Solution of

$$\underline{M}\,\underline{\ddot{u}} + \underline{C}\,\underline{\dot{u}} + \underline{K}\,\underline{U} = \underline{R} \quad ; \quad {}^{0}\underline{u},\ {}^{0}\underline{\dot{u}} \qquad (1)$$

Direct integration : explicit method
Implicit method.

Mode superposition

↑ cheaper than direct integration.

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underset{m\times 1}{\underline{U}(t)} = \underset{n\times n}{\underline{\Phi}}\ \underset{n\times 1}{\underline{X}(t)} \qquad (2) \qquad = \sum_{i=1}^{n} \underline{\phi}_i\, x_i$$

$$\underset{n\times n}{\underline{\Phi}} = [\underline{\phi}_1, \underline{\phi}_2, \cdots \underline{\phi}_n] \qquad (3)$$

$$\underline{K}\,\underline{\phi}_i = \omega_i^2\, \underline{M}\,\underline{\phi}_i \qquad (4) \qquad n \text{ such sol's.}$$

$$0 \le \omega_1^2 \le \omega_2^2 \le \cdots \le \omega_n^2 \qquad (5)$$

$$\underline{\phi}_1 \quad \underline{\phi}_2 \qquad\qquad \underline{\phi}_n$$

Properties of eigen values/vectors — Proofs: 2.5 textbook.

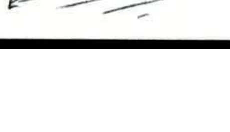
(a) If $\omega_{i-1} \le \omega_i < \underbrace{\omega_{i+1} < \omega_{i+2}}$ $\le \omega_{i+3}$

then $\underline{\phi}_{i+1}$ is unique.

(b) If ω_i has multiplicity m, the the eigenspace of dimension m is unique, but any vector in that space is an eigenvector.

e.g $m = 2$

$$\omega_{i-1} < \omega_i = \omega_{i+1} < \omega_{i+2}$$



Pick "a" $\underline{\phi}_i$, pick "a" $\underline{\phi}_{i+1}$ $(\omega_i = \omega_{i+1})$

$\alpha \cdot | \ \underline{K}\,\underline{\phi}_i = \omega_i^2 \underline{M}\,\underline{\phi}_i$

$\beta \cdot | \ \underline{K}\,\underline{\phi}_{i+1} = \omega_i^2 \underline{M}\,\underline{\phi}_{i+1}$

$$K(\underbrace{\alpha \underline{\phi}_i + \beta \underline{\phi}_{i+1}}) = \omega_i^2 \underline{M}(\underbrace{\alpha \phi_i + \beta \phi_{i+1}})$$

Hence
$(\alpha \underline{\phi}_i + \beta \underline{\phi}_{i+1})$ is also eigen vector.

Key point

$$\underbrace{\underbrace{(\underline{K}\,\underline{x})} = \overset{\text{scalar}}{\lambda} \underbrace{(\underline{M}\,\underline{x})}}_{\text{same direction}}$$

(c) $\underline{\phi}_i^T \underline{M}\,\underline{\phi}_j = \delta_{ij}$

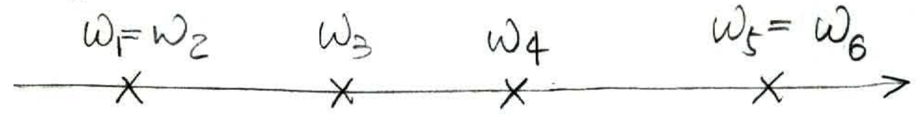
the $\underline{\phi}_i$ are $\underline{M}$ orthonormal

" $\underline{\phi}_i$ " $\underline{K}$ orthogonal.

The eigenspace are orthogonal to each other!
(Ex. 8.32)

geometrically.

Ex

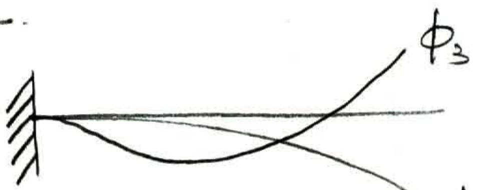


$$\underline{K}\,\underline{\phi}_5 = \omega_5^2\,\underline{M}\,\underline{\phi}_5$$

this not unique.

but orthogonal
to others.

<u>Physically</u>



ϕ_2 : out of the paper.
same shape.

ϕ_4 : out of the paper
same shape.

Substitute (2) into (1)

$$\ddot{\underline{X}} + \underline{\underline{\Phi}}^T \underline{C}\, \underline{\underline{\Phi}}\, \dot{\underline{X}} + \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix} \underline{X} = \underline{\underline{\Phi}}^T \underline{R}$$

Hence, $\underline{\underline{\Phi}}^T \underline{M}\, \underline{\underline{\Phi}} = I$

(6) $\underline{\underline{\Phi}}^T \underline{K}\, \underline{\underline{\Phi}} = \begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix} = \mathrm{diag}(\omega_i^2)$

<u>Assume</u>

(7) $\underline{\underline{\Phi}}^T \underline{C}\, \underline{\underline{\Phi}} = \begin{bmatrix} 2\xi_1\omega_1 & & \\ & \ddots & \\ & & 2\xi_n\omega_n \end{bmatrix} = \mathrm{diag}(2\xi_i\omega_i)$

modal damping.

(6) with (7) is

$$\ddot{x}_i + 2\xi_i\omega_i\dot{x}_i + \omega_i^2 x_i = r_i \qquad r_i = \underline{\phi}_i^T \underline{R}$$

From (2)

$${}^0\underline{U} = \underline{\Phi}\,{}^0\underline{X}$$

$$\underline{\Phi}^T\underline{M}\,{}^0\underline{U} = \underbrace{\underline{\Phi}^T\underline{M}\,\underline{\Phi}}_{I}\,{}^0\underline{X} \rightarrow {}^0\underline{X} = \underline{\Phi}^T\underline{M}\,{}^0\underline{U}$$

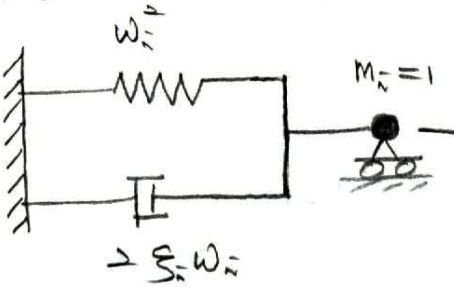
$${}^0x_i = \underline{\phi}_i^T\underline{M}\,{}^0\underline{U}$$

$${}^0\dot{x}_i = \underline{\phi}_i^T\underline{M}\,{}^0\dot{\underline{U}}$$

I.C.

Reformulated Eq. (1) into other basis.

Eqn (8)



r_i, ${}^0x_i$, ${}^0\dot{x}_i$

In practice, we do not need to solve/consider for all (ω_i, ϕ_i)

Case 1 : Assume $\underline{R} = \underline{0} \rightarrow \Gamma_i = 0 \quad \forall i$

Assume $^\circ\underline{u} = \alpha \underline{\phi}_1$, $\underbrace{^\circ\dot{\underline{u}} = \underline{0}}$

$\dot{X}_i = 0 \quad \forall i$

$$^\circ X_i = \underline{\phi}_i^T \underline{M} (\alpha \underline{\phi}_1)$$

$$^\circ X_1 = 1 \cdot \alpha \quad , \quad ^\circ X_2 , \cdots \, ^\circ X_n = 0$$

$$\boxed{\underline{u}(t) = \underline{\phi}_1 X_1}$$

Note

$$^\circ\underline{u} = \sum_{i \in \mathcal{J}} \alpha_i \phi_i \qquad u \in V(\phi_i)_{i \in \mathcal{J}}$$

Case 2 : $^\circ u = {}^\circ\dot{\underline{u}} = 0 \qquad \alpha(t)$

$$\underline{R} = \alpha \underline{M} \phi_1 \qquad \Gamma_i = \underline{\phi}_i^T (\alpha \underline{M} \phi_1)$$

$$^\circ x_i = {}^\circ\dot{x}_i = 0 \quad \forall i \qquad \Gamma_1 = \alpha(t)$$

$$\Gamma_i = 0 \quad (2 \le i \le n)$$

$$\boxed{\underline{u}(t) = \underline{\phi}_1 X_1}$$

Case 3 :