $\underline{\phi}_i^T \underline{M}\,\underline{\phi}_j = \delta_{ij}$

ex

★ also rigid

multiple same frequencies

For distinct eig values

$$(\text{eg. } \omega_{i-1}^2 < \omega_i^2 < \omega_{i+1}^2)$$

$\underline{\phi}_i$ is unique

For eigenvalues of multiplicity m,
the eigen space of dimension m is unique
and we can always choose m eigen
vectors that satisfy (i) & (ii)

<u>take</u> $\quad \underline{P} = [\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_n]$