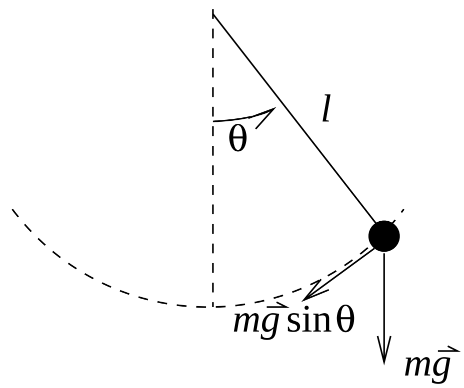
1 Pendulum

1.1 Free oscillator

To introduce dynamical systems, we begin with one of the simplest: a *free oscillator*. Specifically, we consider an unforced, undamped pendulum.



The arc length (displacement) between the pendulum's current position and rest position ($\theta = 0$) is

$$s = l\theta$$

Therefore

$$\begin{aligned} \dot{s} &= l\dot{\theta} \\ \ddot{s} &= l\ddot{\theta} \end{aligned}$$

From Newton's 2nd law,

$$F = ml\ddot{\theta}$$

The restoring force is given by $-mg \sin \theta$. (It acts in the direction opposite to $\operatorname{sgn}(\theta)$). Thus

$$F = ml\ddot{\theta} = -mg \sin \theta$$

or

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l} \sin \theta = 0.$$

Our pendulum equation is idealized: it assumes, e.g., a point mass, a rigid geometry, and most importantly, **no friction**.

The equation is nonlinear, because of the $\sin\theta$ term. Thus the equation is not easily solved.

However for small $\theta \ll 1$ we have $\sin\theta \simeq \theta$. Then

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\theta$$

whose solution is

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t + \phi\right)$$

or

$$\theta = \theta_0 \cos(\omega t + \phi)$$

where the angular frequency is

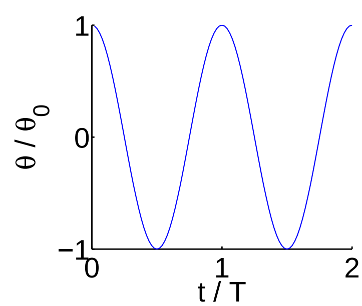
$$\omega = \sqrt{\frac{g}{l}},$$

the period is

$$T = 2\pi\sqrt{\frac{l}{g}},$$

and θ_0 and ϕ come from the initial conditions.

Note that the motion is exactly periodic.
Furthermore, the period T is independent of the amplitude θ_0.



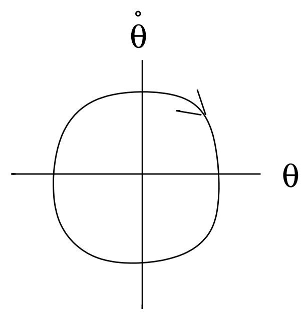
1.2 Global view of dynamics

What do we need to know to completely describe the instantaneous state of the pendulum?

The position θ and the velocity $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$.

Instead of integrating our o.d.e. for the pendulum, we seek a representation of the solution in the plane of θ and $\dot{\theta}$.

Because the solution is periodic, we know that the resulting trajectory must be closed:

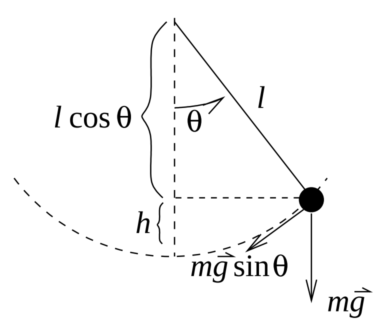


In which direction is the flow?
What shape does the curve take?

To calculate the curve, we note that it should be characterized by constant energy, since no energy is input to the system (it is not driven) and none is dissipated (there is no friction).

Therefore we compute the energy $E(\theta, \dot{\theta})$, and expect the trajectories to be curves of $E(\theta, \dot{\theta}) = \text{const}$.

1.3 Energy in the plane pendulum



The pendulum's height above its rest position is $h = l - l\cos\theta$.
As before, $s = \text{arc length} = l\theta$.

The kinetic energy T is

$$T = \frac{1}{2}m\dot{s}^2 = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

The potential energy U is

$$\begin{aligned} U = mgh &= mg(l - l\cos\theta) \\ &= mgl(1 - \cos\theta) \end{aligned}$$

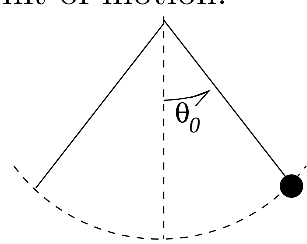
Therefore the energy $E(\theta, \dot{\theta})$ is

$$E(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta)$$

We check that $E(\theta, \dot{\theta})$ is a constant of motion by calculating its time derivative:

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{1}{2}ml^2(2\dot{\theta}\ddot{\theta}) + mgl\dot{\theta}\sin\theta \\ &= ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right) \\ &= 0 \qquad \text{(since the pend. eqn. } \ddot{\theta} = -\frac{g}{l}\sin\theta) \end{aligned}$$

So what do these curves look like?
Take θ_0 to be the highest point of motion.



Then

$$\dot{\theta}(\theta_0) = 0$$

and

$$E(\theta_0, \dot{\theta}\mid_{\theta_0}) = mgl(1 - \cos\theta_0)$$

Since $\cos\theta = 1 - 2\sin^2(\theta/2)$,

$$\begin{aligned} E(\theta_0, \dot{\theta}\mid_{\theta_0}) &= 2mgl\sin^2\left(\frac{\theta_0}{2}\right) \\ &= E(\theta, \dot{\theta}) \text{ in general, since } E \text{ is conserved} \end{aligned}$$

Now write $T = E - U$:

$$\frac{1}{2}ml^2\dot{\theta}^2 = 2mgl\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \quad (1)$$

$$\dot{\theta}^2 = 4\,\frac{g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \quad (2)$$

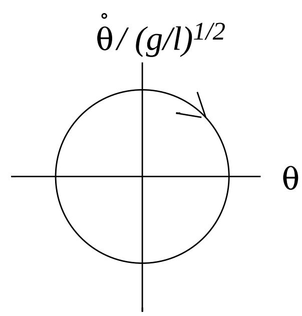
For small θ_0 such that $\theta \ll 1$,

$$\dot{\theta}^2 \simeq 4\,\frac{g}{l}\left(\frac{\theta_0^2}{4} - \frac{\theta^2}{4}\right)$$

or

$$\left(\frac{\dot{\theta}}{\sqrt{g/l}}\right)^2 + \theta^2 \simeq \theta_0^2$$

Thus for small θ the curves are circles of radius θ_0 in the plane of θ and $\dot{\theta}/\sqrt{g/l}$.



What about θ_0 large?
Consider the case $\theta_0 = \pi$.

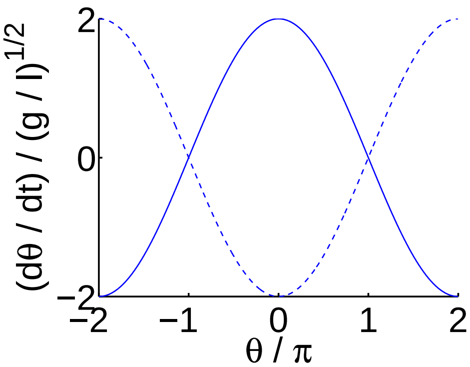
For $\theta_0 = \pi$, $E = 2mgl$, and equation (2) gives

$$\dot{\theta}^2 = 4\,\frac{g}{l}\left[\sin^2\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]$$

$$= 4\,\frac{g}{l}\cos^2\left(\frac{\theta}{2}\right)$$

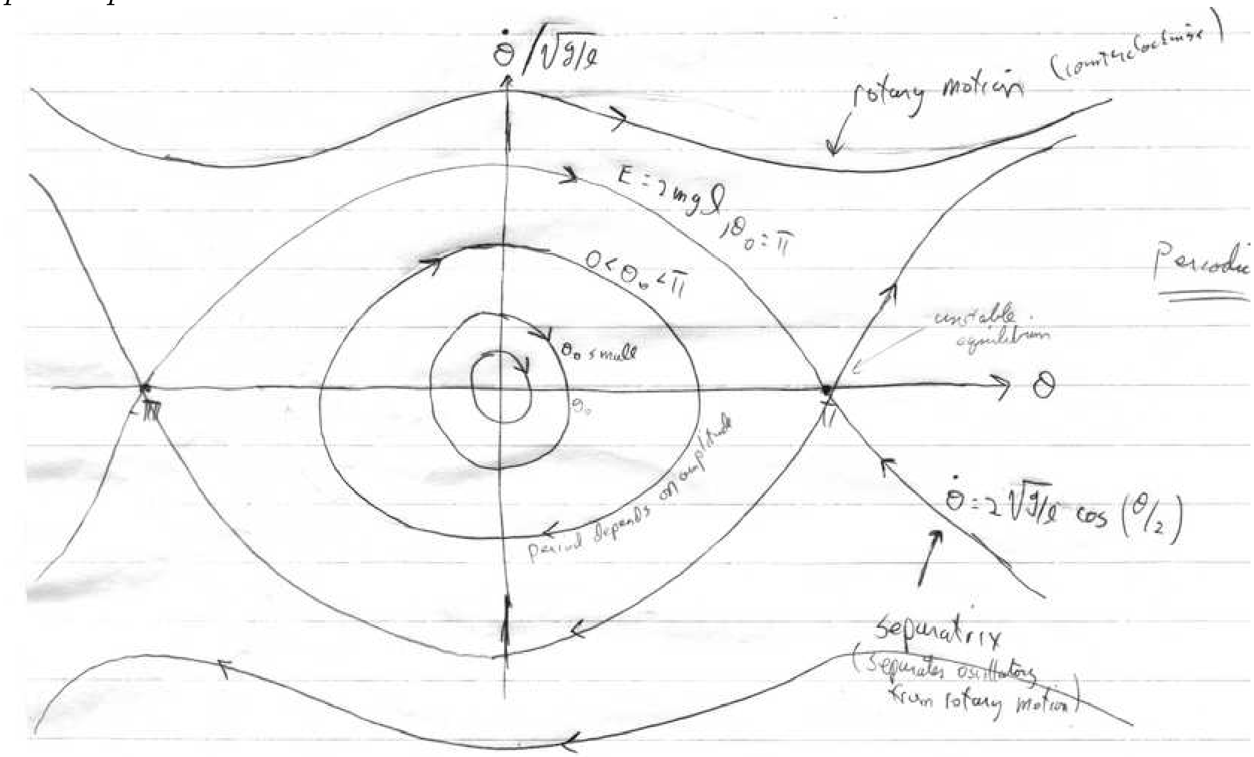
Thus for $\theta_0 = \pi$, the curves are the cosines

$$\dot{\theta} = \pm 2\sqrt{\frac{g}{l}}\,\cos\left(\frac{\theta}{2}\right).$$



Intuitively, we recognize that this curve separates oscillatory motion ($E < 2mgl$) from rotary motion ($E > 2mgl$).

Thus for undampled, nonlinear pendulum we can construct the following *phase portrait*:



The portrait is periodic.

The points $\dot{\theta} = 0$, $\theta = \ldots, -2\pi, 0, 2\pi, \ldots$ are *stable* equilibrium, or fixed, points (actually, *marginally stable*).

The points $\dot{\theta} = 0$, $\theta = \ldots, -3\pi, -\pi, \pi, 3\pi \ldots$ are *unstable* fixed points.

The trajectories appear to cross, but they do not. *Why not?*
(Deterministic trajectories.)

If the trajectories actually arrive to these crossing points, then what happens? (The motion stops, awaiting instability. But we shall see that it would take infinite time to arrive at these points.)