Solution of generalized EVP

$$K\phi = \lambda M \phi \quad (1)$$

(n typically 10,000 or more

We have n solutions

$$K\phi_i = \lambda_i M \phi_i \quad (2) \qquad \lambda_i = \omega_i^2$$

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$
$$\phi_1 \quad \phi_2 \qquad \qquad \phi_n$$

We note

a) λ_i can be zero $i = 1, 2, \ldots$

b) λ_i can be ∞

Recall we can have the consistent mass matrix $\underline{M_c}$, the lumped mass matrix $\underline{M_\ell}$

$$M_\ell = \begin{bmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix} ; \quad \underline{K}\,\underline{\phi_i} = \lambda_i \underline{M}\,\underline{\phi_i}$$

Corresponding to each 0-mass we have

an inf frequency

From (1)

$$\underline{M}\underline{\phi} = \frac{1}{\lambda}\underline{K}\underline{\phi} = \alpha\,\underline{K}\,\underline{\phi} \quad (1a)$$

If we can satisfy (1a) with some nontrivial vector $\underline{\phi}$, then this vector with the associate value α is an eigenvector

Use $\underline{\phi}_i = \underline{e}_i$, i corresponds to the zero mass, $\underline{e}_i$ = vector of zeros with 1 in the i^{th} entry

$$\underline{\phi}_6 = \underline{e}_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \underline{\phi}_7 = \underline{e}_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\alpha_6 = 0 \qquad \alpha_7 = 0$$

$$\lambda_6 = \infty \qquad \lambda_7 = \infty$$

In practice, we sometimes "get rid of" the zero mass dof. Use static condensation (Gauss eli on the zero mass dofs)

Write $\underline{K}\underline{\Phi} = \lambda \underline{M} \underline{\phi}$ as

$$\begin{pmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix} = \lambda \begin{pmatrix} M_a & \\ & \underline{0} \end{pmatrix} \begin{pmatrix} \phi_a \\ \phi_b \end{pmatrix}$$

We eliminate $\underline{\phi}_b$

$$\underline{K}_{ba} \underline{\phi}_a + \underline{K}_{bb} \underline{\phi}_b = 0 \qquad \underline{\phi}_b = -\underline{K}_{bb}^{-1} \underline{K}_{ba} \underline{\phi}_a$$

$$\underbrace{\left(\underline{K}_{aa} - \underline{K}_{ab} \underline{K}_{bb}^{-1} \underline{K}_{ba}\right)}\phi_a = \lambda \underline{M}_a \underline{\phi}_a$$

pull out clamp where there is no mass

We also know that

$$\underline{\Phi}^T \underline{K} \underline{\Phi} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} ; \quad \underline{\Phi}^T \underline{M} \underline{\Phi} = \underline{I} \quad (3)$$

(assume $m_{ii} \neq 0$)

$$\underline{\Phi} = \left[\underline{\phi}_1 \ \underline{\phi}_2 \cdots \phi_n\right]$$

We note from (2)

$$\underbrace{(\underline{K} - \lambda_i \underline{M})}_{\text{singular}} \underline{\phi}_i = 0$$

$\underline{A}\,\underline{x} = 0$

to have n.t sol. $|A| = 0$

$$\det(\underline{K} - \lambda_i \underline{M}) \equiv 0$$

Hence, the λ_i's correspond to the zeros of $p(\lambda) = \det(\underline{K} - \lambda \underline{M})$

Big Conclusion :

1) There are n solutions

2) If $n > \underline{\underline{4}}$ we need iteration to solve for the λ_i's

* closed form : up to 4
* Expensive.

* also Approximate method available don't give the exact solution

* In general, there are 4 categories of Solution Methods

1) Vector iteration methods

e.g. Inverse iteration

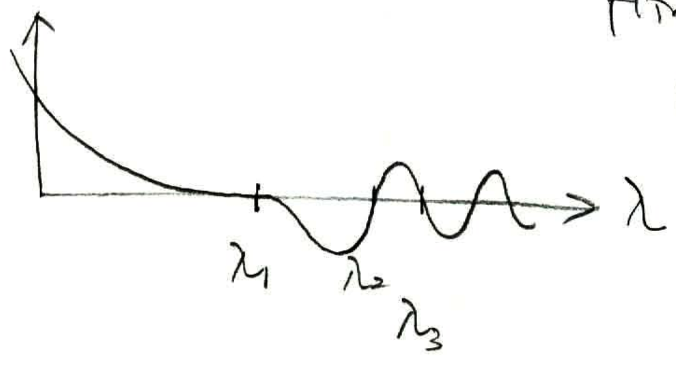
$\underline{K}\,\underline{X}_{k+1} = \underline{M}\,\underline{X}_k$, $\underline{X}_1$ = starting vector

$\underline{X}_k \rightarrow \alpha\,\underline{\phi}_1$ under condition.

2) Transformation methods

Jacobi, QR, ...

3) Find the roots of $p(\lambda)$



e.g.: Determinant Search method.

4) Use the Sturm sequence property (later discussed)

In addition we can combine basic techniques.

Inverse iteration.

$$\underline{K}\,\underline{x}_{k+1} = \underline{M}\,\underline{x}_k \qquad \underline{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \text{ for ex.}$$

$$\underline{x}_{k+1} \to \alpha\,\underline{\Phi}_1$$

R.Q

Aside

$$p(\lambda_i) = \det(\underbrace{\underline{K} - \lambda_i \underline{M}}) = \det(\underline{L}\,\underline{D}\,\underline{L}^T)$$

$$= d_1\, d_2 \cdots d_n$$

$$k_{ii} \sim 10^6$$

then $d_{ii} \sim 10^6 \quad (i=1, \; n-1)$

$d_{nn} \sim 10^{-2}$

① $\sim 10^{60{,}000}$

② $\sim 10^{60{,}000-8}$ ← as 0

Define

$$\underline{x}_k = \underset{n\times n}{\underline{\Phi}}\, \underline{z}_k$$

$$\underline{\Phi}^T \underline{K}\, \underline{\Phi}\, \underline{z}_{k+1} = \underline{\Phi}^T \underline{M}\, \underline{\Phi}\, \underline{z}_k$$

$$\underline{\Lambda}\, \underline{z}_{k+1} = \underline{I}\, \underline{z}_k$$

Now start $\underline{z}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$; $\underline{z}_{k+1} = \begin{pmatrix} (1/\lambda_1)^k \\ (1/\lambda_2)^k \\ \vdots \\ (1/\lambda_n)^k \end{pmatrix}$

Scale z_{k+1} $= \begin{pmatrix} 1 \\ (\lambda_1/\lambda_2)^k \\ \vdots \\ (\lambda_1/\lambda_n)^k \end{pmatrix}$

If $\lambda_2 > \lambda_n$ then

We converge to the 1^{st} eigen vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{\underline{\Lambda}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

%

$\underline{X}_1^T \underline{M} \, \underline{\Phi}_1 \neq 0$