MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.286: The Early Universe — April 28, 1998
Prof. Alan Guth — Corrected Version

QUIZ 3

USEFUL INFORMATION:

COSMOLOGICAL EVOLUTION:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\left(\rho + \frac{3p}{c^2}\right)R$$

EVOLUTION OF A FLAT ($\Omega \equiv \rho/\rho_c = 1$) UNIVERSE:

$$R(t) \propto t^{2/3} \qquad \text{(matter-dominated)}$$

$$R(t) \propto t^{1/2} \qquad \text{(radiation-dominated)}$$

EVOLUTION OF A MATTER-DOMINATED UNIVERSE:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{kc^2}{R^2}$$

$$\ddot{R} = -\frac{4\pi}{3}G\rho R$$

$$\rho(t) = \frac{R^3(t_i)}{R^3(t)}\,\rho(t_i)$$

Closed ($\Omega > 1$):

$$ct = \alpha(\theta - \sin\theta)\ ,$$

$$\frac{R}{\sqrt{k}} = \alpha(1 - \cos\theta)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{k^{3/2}c^2}$$

Open ($\Omega < 1$):

$$ct = \alpha\left(\sinh\theta - \theta\right)$$

$$\frac{R}{\sqrt{\kappa}} = \alpha\left(\cosh\theta - 1\right)\ ,$$

$$\text{where } \alpha \equiv \frac{4\pi}{3}\frac{G\rho R^3}{\kappa^{3/2}c^2}\ ,$$

$$\kappa \equiv -k\ .$$

COSMOLOGICAL REDSHIFT:

$$1 + Z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{R(t_{\text{observed}})}{R(t_{\text{emitted}})}$$

ROBERTSON-WALKER METRIC:

$$ds^2 = -c^2\,d\tau^2 = -c^2\,dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right\}$$

SCHWARZSCHILD METRIC:

$$ds^2 = -c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$$
$$+\, r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \ ,$$

GEODESIC EQUATION:

$$\frac{d}{d\lambda}\left\{g_{ij}\frac{dx^j}{d\lambda}\right\} = \frac{1}{2}\left(\partial_i g_{k\ell}\right)\frac{dx^k}{d\lambda}\,\frac{dx^\ell}{d\lambda}$$

or: $$\frac{d}{d\tau}\left\{g_{\mu\nu}\frac{dx^\nu}{d\tau}\right\} = \frac{1}{2}\left(\partial_\mu g_{\lambda\sigma}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}$$

COSMOLOGICAL CONSTANT:

$$p_{\text{vac}} = -\rho_{\text{vac}}c^2 \qquad \rho_{\text{vac}} = \frac{\Lambda c^2}{8\pi G}$$

where Λ is the cosmological constant.

PHYSICAL CONSTANTS:

$$k = \text{Boltzmann's constant} = 1.381 \times 10^{-16}\,\text{erg}/^\circ\text{K}$$
$$= 8.617 \times 10^{-5}\,\text{eV}/^\circ\text{K} \ ,$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-27}\ \text{erg-sec}$$
$$= 6.582 \times 10^{-16}\ \text{eV-sec} \ ,$$

$$c = 2.998 \times 10^{10}\ \text{cm/sec}$$

$$1\ \text{eV} = 1.602 \times 10^{-12}\ \text{erg} \ .$$

BLACK-BODY RADIATION:

$$u = g\frac{\pi^2}{30}\,\frac{(kT)^4}{(\hbar c)^3} \qquad p = \frac{1}{3}u \qquad \rho = u/c^2$$

$$n = g^*\frac{\zeta(3)}{\pi^2}\,\frac{(kT)^3}{(\hbar c)^3} \qquad s = g\frac{2\pi^2}{45}\,\frac{k^4T^3}{(\hbar c)^3}\ ,$$

where

$$g \equiv \begin{cases} 1 \text{ per spin state for bosons (integer spin)} \\ 7/8 \text{ per spin state for fermions (half-integer spin)} \end{cases}$$

$$g^* \equiv \begin{cases} 1 \text{ per spin state for bosons} \\ 3/4 \text{ per spin state for fermions}\ , \end{cases}$$

and

$$\zeta(3) = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \approx 1.202\ .$$

EVOLUTION OF A FLAT RADIATION-DOMINATED UNIVERSE:

$$kT = \left(\frac{45\hbar^3c^5}{16\pi^3 gG}\right)^{1/4}\ \frac{1}{\sqrt{t}}$$

For $m_\mu = 106$ MeV $\gg kT \gg m_e = 0.511$ MeV, $g = 10.75$ and then

$$kT = \frac{0.860\ \text{MeV}}{\sqrt{t\ (\text{in sec})}}$$

PROBLEM 1: DID YOU DO THE READING? *(20 points)*

In Silk's chapter on "The First Millisecond," he discusses a number of types of particles that may have been produced. One of those particles is a hypothetical entity called a *photino.*

(a) *(5 points)* The photino is related to the photon by a symmetry. It is plausible that the underlying laws of physics exhibit this symmetry, but at the present time there is no firm evidence that this is true. What is the name of this symmetry?

(b) *(5 points)* If photinos do make up a significant amount of the dark matter, they would occasionally undergo annihilation. Silk mentions two possible products of such annihilations that might be detectable. Name one of them.

In Weinberg's chapter titled "A Historical Diversion," he discusses the history of nucleosynthesis.

(c) *(5 points)* In 1964 an important paper was published, concluding that the large amount of helium in the universe could not have been synthesized in ordinary stars, but could have been produced in the early stages of a "big bang." Surprisingly, one of the authors of this paper is better known for being one of the strongest advocates of the steady-state theory. Name either of the two authors of this paper.

Finally, please answer the following question from the Review Problems for Quiz 3:

(d) *(5 points)* In Chapter 6 of **The First Three Minutes**, Steven Weinberg discusses three reasons why the importance of a search for a 3° K microwave radiation background was not generally appreciated in the 1950s and early 1960s. Choose those three reasons from the following list:

  (i) The earliest calculations erroneously predicted a cosmic background temperature of only about 0.1° K, and such a background would be too weak to detect.

  (ii) There was a breakdown in communication between theorists and experimentalists.

  (iii) It was not technologically possible to detect a signal as weak as a 3° K microwave background until about 1965.

  (iv) Since almost all physicists at the time were persuaded by the steady state model, the predictions of the big bang model were not taken seriously.

  (v) It was extraordinarily difficult for physicists to take seriously *any* theory of the early universe.

  (vi) The early work on nucleosynthesis by Gamow, Alpher, Herman, and Follin, et al., had attempted to explain the origin of all complex nuclei by reactions in the early universe. This program was never very successful, and its credibility was further undermined as improvements were made in the alternative theory, that elements are synthesized in stars.

PROBLEM 2: A REVISED THERMAL HISTORY OF THE UNIVERSE *(35 points)*

As promised there is one question verbatim from the homework or review problems—specifically, from Problem Set 4. Part (a) of this question calls for a numerical answer, but since you were not told to bring calculators, you need not carry out the arithmetic. Your answer should be expressed, however, in "calculator-ready" form— that is, it should be an expression involving pure numbers only (no units), with any necessary conversion factors included, and with the units of the final answer specified at the end. (For example, if you were asked how far a light pulse in vacuum travels in 5 minutes, you could express the answer as $2.998 \times 10^8 \times 5 \times 60$ *meters.) Similarly, in part (d) you may express your answer in terms of a trigonometric or exponential function, which you need not evaluate.*

Suppose a New Theory of the Weak Interactions (NTWI) was proposed, in which the weak interactions are somewhat weaker than in the standard model. This problem will deal with the cosmological consequences of such a theory.

The NTWI will predict that the neutrinos in the early universe will decouple at a higher temperature than in the standard model. Suppose that this decoupling takes place at $kT \approx 200$ MeV. This means that when the neutrinos cease to be thermally coupled to the rest of matter, the hot soup of particles would contain not only photons, neutrinos, and e^+-e^- pairs, but also μ^+, μ^-, π^+, π^-, and π^0 particles. (The muon, you will recall, is a particle that behaves almost identically to an electron, except that its rest energy is 106 MeV. The pions are the lightest of the mesons, with zero angular momentum and rest energies of 135 MeV and 140 MeV for the neutral and charged pions, respectively.)

(a) *(10 points)* According to the standard particle physics model, what is the mass density ρ of the universe when $kT \approx 200$ MeV? What is the value of ρ at this temperature, according to NTWI?

(b) *(10 points)* According to the standard model, the temperature today of the thermal neutrino background should be $(4/11)^{1/3}T_\gamma$, where $T_\gamma \approx 2.7°$ K is the temperature of the thermal photon background. What does the NTWI predict for the temperature of the thermal neutrino background?

(c) *(5 points)* According to the standard model, what is the ratio today of the number density of thermal neutrinos to the number density of thermal photons? What is this ratio according to NTWI?

(d) *(10 points)* Since the reactions that interchange protons and neutrons involve neutrinos, these reactions "freeze out" at roughly the same time as the neutrinos decouple. At later times the only reaction which effectively converts neutrons to protons is the free decay of the neutron. Despite the fact that neutron decay is a weak interaction, we will assume that it occurs with the usual 15 minute mean lifetime. Would the helium abundance predicted by the NTWI be higher or lower than the prediction of the standard model? To within 5 or 10%, what would the NTWI predict for the percent abundance (by weight) of helium in the universe?

PROBLEM 3: THE EFFECT OF PRESSURE ON COSMOLOGICAL EVOLUTION *(20 points)*

A radiation-dominated universe behaves differently from a matter-dominated universe because the pressure of the radiation is significant. In this problem we explore the role of pressure for several fictitious forms of matter.

(a) *(10 points)* For the first fictitious form of matter, the mass density ρ decreases as the scale factor $R(t)$ grows, with the relation

$$\rho(t) \propto \frac{1}{R^5(t)} \ .$$

What is the pressure of this form of matter? *[Hint: the answer is proportional to the mass density.]*

(b) *(5 points)* Find the behavior of the scale factor $R(t)$ for a flat universe dominated by the form of matter described in part (a). You should be able to determine the function $R(t)$ up to a constant factor.

(c) *(5 points)* Now consider a universe dominated by a different form of fictitious matter, with a pressure given by

$$p = \frac{1}{6}\rho c^2 \ .$$

As the universe expands, the mass density of this form of matter behaves as

$$\rho(t) \propto \frac{1}{R^n(t)} \ .$$

Find the power n.

PROBLEM 4: PROPERTIES OF BLACK-BODY RADIATION *(25 points)*

In answering the following questions, remember that you can refer to the formulas at the front of the exam. Since you were not asked to bring calculators, you may leave your answers in the form of algebraic expressions, such as $\pi^{32}/\sqrt{5\varsigma(3)}$.

(a) *(5 points)* For the black-body radiation (also called thermal radiation) of photons at temperature T, what is the average energy per photon?

(b) *(5 points)* For the same radiation, what is the average entropy per photon?

(c) *(5 points)* Now consider the black-body radiation of a massless boson which has spin zero, so there is only one spin state. Would the average energy per particle and entropy per particle be different from the answers you gave in parts (a) and (b)? If so, how would they change?

(d) *(5 points)* Now consider the black-body radiation of electron neutrinos. These particles are fermions with spin 1/2, and we will assume that they are massless and have only one possible spin state. What is the average energy per particle for this case?

(e) *(5 points)* What is the average entropy per particle for the black-body radiation of neutrinos, as described in part (d)?