MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — March 10, 1998
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QUIZ 1 SOLUTIONS

PROBLEM 1: DID YOU DO THE READING?

a) Arno A. Penzias and Robert W. Wilson, Bell Telephone Laboratories.

b) Einstein believed that the universe was static, and the cosmological term was necessary to prevent a static universe from collapsing under the attractive force of normal gravity. [The repulsive effect of a cosmological constant grows linearly with distance, so if the coefficient is small it is important only when the separations are very large. Such a term can be important cosmologically while still being too small to be detected by observations of the solar system or even the galaxy. Recent measurements of distant supernovas ($z \approx 1$), which you may have read about in the newspapers, make it look like maybe there is a cosmological constant after all! Since the cosmological constant is the hot issue in cosmology this season, we will want to look at it more carefully. The best time will be after Lecture Notes 7.]

c) At the time of its discovery, de Sitter's model was thought to be static [although it was known that the model predicted a redshift which, at least for nearby galaxies, was proportional to the distance]. From a modern perspective the model is thought to be expanding.

[It seems strange that physicists in 1917 could not correctly determine if the theory described a universe that was static or expanding, but the mathematical formalism of general relativity can be rather confusing. The basic problem is that when space is not Euclidean there is no simple way to assign coordinates to it. The mathematics of general relativity is designed to be valid for any coordinate system, but the underlying physics can sometimes be obscured by a peculiar choice of coordinates. A change of coordinates can not only distort the apparent geometry of space, but it can also mix up space and time. The de Sitter model was first written down in coordinates that made it look static, so everyone believed it was. Later Arthur Eddington and Hermann Weyl (independently) calculated the trajectories of test particles, discovering that they flew apart.]

d) $n_1 = 3$, and $n_2 = 4$.

e)

(i) True. [In 1941, A. McKellar discovered that cyanogen clouds behave as if they are bathed in microwave radiation at a temperature of about 2.3° K, but no connection was made with cosmology.]

(ii) False. [Any radiation reflected by the clouds is far too weak to be detected. It is the bright starlight shining through the cloud that is detectable.]

(iii) True. [Electromagnetic waves at these wavelengths are mostly blocked by the Earth's atmosphere, so they could not be detected directly until high altitude balloons and rockets were introduced into cosmic background radiation research in the 1970s. Precise data was not obtained until the COBE satellite, in 1990.]

(iv) True. [The microwave radiation can boost the CN molecule from its ground state to a low-lying excited state, a state in which the C and N atoms rotate about each other. The population of this low-lying state is therefore determined by the intensity of the microwave radiation. This population is measured by observing the absorption of starlight passing through the clouds, since there are absorption lines in the visible spectrum caused by transitions between the low-lying state and higher energy excited states.]

(v) False. [No chemical reactions are seen.]

f) Above 3,000 K the universe was so hot that the atoms were ionized, dissociated into nuclei and free electrons. At about this temperature, however, the universe was cool enough so that the nuclei and electrons combined to form neutral atoms.

[This process is usually called "recombination," although the prefix "re-" is totally inaccurate, since in the big bang theory these constituents had never been previously combined. As far as I know the word was first used in this context by P.J.E. Peebles, so I once asked him why the prefix was used. He replied that this word is standard terminology in plasma physics, and was carried over into cosmology.]

[Regardless of its name, recombination was crucial for the clumping of matter into galaxies and stars, because the pressure of the photons in the early universe was enormous. When the matter was ionized, the free electrons interacted strongly with the photons, so the pressure of these photons prevented the matter from clumping. After recombination, however, the matter became very transparent to radiation, and the pressure of the radiation became ineffective.]

[Incidentally, at roughly the same time as recombination (with big uncertainties), the mass density of the universe changed from being dominated by radiation (photons and neutrinos) to being dominated by nonrelativistic matter. There is no known underlying connection between these two events, and it seems to be something of a coincidence that they occurred at about the same time. The transition from radiation-domination to matter-domination also helped to promote the clumping of matter, but the effect was much weaker than the effect of recombination—because of the very high velocity of photons and neutrinos, their pressure remained a significant force even after their mass density became much smaller than that of matter.]

PROBLEM 2: ANOTHER FLAT UNIVERSE WITH AN UNUSUAL TIME EVOLUTION

a) The cosmological redshift z is determined by the usual formula,

$$1 + z = \frac{R(t_0)}{R(t_e)} \quad .$$

For light emitted by an object at time t_e, the redshift of the received light can be found from

$$1 + z = \frac{R(t_0)}{R(t_e)} = \left(\frac{t_0}{t_e}\right)^{\gamma} \quad ,$$

which implies that

$$\boxed{z = \left(\frac{t_0}{t_e}\right)^{\gamma} - 1 \quad .}$$

b) The coordinates t_0 and t_e are cosmic time coordinates. The "look-back" time as defined in the exam is then the interval $t_0 - t_e$. We can write this as

$$t_0 - t_e = t_0 \left(1 - \frac{t_e}{t_0}\right) \quad .$$

We can use the result of part (a) to eliminate t_e/t_0 in favor of z. From (a),

$$\frac{t_e}{t_0} = (1 + z)^{-1/\gamma} \quad .$$

Therefore,

$$\boxed{t_0 - t_e = t_0[1 - (1 + z)^{-1/\gamma}] \quad .}$$

c) The present value of the physical distance to the object, $\ell_p(t_0)$, is found from

$$\ell_p(t_0) = R(t_0) \int_{t_e}^{t_0} \frac{c}{R(t)} dt \quad .$$

Calculating this integral gives

$$\ell_p(t_0) = \frac{c t_0^{\gamma}}{1 - \gamma} \left[\frac{1}{t_0^{\gamma-1}} - \frac{1}{t_e^{\gamma-1}}\right] \quad .$$

Pulling $t_0^{\gamma-1}$ out of the parentheses gives

$$\ell_p(t_0) = \frac{ct_0}{1-\gamma}\left[1 - \left(\frac{t_0}{t_e}\right)^{\gamma-1}\right] \quad .$$

This can be rewritten in terms of z and H_0 using the result of part (a) as well as,

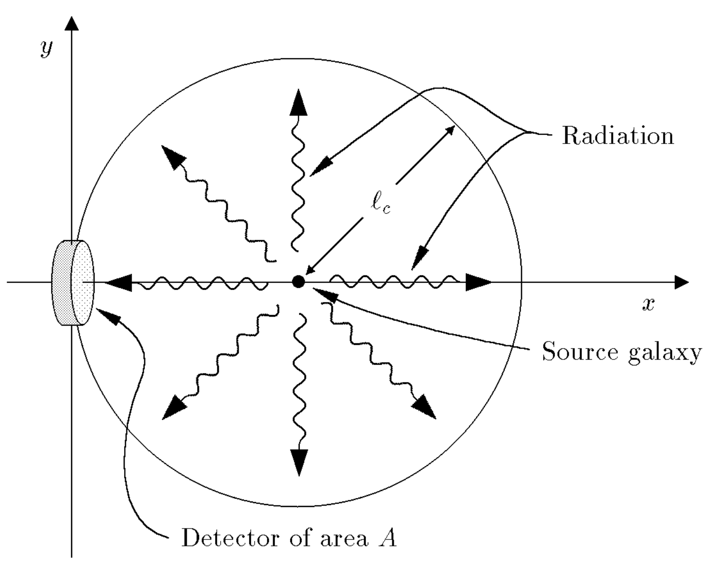
$$H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\gamma}{t_0} \quad .$$

Finally then,

$$\boxed{\ell_p(t_0) = cH_0^{-1}\frac{\gamma}{1-\gamma}\left[1 - (1+z)^{\frac{\gamma-1}{\gamma}}\right] \quad .}$$

d) A nearly identical problem was worked through in Problem 6 of Problem Set 1. The discussion presented here briefly outlines the main elements of the solution and is intended as an indication of what is expected of students in an exam solution.

The energy of the observed photons will be redshifted by a factor of $(1+z)$. In addition the rate of arrival of photons will be redshifted relative to the rate of photon emmission, reducing the flux by another factor of $(1+z)$. Consequently, the observed power will be redshifted by two factors of $(1+z)$ to $P/(1+z)^2$.



Imagine a hypothetical sphere in comoving coordinates as drawn above, centered on the radiating object, with radius equal to the comoving distance ℓ_c. Now consider the photons passing through a patch of the sphere with physical area A. In comoving coordinates the present area of the patch is $A/R(t_0)^2$. Since the object radiates uniformly in all directions, the patch will intercept a fraction $(A/R(t_0)^2)/(4\pi\ell_c^2)$ of the photons passing through the sphere. Thus the power hitting the area A is

$$\frac{(A/R(t_0)^2)}{4\pi\ell_c^2}\frac{P}{(1+z)^2} \ .$$

The radiation energy flux J, which is the received power per area, reaching the earth is then given by

$$J = \frac{1}{4\pi\ell_p(t_0)^2}\frac{P}{(1+z)^2}$$

where we used $\ell_p(t_0) = R(t_0)\ell_c$. Using the result of part (c) to write J in terms of P, H_0, z, and γ gives,

$$\boxed{J = \frac{H_0^2}{4\pi c^2}\left(\frac{1-\gamma}{\gamma}\right)^2 \frac{P}{(1+z)^2\left[1-(1+z)^{\frac{\gamma-1}{\gamma}}\right]^2} \ .}$$

PROBLEM 3: A FLAT UNIVERSE WITH $R(t) \propto t^{3/5}$

a) According to Eq. (3.7) of the Lecture Notes,

$$H(t) = \frac{1}{R(t)}\frac{dR}{dt} \ .$$

For the special case of $R(t) = bt^{3/5}$, this gives

$$H(t) = \frac{1}{bt^{3/5}}\frac{3}{5}bt^{-2/5} = \boxed{\frac{3}{5t} \ .}$$

b) According to Eq. (3.8) of the Lecture Notes, the coordinate velocity of light (in comoving coordinates) is given by

$$\frac{dx}{dt} = \frac{c}{R(t)} \ .$$

Since galaxies A and B have physical separation ℓ_0 at time t_1, their coordinate separation is given by

$$\ell_c = \frac{\ell_0}{bt_1^{3/5}} \ .$$

The radio signal must cover this coordinate distance in the time interval from t_1 to t_2, which implies that

$$\int_{t_1}^{t_2} \frac{c}{R(t)}\, dt = \frac{\ell_0}{b t_1^{3/5}} \ .$$

Using the expression for $R(t)$ and integrating,

$$\frac{5c}{2b}\left(t_2^{2/5} - t_1^{2/5}\right) = \frac{\ell_0}{b t_1^{3/5}} \ ,$$

which can be solved for t_2 to give

$$\boxed{t_2 = \left(1 + \frac{2\ell_0}{5ct_1}\right)^{5/2} t_1 \ .}$$

c) The method is the same as in part (b). The coordinate distance between the two galaxies is unchanged, but this time the distance must be traversed in the time interval from t_2 to t_3. So,

$$\int_{t_2}^{t_3} \frac{c}{R(t)}\, dt = \frac{\ell_0}{b t_1^{3/5}} \ ,$$

which leads to

$$\frac{5c}{2b}\left(t_3^{2/5} - t_2^{2/5}\right) = \frac{\ell_0}{b t_1^{3/5}} \ .$$

Solving for t_3 gives

$$\boxed{t_3 = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 \ .}$$

The above answer is perfectly acceptable, but one could also replace t_2 by using the answer to part (b), which gives

$$\boxed{t_3 = \left(1 + \frac{4\ell_0}{5ct_1}\right)^{5/2} t_1 \ .}$$

[Alternatively, one could have begun the problem by considering the full round trip of the radio signal, which travels a coordinate distance $2\ell_c$ during the time interval from t_1 to t_3. The problem then becomes identical to part (b), except that the coordinate distance ℓ_c is replaced by $2\ell_c$, and t_2 is replaced by t_3. One is led immediately to the answer in the form of the previous equation.]

d) Cosmic time is defined by the reading of suitably synchronized clocks which are each at rest with respect to the matter of the universe at the same location. (For this problem we will not need to think about the method of synchronization.) Thus, the cosmic time interval between the receipt of the message and the response is the same as what is measured on the galaxy B clocks, which is Δt. The response is therefore sent at cosmic time $t_2 + \Delta t$. The coordinate distance between the galaxies is still $\ell_0 / R(t_1)$, so

$$\int_{t_2+\Delta t}^{t_4} \frac{c}{R(t)}\, dt = \frac{\ell_0}{b t_1^{3/5}} \ .$$

Integration gives

$$\frac{5c}{2b}\left[t_4^{2/5} - (t_2 + \Delta t)^{2/5}\right] = \frac{\ell_0}{b t_1^{3/5}} \ ,$$

which can be solved for t_4 to give

$$\boxed{t_4 = \left[\left(\frac{t_2 + \Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{5/2} t_1 \ .}$$

e) From the formula at the front of the exam,

$$1 + z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} = \frac{R(t_4)}{R(t_2 + \Delta t)} = \left(\frac{t_4}{t_2 + \Delta t}\right)^{3/5} \ .$$

So,

$$\boxed{z = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})} = \frac{R(t_4)}{R(t_2 + \Delta t)} = \left(\frac{t_4}{t_2 + \Delta t}\right)^{3/5} - 1 \ .}$$

f) If Δt is small compared to the time that it takes $R(t)$ to change significantly, then the interval between a signal sent at t_3 and a signal sent at $t_3 + \Delta t$ will be received with a redshift identical to that observed between two successive crests of a wave. Thus, the separation between the receipt of the acknowledgement and the receipt of the response will be a factor $(1 + z)$ times longer than the time interval between the sending of the two signals, and therefore

$$\begin{aligned} t_4 - t_3 &= (1 + z)\Delta t + \mathcal{O}(\Delta t^2) \\ &= \left(\frac{t_4}{t_2 + \Delta t}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \ . \end{aligned}$$

Since the answer contains an explicit factor of Δt, the other factors can be evaluated to zeroth order in Δt:

$$\boxed{t_4 - t_3 = \left(\frac{t_4}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \ ,}$$

where to first order in Δt the t_4 in the numerator could equally well have been replaced by t_3.

For those who prefer the brute force approach, the answer to part (d) can be Taylor expanded in powers of Δt. To first order one has

$$t_4 = t_3 + \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t = 0} \Delta t + \mathcal{O}(\Delta t^2) \ .$$

Evaluating the necessary derivative gives

$$\frac{\partial t_4}{\partial \Delta t} = \left[\left(\frac{t_2 + \Delta t}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2 + \Delta t}{t_1}\right)^{-3/5} \ ,$$

which when specialized to $\Delta t = 0$ becomes

$$\left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t = 0} = \left[\left(\frac{t_2}{t_1}\right)^{2/5} + \frac{2\ell_0}{5ct_1}\right]^{3/2} \left(\frac{t_2}{t_1}\right)^{-3/5} \ .$$

Using the first boxed answer to part (c), this can be simplified to

$$\begin{aligned} \left.\frac{\partial t_4}{\partial \Delta t}\right|_{\Delta t = 0} &= \left(\frac{t_3}{t_1}\right)^{3/5} \left(\frac{t_2}{t_1}\right)^{-3/5} \\ &= \left(\frac{t_3}{t_2}\right)^{3/5} \ . \end{aligned}$$

Putting this back into the Taylor series gives

$$t_4 - t_3 = \left(\frac{t_3}{t_2}\right)^{3/5} \Delta t + \mathcal{O}(\Delta t^2) \ ,$$

in agreement with the previous answer.