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# Neo-Fregeanism Reconsidered\*

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## 1 Platonism

Mathematical Platonism is the view that mathematical objects exist. *Traditional* Platonists believe that a world with no mathematical objects is consistent; *subtle* Platonists believe that such a world would be inconsistent.

The easiest way of getting a handle on traditional Platonism is by imagining a creation myth. On the first day God created light; by the sixth day, she had created a large and complex world, including black holes, planets and sea-slugs. But there was something left to be done. So on the seventh day she created mathematical objects. Only *then* did she rest. On this view, it is easy to make sense of a world with no mathematical objects: it is just like the world we are considering, except that God rested on the seventh day.

The crucial feature of this creation myth is that God needed to do something *extra* in order to bring about the existence of mathematical objects: something that wasn't already in place when she created black holes, planets and sea-slugs. According to subtle Platonists, this is a mistake. A subtle Platonist believes that for the number of the Fs to be eight *just is* for there to be eight planets. So when God created eight planets she *thereby* made it the case that the number of the planets was eight. More generally, subtle

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Platonists believe that a world without numbers is *inconsistent*. Suppose, for *reductio*, that there are no numbers. The subtle Platonist thinks that for the number of numbers to be zero *just is* for there to be no numbers. So the number zero must exist after all, contradicting our assumption.

Essential to the subtle Platonist's position is the acceptance of 'just is'-statements. For instance:

For the number of the planets to be eight *just is* for there to be eight planets.

[In symbols:  $\#_x(\text{Planet}(x)) = 8 \equiv \exists!_8x(\text{Planet}(x)).$ ]

This is the sense of 'just is' whereby most of us would wish to claim that for something to be composed of water *just is* for it to be composed of H<sub>2</sub>O [i.e.  $\text{Water}(x) \equiv_x \text{H}_2\text{O}(x)$ ], and that for two people to be siblings *just is* for them to share a parent [i.e.  $\text{Siblings}(x, y) \equiv_{x,y} \exists z(\text{Parent}(z, x) \wedge \text{Parent}(z, y))$ ]. It is also the sense of 'just is' whereby some philosophers (but not all) would wish to claim that for a wedding to take place *just is* for someone to get married [i.e.  $\exists x(\text{Wedding}(x) \wedge \text{TakesPlace}(x)) \equiv \exists x(\text{Married}(x))$ ], or that for there to be a table *just is* for there to be some particles arranged table-wise [i.e.  $\exists x(\text{Table}(x)) \equiv \exists X(\text{ArrTw}(X))$ ].

'Just is'-statements are to be understood as 'no difference'-statements. 'For a wedding to take place *just is* for someone to get married' should be treated as expressing the same thought as 'There is *no difference* between a wedding's taking place and someone's getting married'. Accordingly, I will understand the 'just is' operator ' $\equiv$ ' as reflexive, transitive and symmetric.

One might be tempted to think of 'just is'-statements as expressing identities amongst facts, or identities amongst properties. (The fact that a wedding took place is identical to the fact that someone got married; the property of being composed of water is identical to the property of being composed of H<sub>2</sub>O.) I have no qualms with this way of putting things,

as long as fact-talk and property-talk are understood in a suitably deflationary way. (For the fact that snow is white to obtain *just is* for snow to be white; to have the property of being round *just is* to be round.) But it is important to keep in mind that fact-talk and property-talk are potentially misleading. They might be taken to suggest that one should only accept a ‘just is’-statement if one is prepared to countenance a naïve realism about facts or about properties—the view that even though it is consistent that there be no facts or properties, we are lucky enough to have them. The truth of a ‘just is’-statement, as it will be understood here, is totally independent of such a view.

‘Just is’-statements pervade our pre-theoretic, scientific and philosophical discourse. Yet they have been given surprisingly little attention in the literature, and are in much need of elucidation. I think it would be hopeless to attempt an explicit definition of ‘ $\equiv$ ’, not because true and illuminating equivalences couldn’t be found—I will suggest some below—but because any potential *definiens* can be expected to contain expressions that are in at least as much need of elucidation as ‘ $\equiv$ ’. The right methodology, it seems to me, is to explain how our acceptance of ‘just is’-statements interacts with the rest of our theorizing, and use these interconnections to inform our understanding of ‘ $\equiv$ ’. (I make no claims about conceptual priority: the various interconnections I will discuss are as well-placed to inform our understanding of ‘ $\equiv$ ’ on the basis of other notions as they are to inform our understanding of the other notions on the basis of ‘ $\equiv$ ’.)

## 1.1 ‘Just is’-Statements

In this section I will suggest three different ways in which our acceptance of ‘just is’-statements interacts with the rest of our theorizing. (I develop these connections in greater detail in Rayo (typescript).)

### 1.1.1 Inconsistency

Say that a representation is *inconsistent* if it represents the world as being inconsistent. The following sentence is inconsistent, in this sense:

There is something that is composed of water but is not composed of H<sub>2</sub>O.

For to be composed of water *just is* to be composed of H<sub>2</sub>O. So a world in which something composed of water fails to be composed of H<sub>2</sub>O is a world in which something composed of water fails to be composed of water, which is inconsistent.

There is a different notion that won't be of interest here but is worth mentioning because it is easily conflated with inconsistency. Say that a representation is *conceptually inconsistent* if (a) it is inconsistent (b) if its inconsistency is guaranteed by the concepts employed (plus semantic structure). In the example above, there is no reason to think that the relevant concepts (plus semantic structure) guarantee that the sentence represents the world as being inconsistent. So there is no reason to think that we have a case of conceptual inconsistency. But consider:

There is something that is composed of water but is not composed of water.

This sentence is inconsistent for the same reason as before: it depicts the world as being such that something composed of water fails to be composed of water, which is inconsistent. But in this case it is natural to think that the meanings of relevant terms (plus semantic structure) guarantee that the world will be represented as inconsistent, in which case one will wish to count the sentence as conceptually inconsistent.

I would like to suggest is that there is a tight connection between the notion of inconsistency and 'just is'-statements:

A first-order sentence (or set of first-order sentences) is inconsistent if and only if it is logically inconsistent with the true 'just is'-statements.

The connection between consistency and ‘just is’-statements places constraints on the ‘just is’ operator. It suggests that the role of ‘just is’-statements is very different from the role of other sentences. A sentence like ‘snow is white’ represents the world as being such that snow is white, and in doing so rules out a consistent way for the world to be (namely: such that snow is not white). A true ‘just is’-statement, in contrast, does not rule out a consistent way for the world to be. What it does instead is identify the limits of consistency.

Consider ‘to be composed of water *just is* to be composed of H<sub>2</sub>O’. This statement represents the world as satisfying a certain condition: that of being such that there is no difference between being composed of water and being composed of H<sub>2</sub>O. But the statement is *true*: it is indeed the case that to be composed of water *just is* to be composed of H<sub>2</sub>O (or so we may suppose). So the condition in question is just the condition of being such that there is no difference between being composed of water and being composed of water—something that cannot consistently fail to be satisfied. The result is that—unlike the case of ‘snow is white’, say—our ‘just is’-statement fails to rule out a consistent way for the world to be. It does, however, tell us something important about the limits of consistency. It entails that a scenario whereby something is composed of water but not H<sub>2</sub>O is inconsistent. And to succeed in so delineating the limits of consistency is a non-trivial cognitive accomplishment.

### 1.1.2 Truth-conditions

A sentence’s truth-conditions are usefully thought of as consisting of a *requirement* on the world—the requirement that the world would have to satisfy in order to be as the sentence represents it to be. The truth-conditions of ‘snow is white’, for example, consist of the requirement that snow be white, since that is how the world would have to be in order to be as ‘snow is white’ represents it to be.

Two sentences might have the same truth-conditions even if they have different mean-

ings (in a pre-theoretic sense of ‘meaning’). Consider ‘the glass contains water’ and ‘the glass contains H<sub>2</sub>O’. These two sentences play very different roles in our linguistic practice, so it is natural to describe them as meaning different things. But one should still think that the two sentences have the same truth-conditions. For to be composed of water *just is* to be H<sub>2</sub>O. So there is no difference between what would be required of the world in order to be as ‘the glass contains water’ represents it to be and what would be required of the world in order to be as ‘the glass contains H<sub>2</sub>O’ represents it to be.

I would like to suggest that there is a tight connection between ‘just is’-statements and the notion of sameness of truth-conditions:

The first-order sentences  $\phi$  and  $\psi$  have the same truth-conditions if and only if  
⌈for it to be the case that  $\phi$  *just is* for it to be the case that  $\psi$ ⌋ is true.

Suppose, for example, that you think that for  $A$  to be composed of water *just is* for  $A$  to be composed of H<sub>2</sub>O. Then you should think that ‘ $A$  is composed of water’ and ‘ $A$  is composed of H<sub>2</sub>O’ have the same truth-conditions. For what the former requires of the world is that  $A$  be composed of water. But to be composed of water *just is* to be composed of H<sub>2</sub>O, which is what the latter requires of the world. Conversely: suppose you think ‘ $A$  is composed of water’ and ‘ $A$  is composed of H<sub>2</sub>O’ have the same truth-conditions. Then you think there is no difference between satisfying the requirement that  $A$  be composed of water and satisfying the requirement that  $A$  be composed of H<sub>2</sub>O. You should think, in other words, that for  $A$  to be composed of water *just is* for  $A$  to be composed of H<sub>2</sub>O.

### 1.1.3 Metaphysical Possibility

The use of ‘metaphysical’ in ‘metaphysical possibility’ lends itself to two very different readings. (See Rosen (2006) for illuminating discussion.)

On one reading, ‘metaphysical’ is used as a way of indicating the level of *strictness*

that is to be employed when talking about possibility. Thus, the notion of metaphysical possibility might be thought of as stricter than the notion of conceptual possibility but less strict than the notion of physical possibility. Kment (2006), for example, suggests that we replace “the concept of a possible world with the wider, non-modal notion of a world. Worlds [...] comprise both possible and impossible worlds [and] are ordered by their closeness to the actual world [...] [F]or a proposition to be metaphysically necessary is for it to be true in every world that has at least a certain degree of closeness to the actual world.” (pp.6–7). Nomological and conceptual possibility are said to work the same way, but with standards of closeness that are more or less strict. (A proponent of this sort of view need not think that metaphysical possibility captures a level of strictness that is philosophically significant. Cameron (2010) and Sider (typescript), for example, argue that there’s nothing special about the relevant level of strictness: nothing important would be lost if it had been drawn elsewhere.)

On a different way of reading ‘metaphysical’ in ‘metaphysical possibility’, its role is to determine a *type* of possibility, rather than a level of strictness. It is meant to distinguish between possibility *de mundo* and possibility *de representatione*. The difference, informally, is that whereas possibility *de representatione* is a property of *sentences*, possibility *de mundo* is a property of the *truth-conditions* expressed by such sentences. (Logical consistency, for instance, is a notion of possibility *de representatione* since ‘Hesperus  $\neq$  Phosphorus’ and ‘Hesperus  $\neq$  Hesperus’ differ in terms of logical consistency, even though satisfaction of their truth-conditions imposes the same impossible requirement on the world.) Somewhat more precisely, one might say that whereas possibility *de mundo* applies to ways for the world to be *regardless of how they happen to be represented*, possibility *de representatione* is sensitive to how ways for the world to be happen to be represented.

On this way of seeing things, the role of ‘metaphysical’ in ‘metaphysical possibility’ is to clarify that the notion of possibility in question is to be thought of as a form of possibility



*de mundo*, rather than a form of possibility *de representatione*. The thought, then, is that metaphysical possibility is the most inclusive form of possibility *de mundo* there is. Going beyond metaphysical possibility is not a matter of going beyond a given limit of strictness: it is a matter of lapsing into inconsistency.

I would like to suggest that there is a tight connection between ‘just is’-statements and the notion of metaphysical possibility so understood:

A first-order sentence (or set of first-order sentences) describes a metaphysically possible scenario just in case it is logically consistent with the set of true ‘just is’-statements.

Relatedly,

A ‘just is’-statement  $\lceil \phi(\vec{x}) \equiv_{\vec{x}} \psi(\vec{x}) \rceil$  is true just in case the corresponding modal statement  $\lceil \Box \forall \vec{x} (\phi(\vec{x}) \leftrightarrow \psi(\vec{x})) \rceil$  is true.

So many philosophers have said so many different things about the notion of metaphysical possibility—and so much about the notion is poorly understood—that there are limits to how much light can be shed on the ‘just is’-operator by defining it in terms of metaphysical possibility. It is better to think of the above connections as a two-way street: they use the notion of metaphysical possibility to help explain how the ‘just is’-operator should be understood, but they also use the ‘just is’-operator to help explain how the notion of metaphysical possibility should be understood. Neither of the two notions is being defined in terms of the other, but getting clear about how they are related is a way of shedding light on both.

## 1.2 The Resulting Picture

In the preceding section I suggested several different ways in which our acceptance of ‘just is’-statements interacts with the rest of our theorizing. In doing so, I hope to have shown

that ‘just is’-statements are at the center of our theorizing about consistency, content and metaphysical possibility. The aim of the present section is to make some brief remarks about the philosophical picture that emerges from these theoretical connections.

In theorizing about the world, we do three things at once. First, we develop a language within which to formulate theoretical questions. Second, we endorse a family of ‘just is’-statements, and thereby classify the questions that can be formulated in the language into those we take to be worth answering and those we regard as pointless, at least for the time being. (For instance, by accepting ‘to be composed of water *just is* to be composed of  $H_2O$ ’, we resolve that ‘why does this portion of water contain hydrogen?’ is not the sort of question that is worth answering; had we instead accepted ‘to be composed of water *just is* to be an odorless, colorless liquid with thus-and-such properties’, we would have taken ‘why does this portion of water contain hydrogen?’ to be an interesting question, deserving explanation.) Third, we set forth theoretical claims that address the questions we take to be worth answering. (For instance, having accepted ‘to be composed of water *just is* to be composed of  $H_2O$ ’, we set forth theoretical claims as an answer to the question ‘why is this portion of water an odorless, colorless liquid with thus-and-such properties?’.)

In light of the connection between ‘just is’-statements and possibility, when one endorses a set of ‘just is’-statements one is, in effect, endorsing a space of possible worlds. And in light of the connection between possibility and consistency, a space of possible worlds can be regarded as a framework for ascertaining the limits of consistency. In other words: by endorsing a set of ‘just is’-statements one is, in effect, endorsing a framework for ascertaining the limits of consistency

It is against the background of such a framework that it makes sense to distinguish between ‘factual sentences’—i.e. sentences that are verified by some consistent scenarios but not others—and ‘necessary truths’—i.e. sentences that are verified in every consistent scenario. What one gets is a post-Quinean revival of the position that Carnap advanced

in ‘Empiricism, Semantics and Ontology’. Carnap’s frameworks, like ours, are devices for distinguishing between factual sentences and necessary truths. The big difference is that whereas Carnap’s distinction was tied up with the analytic/synthetic distinction, ours is not.

### 1.3 Back to Platonism

I introduced the difference between traditional and subtle Platonism by saying that whereas traditional Platonists believe that a world with no mathematical objects is consistent, subtle Platonists think one cannot. But if one is prepared to embrace a tight connection between possibility and consistency, as I suggested above, one can also state the difference as follows: traditional Platonists believe that mathematical objects exist contingently; subtle Platonists believe that they exist necessarily.

This way of characterizing the debate might strike you as suspect. If so, it may be because you have a different conception of possibility in mind. In the sense of necessity that is at issue here, the claim that numbers exist necessarily entails that the existence of numbers is no longer a factual question: nothing is required of the world in order for the truth-conditions of a truth of pure mathematics to be satisfied. Satisfaction is guaranteed by the framework with respect to which we have chosen to carry out our theoretical investigation of the world.

Non-factuality, in our sense, does not entail *a priori* knowability. (‘To be composed of water *just is* to be composed of H<sub>2</sub>O’ will count as non-factual even though it is not knowable *a priori*.) But there is still room for mathematical truths to enjoy a distinct epistemological status. For in adopting a framework—i.e. in adopting a family of ‘just is’-statements—one makes decisions that are closely tied to empirical considerations, but also decisions that are more organizational in nature. This is because a delicate balance must be struck. If one accepts too many ‘just is’-statements, one will be committed to treating

as inconsistent scenarios that might have been useful in theorizing about the world. If one accepts too few, one opens the door to a larger range of consistent scenarios, all of them candidates for truth. In discriminating amongst these scenarios one will have to explain why one favors the ones one favors. And although the relevant explanations could lead to fruitful theorizing, they could also prove burdensome.

Logic and mathematics play an important role in finding the right balance between these competing considerations. Take, for example, the decision to treat ' $p \leftrightarrow \neg\neg p$ ' as a logical truth (i.e. the decision to accept every 'just is'-statement of the form 'for  $p$  to be the case *just is* for  $\neg\neg p$  to be the case'). A friend of intuitionistic logic, who denies the logicity of ' $p \leftrightarrow \neg\neg p$ ', thinks it might be worthwhile to ask why it is the case that  $p$  even if you fully understand why it is not the case that  $\neg p$ . In the best case scenario, making room for an answer will lead to fruitful theorizing. But things may not go that well. One might come to see the newfound conceptual space between a sentence and its double negation as a pointless distraction, demanding explanations in places where there is nothing fruitful to be said.

The result is that even if none of the decisions one makes in adopting a family of 'just is'-statements is wholly independent of empirical considerations, some decisions are more closely tied to empirical considerations than others. And when it comes to 'just is'-statements corresponding to logic and mathematics, one would expect the focus to be less on particular empirical matters and more on questions of framework-organization. So there is room for a picture whereby an epistemically responsible subject can believe that numbers exist even if her belief isn't grounded very directly in any sort of empirical investigation.

## 2 Neo-Fregeanism

Hume's Principle is the following sentence:

$$\forall F \forall G (\#_x(F(x)) = \#_x(G(x)) \leftrightarrow F(x) \approx_x G(x))$$

[*Read:* the number of the Fs equals the number of the Gs just in case the Fs are in one-one correspondence with the Gs.]

Neo-Fregeanism is the view that when Hume’s Principle is set forth as an implicit definition of ‘ $\#_x(F(x))$ ’, one gets the following two results: (1) the truth of Hume’s Principle is knowable *a priori*, and (2) the referents of number-terms constitute a realm of mind-independent objects which is such as to render mathematical Platonism true. (Neo-Fregeanism was first proposed in Wright (1983), and has since been championed by Bob Hale, Crispin Wright and others. For a collection of relevant essays, see Hale and Wright (2001).)

Just as one can distinguish between two different varieties of mathematical Platonism, one can distinguish between two different varieties of neo-Fregeanism. Traditional and subtle neo-Fregeanism agree that numbers—the referents of numerical-terms—constitute a realm of mind-independent objects. But they disagree about whether this realm of objects can consistently fail to exist.

Subtle neo-Fregeans go beyond mere acceptance of Hume’s Principle; they accept a ‘*just is*’-statement corresponding to Hume’s Principle:

$$\#_x(F(x)) = \#_x(G(x)) \equiv_{F,G} F(x) \approx_x G(x)$$

[*Read:* for the number of the Fs to equal the number of the Gs *just is* for the Fs to be in one-one correspondence with the Gs.]

A consequence of this ‘just is’-statement is that it is inconsistent that there be no numbers. For it is trivially true that, e.g. the planets are in one-one correspondence with themselves. But for the planets to be in one-one correspondence with themselves *just is* for the number of the planets to be self-identical. So numbers exist after all.

Traditional neo-Fregeans, in contrast, accept Hume’s Principle, but shy away from accepting the corresponding ‘just is’-statement. Accordingly, they take themselves to be

able to make sense of a world with no numbers.

It seems to me that there is some confusion in the literature about which of the two version of the neo-Fregeanism is being discussed. Critics of neo-Fregeanism have sometimes interpreted the program as a version of traditional neo-Fregeanism—the author of Rayo (2003) and Rayo (2005), for instance. (For a survey of the literature, see MacBride (2003).) But it is not clear that this is what proponents of neo-Fregeanism have in mind. There are strong indications that the subtle variety is closer to the mark. One such indication is the use of ‘neo-Fregeanism’ as a name for the program—for there is good reason to think that Frege himself was a proponent of subtle Platonism.

When Frege claims, for example, that the sentence ‘there is at least one square root of 4’ expresses the same thought as ‘the concept *square root of 4* is realized’, and adds that “a thought can be split up in many ways, so that now one thing, now another, appears as subject or predicate” (Frege (1892) p. 199), it is natural to interpret him as embracing the ‘just is’-statement:

For the concept *square root of 4* to be realized *just is* for there to be at least one square root of 4.

And when he claims, in *Grundlagen* §64, that in treating the judgement ‘line *a* is parallel to line *b*’ as a ‘just is’-statement, so as to obtain ‘the direction of line *a* is identical to the direction of line *b*’, we “carve up the content in a way different from the original way”, it is natural to interpret him as embracing the ‘just is’-statement:

For the direction of line *a* to equal the direction of line *b* *just is* for *a* and *b* to be parallel.

In both instances, Frege puts the point in terms of content-recarving, rather than as a ‘just is’-statement. But, as emphasized above, one’s views about truth-conditions are tightly correlated with the ‘just is’-statements one accepts.

Neo-Fregeans are evidently sympathetic towards Frege’s views on content-recarving. (See, for instance, Wright (1997).) And even though talk of content-recarving has become less prevalent in recent years, with more of the emphasis on implicit definitions, a version of neo-Fregeanism rooted in subtle Platonism is clearly on the cards. In the remainder of the paper I will argue that such an interpretation of the program would be decidedly advantageous.

## 2.1 Objects

Suppose you introduce the verb ‘to tableize’ into your language, and accept ‘for it to tableize *just is* for there to be a table’ (where the ‘it’ in ‘it tableizes’ is assumed to play the same dummy role as the ‘it’ in ‘it is raining’). Then you will think that ‘it tableizes’ and ‘there is a table’ have the same truth-conditions. In each case, what is required in order for the truth-conditions to be satisfied is that there be a table (equivalently: that it tableize). So you will think that—for the purposes of stating that there is a table—object-talk is *optional*. One can state that there is a table by employing a quantifier that binds singular term positions—as in ‘there is a table’—but also by employing an essentially different syntactic structure—as in ‘it tableizes’.

If object-talk is optional, what is the point of giving it a place in our language? According to *compositionality*, as I shall call it, the answer is “compositionality”. A language involving object-talk—that is, a language including singular terms and quantifiers binding singular term positions—is attractive because it enables one to give a recursive specification of truth-conditions for a class of sentences rich in expressive power. But there is not much more to be said on its behalf. If one could construct a language that never indulged in object-talk, and was able to do so without sacrificing compositionality or expressive power, there would be no immediate reason to think it inferior to our own. Whether or not we choose to adopt it should turn entirely on matters of convenience. (For an example

of such a language, and illuminating discussion, see Burgess (2005).)

Proponents of the compositionism believe that it takes very little for a singular term  $t$  to refer. All it takes is an assignment of truth-conditions to whichever sentences involving  $t$  one wishes to make available for use with the following two features: (1) the assignment respects compositionality in the sense that if  $\psi$  is a syntactic consequence of  $\phi$  then the truth conditions assigned to  $\phi$  are strictly stronger than the truth conditions assigned to  $\psi$ ; and (2) the world is such as to satisfy the truth-conditions that have been associated with ‘ $\exists x(x = t)$ ’.

The reason there is nothing more that needs to be done is that there was nothing special about using singular terms to begin with. In setting forth a language, all we wanted was the ability to express a suitably rich range of truth-conditions. If we happened to carry out this aim by bringing in singular terms, it was because they supplied a convenient way of specifying the right range of truth-conditions, not because they had some further virtue.

Proponents of *Tractarianism*, as I shall call the opposing view, believe that object-talk is subject to a further constraint: there needs to be a certain kind of correspondence between the semantic structure of our sentences and the ‘metaphysical structure of reality’. In particular, they presuppose the following: (1) there is a particular carving of reality into objects which is more apt, in some metaphysical sense, than any potential rival—the one and only carving that is in accord with reality’s true metaphysical structure; (2) to each legitimate singular term there must correspond an object carved out by this metaphysical structure; and (3) satisfaction of the truth-conditions of a sentence of the form ‘ $P(t_1, \dots, t_n)$ ’ requires that the objects paired with  $t_1, \dots, t_n$  bear to each other the property expressed by  $P$ .

A consequence of Tractarianism is that one cannot accept ‘for it to tableize *just is* for there to be a table’. (Since ‘it tableizes’ and ‘there is a table’ have different semantic structures, there can’t be a single feature of reality they are both accurate descriptions of



when it is presupposed that correspondence with the one and only structure of reality is a precondition for accuracy.) For similar reasons, one cannot accept ‘for some things to be arranged tablewise *just is* for there to be a table’, or ‘for the property of tablehood to be instantiated *just is* for there to be a table’, or, indeed, any ‘just is’ statement  $\lceil \phi \equiv \psi \rceil$  where  $\phi$  and  $\psi$  are atomic sentences with different semantic structures. Accordingly, Tractarians takes themselves to be in a position to make distinctions that a compositionalist would fail to make room for. For instance, they might take themselves to be in a position to make sense of a scenario in which some things are arranged tablewise but there is no table.

I am ignoring a complication. Tractarians might favor a *moderate* version of the proposal according to which only a subset of our discourse is subject to the constraint that there be a correspondence between semantic structure and the metaphysical structure of reality. (See, for instance, Sider (typescript).) One could claim, for example, that the constraint only applies when one is in the ‘ontology room’. Accordingly, friends of the moderate view would be free to accept a version of, e.g. ‘for there to be some things arranged tablewise *just is* for there to be a table’ by arguing that only ‘there are some things arranged tablewise’ is to be understood in an ontology-room spirit. I will not be concerned with moderate Tractarianism here.

The difference between compositionism and (non-moderate) Tractarianism is subtle but important. For compositionism leaves room for meaninglessness where Tractarianism does not. Suppose it is agreed on all sides that the singular terms  $t_1$  and  $t_2$  both have referents, and figure meaningfully in sentences with well-defined truth-conditions. A Tractarian is, on the face of it, committed to the claim that it must be possible to meaningfully ask whether  $\lceil t_1 = t_2 \rceil$  is true. For she believes that each of  $t_1$  and  $t_2$  is paired with one of the objects carved out by the metaphysical structure of reality. So the question whether  $\lceil t_1 = t_2 \rceil$  is true can be cashed out as the question whether  $t_1$  and  $t_2$  are paired with the same such object. (Tractarians could, of course, deny that *this* is a meaningful question.

But such a move would come at a cost, since it would make it hard to understand what is meant by ‘carving reality into objects’.)

For a compositionalist, in contrast, there is no tension between thinking that  $t_1$  and  $t_2$  have referents (and figure meaningfully in sentences with well-defined truth conditions), and denying that one has asked a meaningful question when one asks whether  $\lceil t_1 = t_2 \rceil$  is true. For according to the compositionalist, all it takes for a singular term to be in good order is for there to be a suitable specification of truth-conditions for whichever sentences involving the term one wishes to make available for use. And there is no reason one couldn’t have a suitable specification of truth-conditions for a large range of sentences involving  $t_1$  and  $t_2$  without thereby specifying truth-conditions for  $\lceil t_1 = t_2 \rceil$ .

Arithmetic is a case in point. The subtle Platonist has a straightforward way of specifying the right sort of assignment of truth-conditions to arithmetical sentences. (See appendix A.) On this assignment, every arithmetical sentence a non-philosopher would care about gets well-defined truth-conditions, as does every sentence in the non-arithmetical fragment of the language. But no truth-conditions are supplied for mixed identity-statements, such as ‘the number of the planets = Julius Caesar’. And for good reason: there is no natural way of extending the relevant semantic clauses to cover these cases.

Tractarians will claim that something important has been left out. For in the absence of well-defined truth-conditions for ‘the number of the planets = Julius Caesar’, it is unclear which of the objects carved out by the metaphysical structure of reality has been paired with ‘the number of the planets’. But compositionlists would disagree: it is simply a mistake to think that such pairings are necessary to render a singular term meaningful.

“If compositionism is right”—you might be tempted to ask—“in what sense are we committed to the existence of numbers when we say that the number of the planets is eight?” In the usual sense, say I. In order for the truth-conditions of ‘the number of the planets is eight’ to be satisfied the number of the planets must be eight, and in order for

the number of the planets to be eight there have to be numbers. What else could it take for someone who asserts ‘the number of the planets is eight’ to be committed to the existence of numbers?

Of course, the subtle Platonist will make a further claim. She will claim that it is *also* true that all it takes for the truth-conditions of ‘the number of the planets is eight’ to be satisfied is that there be eight planets. To think that this contradicts what was said in the preceding paragraph is to fail to take seriously the idea that for the number of the planets to be eight *just is* for there to be eight planets. It is true that ‘the number of the planets is eight’ carries commitment to numbers, but it is also true that commitment to numbers is no commitment at all. (If you have trouble getting your head around this, consider the following. The subtle Platonist accepts ‘For the number of the planets to be zero *just is* for there to be no planets’. She also accepts ‘For the number of the planets to be distinct from zero *just is* for there to be planets’. Putting the two together, she accepts ‘For the number of planets to be either identical to or distinct from zero *just is* for there to be some planets or none’; equivalently: ‘For the number of the planets to exist *just is* for there to be some planets or none’. But ‘there are some planets or none’ has *trivial* truth-conditions—truth-conditions whose non-satisfaction would be inconsistent—so ‘the number of the planets exists’ must also have trivial truth-conditions.)

The moral I hope to draw from the discussion this section is that subtle Platonists have reason to embrace compositionality. This goes, in particular, for subtle neo-Fregeans—neo-Fregeans who favor subtle Platonism. But once one embraces compositionality, there is no pressure for thinking that a mixed identity-statement such as ‘the number of the planets = Julius Caesar’ should have well-defined truth-conditions. So there is no reason to think—in spite of what Frege suggests in §66 of the *Grundlagen* and what proponents of neo-Fregeanism tend to presuppose—that a characterization of the concept of number will be unacceptable unless it settles the truth-value of mixed identity-statements. In short: if

you are a subtle neo-Fregean, you should be resolute about it, and stop worrying about mixed identities.

## 2.2 Abstraction Principles

I have never been able to understand why a *traditional* neo-Fregean would think it important to use Hume's Principle to characterize the meaning of arithmetical vocabulary, instead of using, e.g. the (second-order) Dedekind Axioms. In the case of *subtle* neo-Fregeans, on the other hand, I can see a motivation. Hume's Principle, and abstraction principles more generally, might be thought to be important because they are seen as capturing the difference between setting forth a mere quantified biconditional as an implicit definition of mathematical terms:

$$\forall\alpha\forall\beta(f(\alpha) = f(\beta) \leftrightarrow R(\alpha, \beta))$$

and setting forth the corresponding 'just is'-statement:

$$f(\alpha) = f(\beta) \equiv_{\alpha, \beta} R(\alpha, \beta).$$

And this is clearly an important difference. Only the latter delivers subtle Platonism, and only the latter promises to deliver an account for the special epistemic status of mathematical truths.

If that's what's intended, however, it seems to me that there are better ways of doing the job. Consider the case of Hume's Principle. In order to convince oneself that it succeeds in characterizing the meaning of our arithmetical vocabulary, one needs to prove a certain kind of completeness result: one needs to show that setting forth Hume's Principle as an implicit definition of numerical terms is enough to pin down the truth-conditions of every arithmetical sentence one wishes to have available for use. This, in turn, involves proving the following two results:

1. Given suitable definitions, every true sentence in the language of pure arithmetic is a logical consequence of Hume's Principle.

[This result has come to be known as *Frege's Theorem*. It was originally proved in Frege (1893 and 1903), by making what Heck (1993) identified as a non-essential use of Basic Law V. That the result could be established without using Basic Law V was noted in Parsons (1965) and proved in Wright (1983).]

2. Given suitable definitions, every true sentence in the (two-sorted) language of applied arithmetic is a logical consequence of Hume's Principle, together with the set of true sentences containing no arithmetical vocabulary.

Both of these results are true, but neither of them is trivial.

Now suppose that one characterizes the meaning of arithmetical vocabulary by using the compositional semantics in appendix A. The assignment of truth-conditions one gets is exactly the same as the one one would get by setting forth the 'just is'-statement corresponding to Hume's Principle as an implicit definition of arithmetical terms—but with the advantage that establishing completeness, in the relevant sense, turns out to be absolutely straightforward. One can simply read it off from one's semantic clauses. (In fact, the easiest method I know of for proving the second of the two results above, and thereby establishing the completeness of Hume's Principle, proceeds via the appendix A semantics.) None of this should come as a surprise. When one uses the method in the appendix to specify truth-conditions for arithmetical sentences one avails oneself of all the advantages of a compositional semantics. Not so when one sets forth Hume's Principle as an implicit definition.

There is, however, a much more important reason for preferring the method in the appendix over specifications of truth-conditions based on abstraction principles. Neo-Fregeans have found it difficult to identify abstraction principles that can do for set-

theory what Hume's Principle does for arithmetic. (For a selection of relevant literature, see Cook (2007); see also contributions to this volume by Cook, and by Shapiro and Uzquiano.) But if one waives the requirement that the meaning-fixation work be done by abstraction principles, the difficulties vanish. The subtle Platonist has a straightforward way of specifying a compositional assignment of truth-conditions to set-theoretic sentences. (See appendix B for details.) As in the case of arithmetic, one gets the result that every consequence of the standard set-theoretic axioms has well-defined truth-conditions. And, as in the case of arithmetic, one gets subtle Platonism. In particular, it is a consequence of the semantics that for the set of Romans to contain Julius Caesar *just is* for Julius Caesar to be a Roman.

Perhaps there is some other reason to insist on using abstraction principles to fix the meanings of set-theoretic terms. But if the motivation is simply to secure subtle Platonism, it seems to me that neo-Fregeans would do better by using a compositional semantics instead.

## A A Semantics for the Language of Arithmetic

Consider a two-sorted first-order language with identity,  $\mathcal{L}$ . It contains *arithmetical* variables ( $'n_1', 'n_2', \dots$ ), constants ( $'0'$ ) and function-letters ( $'S', '+'$  and  $'\times'$ ), and *non-arithmetical* variables ( $'x_1', 'x_2', \dots$ ), constants ( $'Caesar'$  and  $'Earth'$ ) and predicate-letters ( $'Human(\dots)'$  and  $'Planet(\dots)'$ ). In addition,  $\mathcal{L}$  has been enriched with the function-letter  $'\#_v(\dots)'$  which takes a first-order predicate in its single argument-place to form a first-order arithmetical term (as in  $'\#_{x_1}(Planet(x_1))'$ , which is read 'the number of the planets').

If  $\sigma$  is a variable assignment and  $w$  is a world, truth and denotation in  $\mathcal{L}$  relative to  $\sigma$  and  $w$  can be characterized as follows:

### Denotation of arithmetical terms:

1.  $\delta_{\sigma,w}(\ulcorner n_i \urcorner) = \sigma(\ulcorner n_i \urcorner)$
2.  $\delta_{\sigma,w}(\ulcorner 0 \urcorner) = 0$
3.  $\delta_{\sigma,w}(\ulcorner S(t) \urcorner) = \delta_{\sigma,w}(t) + 1$
4.  $\delta_{\sigma,w}(\ulcorner t_1 + t_2 \urcorner) = \delta_{\sigma,w}(t_1) + \delta_{\sigma,w}(t_2)$
5.  $\delta_{\sigma,w}(\ulcorner t_1 \times t_2 \urcorner) = \delta_{\sigma,w}(t_1) \times \delta_{\sigma,w}(t_2)$
6.  $\delta_{\sigma,w}(\ulcorner \#_{x_i}(\phi(x_i)) \urcorner) =$  the number of  $zs$  such that  $Sat(\ulcorner \phi(x_i) \urcorner, \sigma^{z/\ulcorner x_i \urcorner}, w)$
7.  $\delta_{\sigma,w}(\ulcorner \#_{n_i}(\phi(n_i)) \urcorner) =$  the number of  $ms$  such that  $Sat(\ulcorner \phi(n_i) \urcorner, \sigma^{m/\ulcorner n_i \urcorner}, w)$

### Denotation of non-arithmetical terms:

1.  $\delta_{\sigma,w}(\ulcorner x_i \urcorner) = \sigma(\ulcorner x_i \urcorner)$
2.  $\delta_{\sigma,w}(\ulcorner Caesar \urcorner) =$  Julius Caesar
3.  $\delta_{\sigma,w}(\ulcorner Earth \urcorner) =$  the planet Earth

**Satisfaction:**

Where  $\ulcorner \phi \urcorner_w$  is read ‘it is true at  $w$  that  $\phi$ ’,

1.  $Sat(\ulcorner t_1 = t_2 \urcorner, \sigma, w) \leftrightarrow \delta_{\sigma, w}(t_1) = \delta_{\sigma, w}(t_2)$  (for  $t_1, t_2$  arithmetical terms)
2.  $Sat(\ulcorner t_1 = t_2 \urcorner, \sigma, w) \leftrightarrow [\delta_{\sigma, w}(t_1) = \delta_{\sigma, w}(t_2)]_w$  (for  $t_1, t_2$  non-arithmetical terms)
3.  $Sat(\ulcorner \text{Human}(t) \urcorner, \sigma, w) \leftrightarrow [\delta_{\sigma, w}(t) \text{ is human}]_w$  (for  $t$  a non-arithmetical term)
4.  $Sat(\ulcorner \text{Planet}(t) \urcorner, \sigma, w) \leftrightarrow [\delta_{\sigma, w}(t) \text{ is a planet}]_w$  (for  $t$  a non-arithmetical term)
5.  $Sat(\ulcorner \exists n_i \phi \urcorner, \sigma, w) \leftrightarrow$  there is a number  $m$  such that  $Sat(\phi, \sigma^{m/\ulcorner n_i \urcorner}, w)$
6.  $Sat(\ulcorner \exists x_i \phi \urcorner, \sigma, w) \leftrightarrow$  there is a  $z$  such that  $([\exists y(y = z)]_w \wedge Sat(\phi, \sigma^{z/\ulcorner x_i \urcorner}, w))$
7.  $Sat(\ulcorner \phi \wedge \psi \urcorner, \sigma, w) \leftrightarrow Sat(\phi, \sigma, w) \wedge Sat(\psi, \sigma, w)$
8.  $Sat(\ulcorner \neg \phi \urcorner, \sigma, w) \leftrightarrow \neg Sat(\phi, \sigma, w)$

**Philosophical comment:**

As argued in the main text, a sentence’s truth-conditions can be modeled by a set of consistent scenarios. Accordingly, when  $w$  is taken to range over consistent scenarios—as is intended above—the characterization of truth at  $w$  yields a specification of truth-conditions for every sentence in the language.

The proposed semantics makes full use of arithmetical vocabulary. So it can only be used to explain the truth-conditions of arithmetical sentences to someone who is already in a position to use arithmetical vocabulary. What then is the point of the exercise? How is it an improvement over a homophonic specification of truth-conditions?

The first thing to note is that the present proposal has consequences that a homophonic specification lacks. Conspicuously, a homophonic specification would be compatible with



both traditional and subtle Platonism, but the present proposal is only compatible with subtle Platonism (since it entails that there is no consistent scenario with no numbers). And unlike a homophonic specification, the present proposal can be used to go from the assumption that every sentence in the purely mathematical fragment of one’s metalanguage is either true or false to the assumption that every sentence in the purely mathematical fragment of the object-language is either necessarily true or necessarily false.

The reason the present proposal has non-trivial consequences is that mathematical vocabulary never occurs within the scope of ‘[. . .]<sub>w</sub>’. So even though mathematical vocabulary is used to specify the satisfaction clauses, the *worlds* in the range of *w* can be characterized entirely in non-mathematical terms. As a result, the formal semantics establishes a connection between mathematical and non-mathematical descriptions of the world. It entails, for example, that the consistent scenarios at which the number of the planets is eight are precisely the consistent scenarios at which there are eight planets—from which it follows that for the number of the planets to be eight *just is* for there to be eight planets.

Why not make this connection more explicit still? Why not specify a translation-function that maps each arithmetical sentence  $\phi$  to a sentence  $\phi^*$  such that (i)  $\phi^*$  contains no mathematical vocabulary and (ii) the subtle Platonist would accept  $\ulcorner \phi \equiv \phi^* \urcorner$ ? Although doing so would certainly be desirable, it is provably beyond our reach: one can show that no such function is finitely specifiable. (See Rayo (2008) for details.)

**Technical comments:**

1. Throughout the definition of dentation and satisfaction I assume that the range of the metalinguistic variables includes merely possible objects. This is a device to simplify the exposition, and could be avoided by appeal to the technique described in Rayo (2008) and Rayo (typescript).
2. In clause 5 of the definition of satisfaction I assume that ‘number’ includes infinite

numbers. This is done in order to ensure that there are no empty terms in the language. If one wanted to restrict one's attention to the natural numbers one could do so by working in a free logic.

## B A Semantics for the Language of Set-Theory

Consider a two-sorted first-order language with identity,  $\mathcal{L}$ . It contains *set-theoretic* variables ( $'\alpha_1', ' \alpha_2', \dots$ ), and *non-set-theoretic* variables ( $'x_1', 'x_2', \dots$ ), and predicate-letters ( $'\text{Human}(\dots)'$  and  $'\text{Planet}(\dots)'$ ). If  $\sigma$  is a variable assignment and  $w$  is a world, truth in  $\mathcal{L}$  relative to  $\sigma$  and  $w$  can be characterized as follows:

1.  $Sat(\ulcorner x_i = x_j \urcorner, \sigma, w) \leftrightarrow [\sigma(\ulcorner x_i \urcorner) = \sigma(\ulcorner x_j \urcorner)]_w$
2.  $Sat(\ulcorner x_i \in \alpha_j \urcorner, \sigma, w) \leftrightarrow \sigma(\ulcorner x_i \urcorner) \in \sigma(\ulcorner \alpha_j \urcorner)$
3.  $Sat(\ulcorner \alpha_i \in \alpha_j \urcorner, \sigma, w) \leftrightarrow \sigma(\ulcorner \alpha_i \urcorner) \in \sigma(\ulcorner \alpha_j \urcorner)$
4.  $Sat(\ulcorner \text{Human}(x_i) \urcorner, \sigma, w) \leftrightarrow [\sigma(\ulcorner x_i \urcorner) \text{ is human}]_w$
5.  $Sat(\ulcorner \text{Planet}(x_i) \urcorner, \sigma, w) \leftrightarrow [\sigma(\ulcorner x_i \urcorner) \text{ is a planet}]_w$
6.  $Sat(\ulcorner \exists \alpha_i \phi \urcorner, \sigma, w) \leftrightarrow \text{there is a set } \beta \text{ such that } (\text{Good}_w(\beta) \wedge Sat(\phi, \sigma^{\beta/\ulcorner x_i \urcorner}, w))$
7.  $Sat(\ulcorner \exists x_i \phi \urcorner, \sigma, w) \leftrightarrow \text{there is a } z \text{ such that } ([\exists y(y = z)]_w \wedge Sat(\phi, \sigma^{z/\ulcorner x_i \urcorner}, w))$
8.  $Sat(\ulcorner \phi \wedge \psi \urcorner, \sigma, w) \leftrightarrow Sat(\phi, \sigma, w) \wedge Sat(\psi, \sigma, w)$
9.  $Sat(\ulcorner \neg \phi \urcorner, \sigma, w) \leftrightarrow \neg Sat(\phi, \sigma, w)$

where a set is  $\text{good}_w$  just in case it occurs at some stage of the iterative hierarchy built up from objects  $x$  such that  $[\text{Urelement}(x)]_w$ . (For philosophical and technical comments, see appendix A.)

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