7 Fourier transforms

Except in special, idealized cases (such as the linear pendulum), the precise oscillatory nature of an observed time series $x(t)$ may not be identified from $x(t)$ alone.

We may ask

- How well-defined is the the dominant frequency of oscillation?
- How many frequencies of oscillation are present?
- What are the relative contributions of all frequencies?

The analytic tool for answering these and myriad related questions is the *Fourier transform.*

7.1 Continuous Fourier transform

We first state the Fourier transform for functions that are continuous with time.

The Fourier transform of some function $f(t)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \mathrm{d}t$$

Similarly, the inverse Fourier transform is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \mathrm{d}\omega.$$

That the second relation is the inverse of the first may be proven, but we save that calculation for the discrete transform, below.

7.2 Discrete Fourier transform

We are interested in the analysis of experimental (or numerical) data, which is almost always discrete. Thus we specialize to *discrete Fourier transforms*.

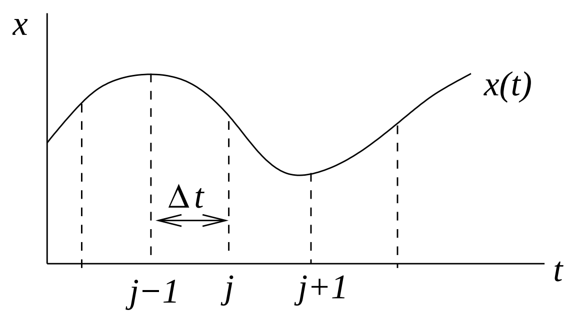
In modern data, one almost always observes a discretized signal

$$x_j, \qquad j = \{0, 1, 2, \ldots, n-1\}$$

We take the *sampling interval*—the time between samples—to be Δt. Then

$$x_j = x(j\Delta t).$$

The discretization process is pictured as

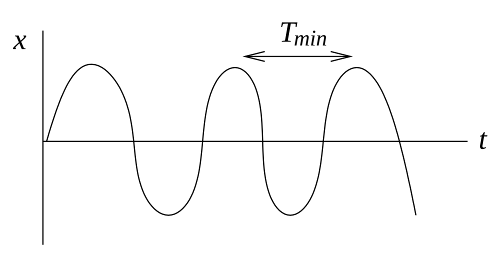


A practical question concerns the choice of Δt. To choose it, we must know the highest frequency, f_{max}, contained in $x(t)$.

The shortest period of oscillation is

$$T_{\min} = 1/f_{\max}$$

Pictorially,



We require at least two samples per period. Therefore

$$\Delta t \leq \frac{T_{\min}}{2} = \frac{1}{2f_{\max}}.$$

The discrete Fourier transform (DFT) of a time series $x_j, j = 0, 1, \ldots, n-1$ is

$$\hat{x}_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(-i\frac{2\pi jk}{n}\right) \qquad k = 0, 1, \ldots, n-1$$

To gain some intuitive understanding, consider the range of the exponential multiplier.

- $k = 0 \Rightarrow \exp(-i2\pi jk/n) = 1$. Then

  $$\hat{x}_0 = \frac{1}{\sqrt{n}} \sum_j x_j$$

  Thus $\hat{x}_0$ is, within a factor of $1/\sqrt{n}$, equal to the mean of the x_j's.

  This is the "DC" component of the transform.

  Question: Suppose a seismometer measures ground motion. What would $\hat{x}_0 \neq 0$ mean?

- $k = n/2 \Rightarrow \exp(-i2\pi jk/n) = \exp(-i\pi j)$. Then

  $$\hat{x}_{n/2} = \frac{1}{\sqrt{n}} \sum_j x_j (-1)^j \qquad (19)$$

  $$= x_0 - x_1 + x_2 - x_3 \ldots \qquad (20)$$

  Frequency index $n/2$ is clearly the highest accessible frequency.

- The frequency indices $k = 0, 1, \ldots, n/2$ correspond to frequencies

  $$f_k = k/t_{\max},$$

  i.e., k oscillations per $t_{\max}$, the period of observation.
  Index $k = n/2$ then corresponds to

  $$f_{\max} = \left(\frac{n}{2}\right)\left(\frac{1}{n\Delta t}\right) = \frac{1}{2\Delta t}$$

But if $n/2$ is the highest frequency that the signal can carry, what is the significance of $\hat{x}_k$ for $k > n/2$?

For real x_j, frequency indicies $k > n/2$ are *redundant*, being related by

$$\hat{x}_k = \hat{x}^*_{n-k}$$

where z^* is the complex conjugate of z (i.e., if $z = a + ib, z^* = a - ib$).

We derive this relation as follows. From the definition of the DFT, we have

$$\begin{aligned}
\hat{x}^*_{n-k} &= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(+i\frac{2\pi j(n-k)}{n}\right) \\
&= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \underbrace{\exp(i2\pi j)}_{1} \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x_j \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \hat{x}_k
\end{aligned}$$

where the + in the first equation derives from the complex conjugation, and the last line again employs the definition of the DFT.

Note that we also have the relation

$$\hat{x}^*_{-k} = \hat{x}^*_{n-k} = \hat{x}_k.$$

The frequency indicies $k > n/2$ are therefore sometimes referred to as *negative frequencies*

7.3 Inverse DFT

The inverse DFT is given by

$$x_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \hat{x}_k \exp\left(+i\frac{2\pi jk}{n}\right) \qquad j = 0, 1, \ldots, n-1$$

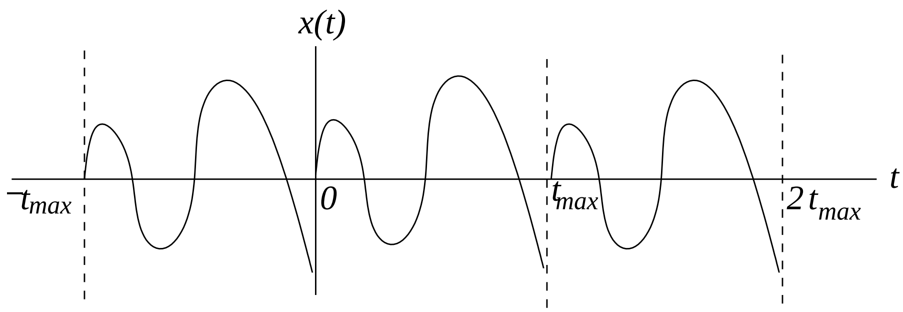
We proceed to demonstrate this inverse relation. We begin by substituting the DFT for $\hat{x}_k$, using dummy variable j':

$$\begin{aligned}
x_j &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n}} \left[\sum_{j'=0}^{n-1} x_{j'} \exp\left(-i\frac{2\pi j'k}{n}\right) \right] \exp\left(+i\frac{2\pi kj}{n}\right) \\
&= \frac{1}{n} \sum_{j'=0}^{n-1} x_{j'} \sum_{k=0}^{n-1} \exp\left(-i\frac{2\pi k(j'-j)}{n}\right) \\
&= \frac{1}{n} \sum_{j'=0}^{n-1} x_{j'} \times \begin{cases} n, & j'=j \\ 0, & j' \neq j \end{cases} \\
&= \frac{1}{n}(nx_j) \\
&= x_j
\end{aligned}$$

The third relation derives from the fact that the previous $\sum_k$ amounts to a sum over the unit circle in the complex plane, except when $j'=j$. The sum over the circle always sums to zero. For example, consider $j'-j=1, n=4$. The elements of the sum are then just the four points on the unit circle that intersect the real and imaginary axes, i.e., the

$$\begin{aligned}
\sum_{k=0}^{n-1} \exp\left(-i\frac{2\pi k(j'-j)}{n}\right) &= e^0 + e^{-i\pi/2} + e^{-i\pi} + e^{-i3\pi/2} \\
&= 1 + i - 1 - i \\
&= 0.
\end{aligned}$$

Finally, note that the DFT relations imply that x_j is periodic in n, so that $x_{j+n} = x_j$. This means that a finite time series is treated precisely as if it were recurring, as illustrated below:



7.4 Autocorrelations, power spectra, and the Wiener-Khintchine theorem

Assume that the time series x_j has zero mean and that it is periodic, i.e., $x_{j+n} = x_j$.

Define the **autocorrelation function** ψ:

$$\psi_m = \frac{1}{n} \sum_{j=0}^{n-1} x_j x_{j+m}$$

where

$$\psi_m = \psi(m\Delta t)$$

The autocorrelation function measures the degree to which a signal resembles itself over time. Thus it measures the predictability of the future from the past. Some intuition may be gained as follows:

- Consider, for example, $m = 0$. Then

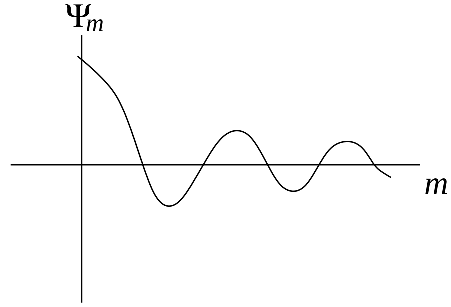
  $$\psi_0 = \frac{1}{n} \sum_{j=0}^{n-1} x_j^2,$$

  which is the mean squared value of x_j (i.e., its variance).

- Alternatively, if $m\Delta t$ is much less than the dominant period of the data, ψ_m should not be too much less than ψ_0.

- Last, if $m\Delta_t$ is much greater than the dominant period of the data, $|\psi_m|$ is relatively small.

A typical ψ_m looks like



Define the **power spectrum** to be the magnitude squared of the Fourier transform; i.e.,

$$|\hat{x}_k|^2 = \frac{1}{n}\left|\sum_{j=0}^{n-1} x_j \exp\left(-i\frac{2\pi jk}{n}\right)\right|^2.$$

We proceed to show that for real time series x_j,

autocorrelation $\propto$ Fourier transform of the power spectrum.

This is called the the *Wiener-Khintchine theorem.* We proceed to derive this relation.

Substitute the inverse DFT for x_j in ψ_m:

$$\begin{aligned}
\psi_m &= \frac{1}{n}\sum_{j=0}^{n-1}\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\hat{x}_k \exp\left(i\frac{2\pi kj}{n}\right)\right]\left[\frac{1}{\sqrt{n}}\sum_{k'=0}^{n-1}\hat{x}_{k'} \exp\left(i\frac{2\pi k'(j+m)}{n}\right)\right] \\
&= \frac{1}{n^2}\sum_{k=0}^{n-1}\sum_{k'=0}^{n-1}\hat{x}_k\hat{x}_{k'}\exp\left(i\frac{2\pi mk'}{n}\right)\underbrace{\sum_{j=0}^{n-1}\exp\left(i\frac{2\pi j(k+k')}{n}\right)}_{\substack{=n,\quad k'=n-k \\ =0,\quad k'\neq n-k}} \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\hat{x}_k\hat{x}_{n-k}\exp\left(i\frac{2\pi m(n-k)}{n}\right) \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\hat{x}_k\hat{x}_k^*\exp\left(-i\frac{2\pi mk}{n}\right)
\end{aligned}$$

In the last line we have used the redundancy relation $\hat{x}_k^* = \hat{x}_{n-k}$.

We thus find that

$\psi_m \propto$ Fourier transform of the power spectrum $\hat{x}_k\hat{x}_k^* = |\hat{x}_k|^2$

Of course the inverse relation holds also.

For real time series $\{x_j\}$, the power spectrum contains redundant information that is similar to that of the Fourier transform but more severe:

$$|\hat{x}_k|^2 \;=\; \hat{x}_k\hat{x}_k^* \;=\; \hat{x}_k\hat{x}_{n-k} \;=\; \hat{x}_{n-k}^*\hat{x}_{n-k} \;=\; |\hat{x}_{n-k}|^2.$$

This redundancy results from the fact that neither the autorcorrelation nor the power spectra contain information on any "phase lags" in either x_j or its individual frequency components.

Thus while the DFT of an n-point time series results in n independent quantities ($2 \times n/2$ complex numbers), the power spectrum yields only $n/2$ independent quantities.

One may therefore show that there are an infinite number of time series that have the same power spectrum, but that each time series uniquely defines its Fourier transform, and vice-versa.

Consequently a time series cannot be reconstructed from its power spectrum or autocorrelation function.

7.5 Power spectrum of a periodic signal

Consider a periodic signal

$$x(t) = x(t+T) = x\left(t + \frac{2\pi}{\omega}\right)$$

Consider the extreme case where the period T is equal to the duration of the signal:

$$T = t_{\max} = n\triangle t$$

The Fourier components are separated by

$$\Delta f = \frac{1}{t_{\max}}$$

i.e. at frequencies

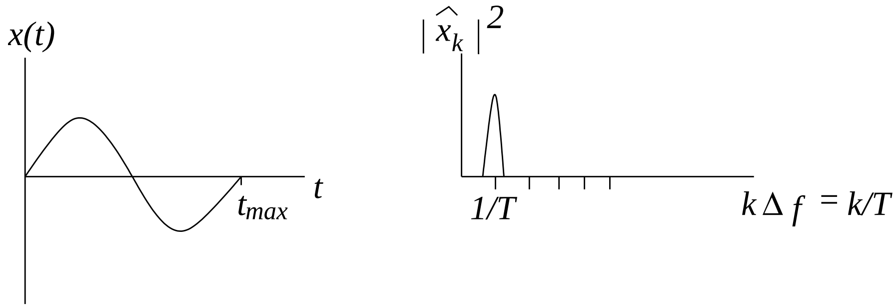
$$0,\ 1/T,\ 2/T, \ldots, (n-1)/T.$$

7.5.1 Sinusoidal signal

In the simplest case, $x(t)$ is a sine or cosine, i.e.,

$$x(t) = \sin\left(\frac{2\pi t}{t_{\max}}\right).$$

What is the Fourier tranform? Pictorially, we expect



We proceed to calculate the power spectrum analytically, beginning with the DFT:

$$
\begin{aligned}
\hat{x}_k &= \frac{1}{\sqrt{n}} \sum_j x_j \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \frac{1}{\sqrt{n}} \sum_j \sin\left(\frac{2\pi j \Delta t}{t_{\max}}\right) \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \frac{1}{2i\sqrt{n}} \sum_j \left[\exp\left(\frac{i2\pi j \Delta t}{t_{\max}}\right) - \exp\left(\frac{-i2\pi j \Delta t}{t_{\max}}\right)\right] \exp\left(\frac{-i2\pi jk}{n}\right) \\
&= \frac{1}{2i\sqrt{n}} \sum_j \left[\exp\left\{i2\pi j\left(\frac{\Delta t}{t_{\max}} - \frac{k}{n}\right)\right\} - \exp\left\{-i2\pi j\left(\frac{\Delta t}{t_{\max}} + \frac{k}{n}\right)\right\}\right] \\
&= \pm\frac{\sqrt{n}}{2i} \qquad \text{when } k = \frac{\pm n \Delta t}{t_{\max}}
\end{aligned}
$$

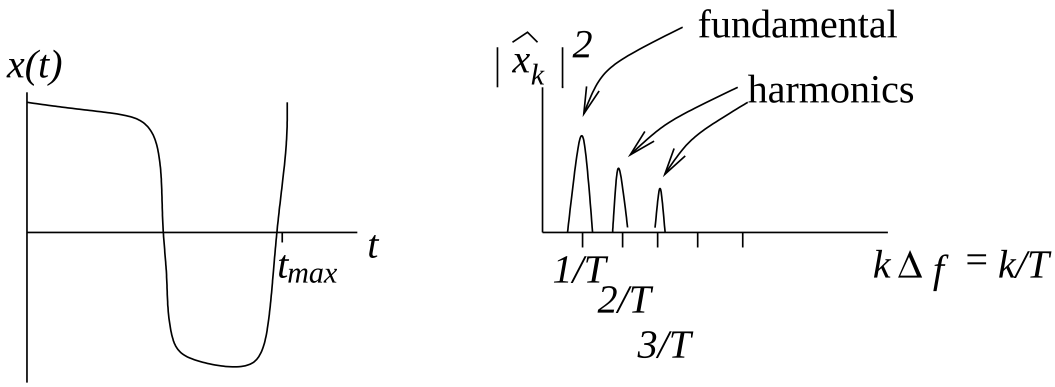
Thus

$$
|\hat{x}_j|^2 = \frac{n}{4} \qquad \text{for } k = \pm 1.
$$

7.5.2 Non-sinusoidal signal

Consider now a non-sinusoidal yet periodic signal, e.g., a relaxation oscillation as obtained from the van der Pol system.

The non-sinusoidal character of the relaxation oscillation implies that it contains higher-order **harmonics**, i.e., integer multiples of the **fundamental frequency** $1/T$. Thus, pictorially, we expect



Now suppose $t_{\max} = pT$, where p is an integer. The non-zero components of the power spectrum must still be at frequencies

$$1/T, 2/T, \ldots.$$

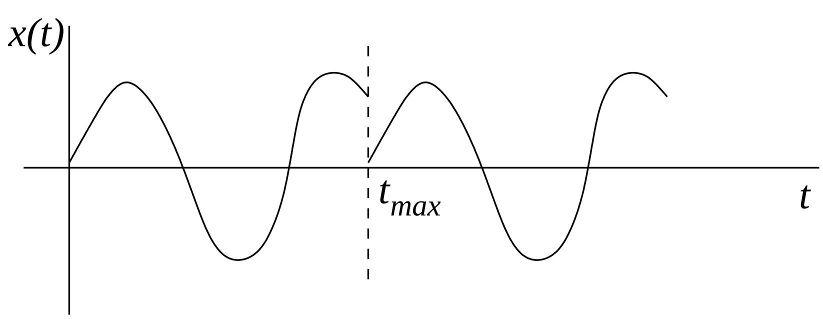
But since

$$\Delta f = \frac{1}{t_{\max}} = \frac{1}{pT}$$

the frequency resolution is p times greater. Contributions to the power spectrum would remain at integer multiples of the frequency $1/T$, but spaced p samples apart on the frequency axis.

7.5.3 $t_{\max}/T \neq$ integer

If $t_{\max}/T$ is not an integer, the (effectively periodic) signal looks like



We proceed to calculate the power spectrum of such a signal. Assume the sinusoidal function

$$x(t) = \exp\left(i\frac{2\pi t}{T}\right)$$

which yields

$$x_j = \exp\left(i\frac{2\pi j\Delta t}{T}\right)$$

The DFT is

$$\hat{x}_k = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\exp\left(i\frac{2\pi j\Delta t}{T}\right)\exp\left(-i\frac{2\pi jk}{n}\right)$$

Set

$$\phi_k = \frac{\Delta t}{T} - \frac{k}{n}.$$

Then

$$\hat{x}_k = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}\exp\left(i2\pi\phi_k j\right)$$

Recall the identity

$$\sum_{j=0}^{n-1} x^j = \frac{x^n - 1}{x - 1}$$

Then

$$\hat{x}_k = \frac{1}{\sqrt{n}}\,\frac{\exp(i2\pi\phi_k n) - 1}{\exp(i2\pi\phi_k) - 1}$$

The power spectrum is

$$\begin{aligned} |\hat{x}_k|^2 = \hat{x}_k\hat{x}_k^* &= \frac{1}{n}\left(\frac{1-\cos(2\pi\phi_k n)}{1-\cos(2\pi\phi_k)}\right) \\ &= \frac{1}{n}\left(\frac{\sin^2(\pi\phi_k n)}{\sin^2(\pi\phi_k)}\right) \end{aligned}$$

Note that

$$n\phi_k = \frac{n\Delta t}{T} - k = \frac{t_{\max}}{T} - k$$

is the difference between a DFT index k and the "real" non-integral frequency index $t_{\max}/T$.

Assume that n is large and k is close to that "real" frequency index such that

$$n\phi_k = \frac{n\Delta t}{T} - k \ll n.$$

Consequently $\phi_k \ll 1$, so we may also assume

$$\pi\phi_k \ll 1.$$

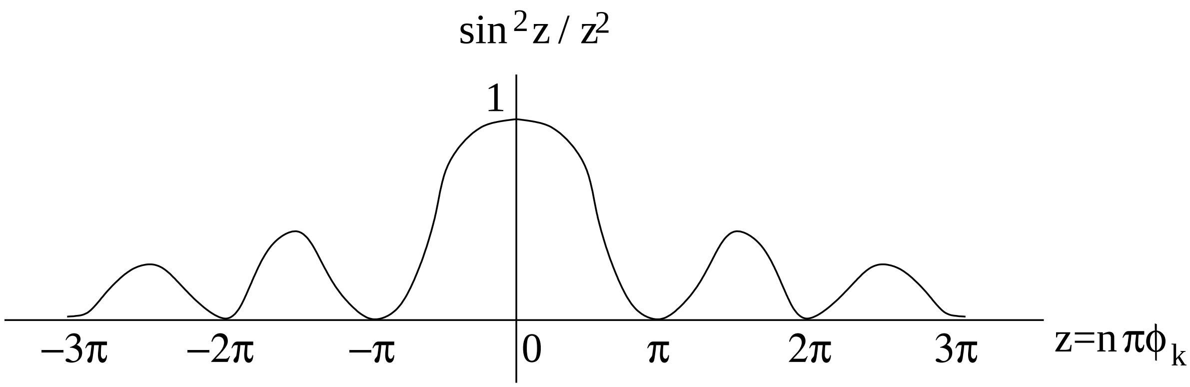
Then

$$\begin{aligned} |\hat{x}_k|^2 &\simeq \frac{1}{n}\,\frac{\sin^2(\pi\phi_k n)}{(\pi\phi_k)^2} \\ &= n\,\frac{\sin^2(\pi\phi_k n)}{(\pi\phi_k n)^2} \\ &\propto \frac{\sin^2 z}{z^2} \end{aligned}$$

where

$$z = n\pi\phi_k = \pi\left(\frac{n\Delta t}{T} - k\right) = \pi\left(\frac{t_{\max}}{T} - k\right)$$

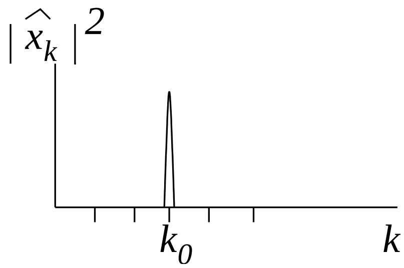
Thus $|\hat{x}_k|^2$ is no longer a simple spike. Instead, as a function of $z = n\pi\phi_k$ it appears as

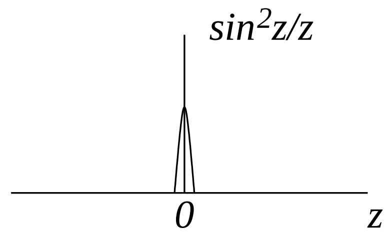


The plot gives the kth component of the power spectrum of $e^{i2\pi t/T}$ as a function of $\pi(t_{\max}/T - k)$.

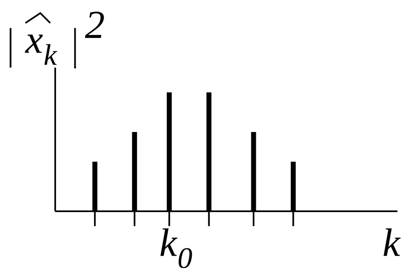
To interpret the plot, let k_0 be the integer closest to $t_{\max}/T$. There are then two extreme cases:

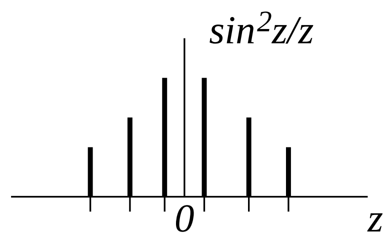
1. $t_{\max}$ is an integral multiple of T:
$$\frac{t_{\max}}{T} - k_0 = 0.$$
The spectrum is perfectly sharp:





2. $t_{\max}/T$ falls midway between two frequencies. Then
$$\frac{t_{\max}}{T} - k_0 = \frac{1}{2}.$$
The spectrum is smeared:





The smear decays like
$$\frac{1}{(k - t_{\max}/T)^2} \sim \frac{1}{k^2}$$

7.5.4 Conclusion

The power spectrum of a periodic signal of period T is composed of:

1. a peak at the frequency $1/T$
2. a smear (sidelobes) near $1/T$
3. possibly harmonics (integer multiples) of $1/T$
4. smears near the harmonics.

7.6 Quasiperiodic signals

Let y be a function of r independent variables:

$$y = y(t_1, t_2, \ldots, t_r).$$

y is **periodic**, of period 2π in *each* argument, if

$$y(t_1, t_2, \ldots, t_j + 2\pi, \ldots, t_r) = y(t_1, t_2, \ldots, t_j, \ldots, t_r), \quad j = 1, \ldots, r$$

y is called **quasiperiodic** if each t_j varies with time at a different rate (i.e., different "clocks"). We have then

$$t_j = \omega_j t, \qquad j = 1, \ldots, r.$$

The quasiperiodic function y has r fundamental frequencies:

$$f_j = \frac{\omega_j}{2\pi}$$

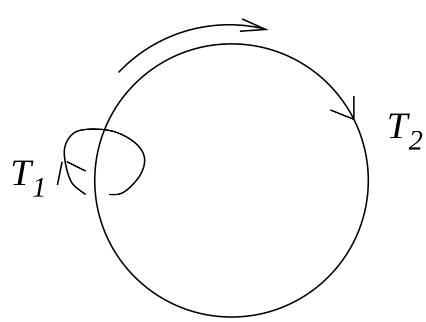
and r periods

$$T_j = \frac{1}{f_j} = \frac{2\pi}{\omega_j}.$$

Example: The astronomical position of a point on Earth's surface changes due to

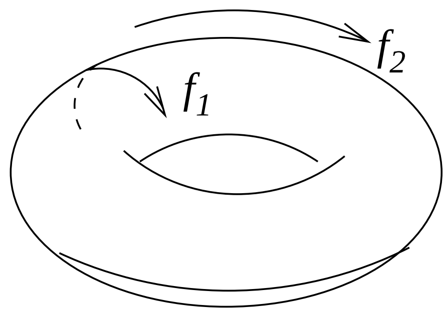
- rotation of Earth about axis ($T_1 = 24$ hours).
- revolution of Earth around sun ($T_2 \simeq 365$ days).
- we ignore precession and other orbital changes.

Mathematically, we can conceive of such a function on a 2-D torus T^2, existing in a 3-D space.



Here we think of a disk spinning with period T_1 while it revolves along the circular path with period T_2.

Such behavior can be conceived as a trajectory on the *surface* of a doughnut or inner tube, or a torus T_2 in $\mathbb{R}^3$.



What is the power spectrum of a quasiperiodic signal $x(t)$? There are two possibilities:

1. The quasiperiodic signal is a *linear* combination of independent periodic functions. For example:

$$x(t) = \sum_{i=1}^{r} x_i(\omega_i t).$$

   Because the Fourier transform is a linear transformation, the power spectrum of $x(t)$ is a set of peaks at frequencies

$$f_1 = \omega_1/2\pi,\ f_2 = \omega_2/2\pi, \ldots$$

   and their harmonics

$$m_1 f_1,\ m_2 f_2, \ldots \qquad (m_1, m_2, \ldots \text{ positive integers}).$$

2. The quasiperiodic signal $x(t)$ depends nonlinearly on periodic functions. For example,

$$x(t) = \sin(2\pi f_1 t)\sin(2\pi f_2 t) = \frac{1}{2}\cos(|f_1 - f_2|2\pi t) - \frac{1}{2}\cos(|f_1 + f_2|2\pi t).$$

   The fundamental frequencies are

$$|f_1 - f_2| \qquad \text{and} \qquad |f_1 + f_2|.$$

The harmonics are

$$m_1|f_1 - f_2| \qquad \text{and} \qquad m_2|f_1 + f_2|, \qquad m_1, m_2 \text{ positive integers.} \quad (21)$$

The nonlinear case requires more attention. In general, if $x(t)$ depends nonlinearly on r periodic functions, then the harmonics are

$$|m_1 f_1 + m_2 f_2 + \ldots + m_r f_r|, \qquad m_i \text{ arbitrary integers.}$$

(This is the most general case, for which equation (21) is a specific example. The expression above derives from $m_1 f_1 \pm m_2 f_2 \pm \ldots$, with m_i positive)

We proceed to specialize to $r = 2$ frequencies, and forget about finite Δf.

Each nonzero component of the spectrum of $x(\omega_1 t, \omega_2 t)$ is a peak at

$$f = |m_1 f_1 + m_2 f_2|, \qquad m_1, m_2 \text{ integers .}$$

There are two cases:

1. f_1/f_2 rational $\Rightarrow$ *sparse spectrum.*
2. f_1/f_2 irrational $\Rightarrow$ *dense spectrum.*

To understand this, rewrite f as

$$f = f_2 \left| m_1 \frac{f_1}{f_2} + m_2 \right| .$$

In the rational case,

$$\frac{f_1}{f_2} = \frac{\text{integer}}{\text{integer}}.$$

Then

$$\left| m_1 \frac{f_1}{f_2} + m_2 \right| = \left| \frac{\text{integer}}{f_2} + \text{integer} \right| = \text{ integer multiple of } \frac{1}{f_2}.$$

Thus the peaks of the spectrum must separated (i.e., sparse).

Alternatively, if f_1/f_2 is irrational, then m_1 and m_2 may always be chosen so that

$$\left| m_1 \frac{f_1}{f_2} + m_2 \right| \text{ is not similarly restricted.}$$

These distinctions have further implications.

In the rational case,

$$\frac{f_1}{f_2} = \frac{n_1}{n_2}, \qquad n_1, n_2 \text{ integers.}$$

Since

$$\frac{n_1}{f_1} = \frac{n_2}{f_2}$$

the quasiperiodic function is *periodic* with period

$$T = n_1 T_1 = n_2 T_2.$$

All spectral peaks must then be harmonics of the fundamental frequency

$$f_0 = \frac{1}{T} = \frac{f_1}{n_1} = \frac{f_2}{n_2}.$$

Thus the rational quasiperiodic case is in fact periodic, and some writers restrict quasiperiodicity to the irrational case.

Note further that, in the irrational case, the signal never exactly repeats itself.

One may consider, as an example, the case of a child walking on a sidewalk, attempting with uniform steps to never step on a crack (and breaking his mother's back...).

Then if $x(t)$ were the distance from the closest crack at each step, it would only be possible to avoid stepping on a crack if the ratio

$$\frac{\text{step size}}{\text{crack width}}$$

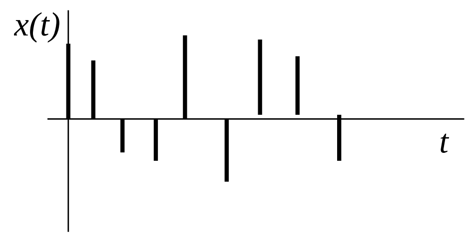
were rational.

7.7 Aperiodic signals

Aperiodic signals are neither periodic nor quasiperiodic.

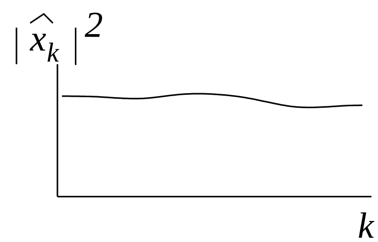
Aperiodic signals appear random, though they may have a deterministic foundation.

An example is white noise, which is a signal that is "new" and unpredictable at each instant, e.g.,



Statistically, each sample of a white-noise signal is independent of the others, and therefore uncorrelated to them.

The power spectrum of white noise is, on average, flat:



The flat spectrum of white noise is a consequence of its lack of harmonic structure (i.e., one cannot recognize any particular tone, or dominant frequency).

We proceed to derive the spectrum of a white noise signal $x(t)$.

Rather than considering only one white-noise signal, we consider an **ensemble** of such signals, i.e.,

$$x^{(1)}(t), x^{(2)}(t), \ldots$$

where the superscipt denotes the particular realization within the ensemble. Each realization is independent of the others.

Now discretize each signal so that

$$x_j = x(j\Delta t), \qquad j = 0, \ldots, n-1$$

We take the signal to have finite length n but consider the ensemble to contain an infinite number of realizations.

We use angle brackets to denote **ensemble averages** (i.e., averages taken over the ensemble).

The ensemble-averaged mean of the jth sample is then

$$\langle x_j \rangle = \lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} x_j^{(i)}$$

Similarly, the mean-square value of the jth sample is

$$\langle x_j^2 \rangle = \lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} (x_j^{(i)})^2$$

Now assume *stationarity*: $\langle x_j \rangle$ and $\langle x_j^2 \rangle$ are independent of j. We take these mean values to be $\langle x \rangle$ and $\langle x^2 \rangle$, respectively, assume $\langle x \rangle = 0$.

Recall the autocorrelation ψ_m:

$$\psi_m = \frac{1}{n} \sum_{j=0}^{n-1} x_j x_{j+m}.$$

By definition, each sample of white noise is uncorrelated with its past and future. Therefore

$$\begin{aligned} \langle \psi_m \rangle &= \left\langle \frac{1}{n} \sum_j x_j x_{j+m} \right\rangle \\ &= \langle x^2 \rangle \delta_m \end{aligned}$$

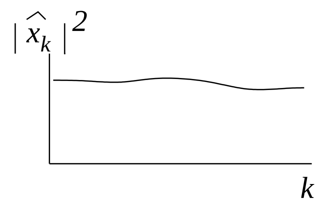
where

$$\delta_m = \begin{cases} 1 & m = 0 \\ 0 & \text{else} \end{cases}$$

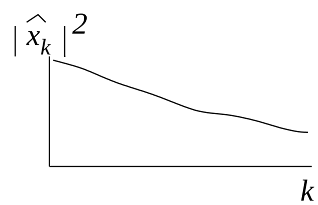
We obtain the power spectrum from the autocorrelation function by the Wiener-Khintchine theorem:

$$
\begin{aligned}
\langle |\hat{x}_k|^2 \rangle &= \sum_{m=0}^{n-1} \langle \psi_m \rangle \exp\left(i \frac{2\pi mk}{n} \right) \\
&= \sum_{m=0}^{n-1} \langle x^2 \rangle \delta_m \exp\left(i \frac{2\pi mk}{n} \right) \\
&= \langle x^2 \rangle \\
&= \text{constant.}
\end{aligned}
$$

Thus for white noise, the spectrum is indeed flat, as previously indicated:

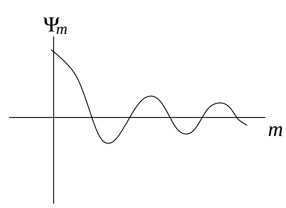


A more common case is "colored" noise: a continuous spectrum, but not constant:



In such (red) colored spectra, there is a relative lack of high frequencies. The signal is still apparently random, but only beyond some interval Δt.

The autocorrelation of colored noise is broader, e.g.,



Finally, we note a problem: power spectra can recognize a signal that is approximately aperiodic, but they cannot distinguish between **deterministic** systems and statistical, **random** systems.

Thus we turn to Poincaré sections.