MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 12, 1996
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QUIZ 1

USEFUL INFORMATION:

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3} G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Flat ($\Omega \equiv \rho/\rho_c = 1$), matter-dominated universe:

$$R(t) \propto t^{2/3}$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

NOTE: Any answer may be expressed in terms of symbols representing the answers to previous parts of the same question.

PROBLEM 1: DID YOU DO THE READING? *(25 points)*

The following questions are worth 5 points each.

a) In 1814-1815, the Munich optician Joseph Frauenhofer allowed light from the sun to pass through a slit and then through a glass prism. The light was spread into a spectrum of colors, showing lines that could be identified with known elements — sodium, iron, magnesium, calcium, and chromium. Were these lines dark, or bright *(2 points)*? Why *(3 points)*?

b) The Andromeda Nebula was shown conclusively to lie outside our own galaxy when astronomers acquired telescopes powerful enough to resolve the individual stars of Andromeda. Was this feat accomplished by Galileo in 1609, by Immanuel Kant in 1755, by Henrietta Swan Leavitt in 1912, by Edwin Hubble in 1923, or by Walter Baade and Allan Sandage in the 1950s?

c) Some of the earliest measurements of the cosmic background radiation were made indirectly, by observing interstellar clouds of a molecule called cyanogen (CN). State whether each of the following statements is true or false *(1 point each)*:

(i) The first measurements of the temperature of the interstellar cyanogen were made over twenty years before the cosmic background radiation was directly observed.

(ii) Cyanogen helps to measure the cosmic background radiation by reflecting it toward the earth, so that it can be received with microwave detectors.

(iii) One reason why the cyanogen observations were important was that they gave the first measurements of the equivalent temperature of the cosmic background radiation at wavelengths shorter than the peak of the black-body spectrum.

(iv) By measuring the spectrum of visible starlight that passes through the cyanogen clouds, astronomers can infer the intensity of the microwave radiation that bathes the clouds.

(v) By observing chemical reactions in the cyanogen clouds, astronomers can infer the temperature of the microwave radiation in which they are bathed.

d) In about 280 B.C., a Greek philosopher proposed that the Earth and the other planets revolve around the sun. What was the name of this person?

e) In 1832 Heinrich Wilhelm Olbers presented what we now know as "Olbers' Paradox," although a similar argument had been discussed as early as 1610 by Johannes Kepler. Olbers argued that if the universe were transparent, static, infinitely old, and was populated by a uniform density of stars similar to our sun, then one of the following consequences would result:

(i) The brightness of the night sky would be infinite.

(ii) Any patch of the night sky would look as bright as the surface of the sun.

(iii) The total energy flux from the night sky would be about equal to the total energy flux from the sun.

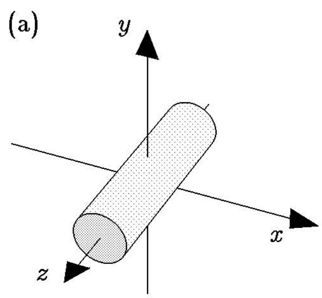
(iv) Any patch of the night sky would look as bright as the surface of the moon.

Which one of these statements is the correct statement of Olbers' paradox?

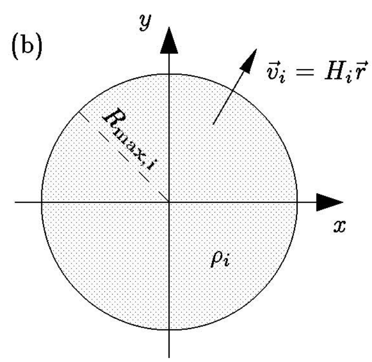
PROBLEM 2: A CYLINDRICAL UNIVERSE *(30 points)*

The following problem was Problem 4, Quiz 2, 1994, and was on the Review Problems for Quiz 1:

The lecture notes showed a construction of a Newtonian model of the universe that was based on a uniform, expanding, sphere of matter. In this problem we will construct a model of a cylindrical universe, one which is expanding in the x and y directions but which has no motion in the z direction. Instead of a sphere, we will describe an infinitely long cylinder of radius $R_{\rm max,i}$, with an axis coinciding with the z-axis of the coordinate system:



3D Perspective



2D Cross Section
(Different scale from (a))

We will use cylindrical coordinates, so

$$r = \sqrt{x^2 + y^2}$$

and

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \ ; \qquad \hat{r} = \frac{\vec{r}}{r} \ ,$$

where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the usual unit vectors along the x, y, and z axes. We will assume that at the initial time t_i, the initial density of the cylinder is ρ_i, and the initial velocity of a particle at position $\vec{r}$ is given by the Hubble relation

$$\vec{v}_i = H_i\vec{r} \ .$$

a) *(5 points)* By using Gauss' law of gravity, it is possible to show that the gravitational acceleration at any point is given by

$$\vec{g} = -\frac{A\mu}{r}\hat{r} \ ,$$

where A is a constant and μ is the total mass per length contained within the radius r. Evaluate the constant A.

b) *(5 points)* As in the lecture notes, we let $r(r_i, t)$ denote the trajectory of a particle that starts at radius r_i at the initial time t_i. Find an expression for $\ddot{r}(r_i, t)$, expressing the result in terms of r, r_i, ρ_i, and any relevant constants. (Here an overdot denotes a time derivative.)

c) *(8 points)* Defining

$$u(r_i, t) \equiv \frac{r(r_i, t)}{r_i} ,$$

show that $u(r_i, t)$ is in fact independent of r_i. This implies that the cylinder will undergo uniform expansion, just as the sphere did in the case discussed in the lecture notes. As before, we define the scale factor $R(t) \equiv u(r_i, t)$.

d) *(5 points)* Express the mass density $\rho(t)$ in terms of the initial mass density ρ_i and the scale factor $R(t)$. Use this expression to obtain an expression for $\ddot{R}$ in terms of R, ρ, and any relevant constants.

e) *(7 points)* Find an expression for a conserved quantity of the form

$$E = \frac{1}{2}\dot{R}^2 + V(R) .$$

What is $V(R)$? Will this universe expand forever, or will it collapse?

PROBLEM 3: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$ *(45 points)*

Consider a ***flat*** universe which is filled with some peculiar form of matter, so that the Robertson–Walker scale factor behaves as

$$R(t) = bt^{3/5} \ ,$$

where b is a constant.

a) *(5 points)* Find the Hubble constant H at an arbitrary time t.

b) *(5 points)* What is the horizon distance at time t?

c) *(5 points)* Suppose a light pulse leaves galaxy A at time t_A and arrives at galaxy B at time t_B. What is the coordinate distance between these two galaxies?

d) *(5 points)* What is the physical separation between galaxy A and galaxy B at time t_A? At time t_B?

e) *(5 points)* At what time is the light pulse equidistant from the two galaxies?

f) *(5 points)* What is the speed of B relative to A at the time t_A? (By "speed," I mean the rate of change of the physical distance with respect to cosmic time, $d\ell_p/dt$.)

g) *(5 points)* For observations made at time t, what is the present value of the physical distance as a function of the redshift Z (and the time t)? What physical distance corresponds to $Z = \infty$? How does this compare with the horizon distance?

h) *(5 points)* Suppose the radiation from galaxy A is emitted with total power P. What is the power per area received at galaxy B?

i) *(5 points)* When the light pulse is received by galaxy B, a pulse is immediately sent back toward galaxy A. At what time does this second pulse arrive at galaxy A?